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ARE ONE-SIDED INVERSES TWO-SIDED INVERSES IN A MATRIX RING OVER A GROUP RING?

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§1. **Introduction.** A ring R with identity element is n-finite(1) if for any pair A, B of $n \times n$ matrices over R, $AB = I_n$ implies $BA = I_n$. In module theoretic terms, R is n-finite if and only if in a free R-module of rank n any generating set of n elements is free. If R is n-finite for all positive integers n then R is said to be *strongly finite*. It is known that all commutative rings, all Artinian rings and all Noetherian rings are strongly finite. These and many other interesting results appear in a paper of P. M. Cohn [1]. In that paper there is a conjecture, attributed to I. Kaplansky, that:

(C_1) The group algebra of any group over any field is strongly finite. A proof of this conjecture for the field of complex numbers appears in [4].

In §2 of this paper an apparent generalization of this conjecture is considered, namely:

(C_2) The group ring of any group over any commutative ring is strongly finite.

It is shown (Theorem 1) that, in fact, (C_1) and (C_2) are equivalent.

A broader generalization, but one which seems to be easier to handle, is:

(C_3) The group ring of any group over any strongly finite ring is strongly finite.

Denote by $\mathscr{F}$ the class of all groups G having the property that the group ring RG is strongly finite for any strongly finite ring R. If $G \in \mathscr{F}$ we say that G is an $\mathscr{F}$-group. Then (C_3) is equivalent to the assertion: $\mathscr{F}$ is the class of all groups. In §3 it is shown that the class $\mathscr{F}$ is closed under taking subgroups and formation of (complete) direct products, that $\mathscr{F}$ contains all finite groups, abelian groups, nilpotent groups and free groups and that any group which is locally or residually an $\mathscr{F}$-group is an $\mathscr{F}$-group.

All rings R appearing in this paper are assumed to have an identity element and any subring S of R is assumed to contain the identity element of R.

The following easily proved results will frequently be used in what follows:

(I) Any subring of an n-finite (strongly finite) ring is n-finite (strongly finite).

(II) A ring is n-finite (strongly finite) if and only if every finitely generated subring is n-finite (strongly finite).

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(1) The terminology is a modification of that in [3]. Also, R is n-finite if, in the terminology of [1], R does not have property γ_n.

(III) If $\{R_\alpha\}_{\alpha\in A}$ is a family of n-finite (strongly finite) rings then $\prod_{\alpha\in A} R_\alpha$, the complete direct product of the R_α, is n-finite (strongly finite).

(IV) If R is strongly finite then so is $(R)_n$, the ring of $n\times n$ matrices over R.

The group ring of the group G over the ring R is denoted by RG. If $\sigma: G\to H$ is a homomorphism of groups with kernel K then σ can be extended by linearity to a ring homomorphism $\bar{\varphi}: RG\to RH$ with kernel $I_G(K)$, the ideal of RG generated by the elements $k-1$ for all $k\in K$ (cf. [2]).

§2. **The Equivalence of** (C_1) **and** (C_2). We first prove

LEMMA 1. *Let R be a ring and $\{J_\alpha\}_{\alpha\in A}$ a family of ideals of R such that*

(i) *R/J_α is n-finite for all $\alpha\in A$,*

(ii) *$J=\bigcap_{\alpha\in A} J_\alpha$ is locally nilpotent.*

Then R is a n-finite.

Proof. Let $\hat{R}$ be the complete direct product of the rings R/J_α, $\alpha\in A$. By (III), §1, $\hat{R}$ is n-finite. The canonical homomorphism $\varphi: R\to\hat{R}$ sending r onto $(r+J_\alpha)_{\alpha\in A}$ has kernel J. Extend φ in the natural manner to a homomorphism $\tilde{\varphi}: (R)_n\to(\hat{R})_n$ of the corresponding matrix rings: $\tilde{\varphi}([a_{ij}])=[\varphi(a_{ij})]$. The kernel of $\tilde{\varphi}$ is $(J)_n$. Since J is locally nilpotent so is $(J)_n$.

Let A, $B\in(R)_n$ and assume $AB=I_n$. Set $D=I_n-BA$. Then $AD=A-ABA=A-I_nA=0$ and so $D^2=(I_n-BA)D=D-BAD=D$. Thus $D^m=D$ for all positive integers m. Now $\tilde{\varphi}(A)\tilde{\varphi}(B)=\tilde{\varphi}(AB)=\tilde{\varphi}(I_n)=I_n$ and so $\tilde{\varphi}(B)\tilde{\varphi}(A)=I_n$ since $\hat{R}$ is n-finite. Hence, $\tilde{\varphi}(D)=\tilde{\varphi}(I_n-BA)=0$ and $D\in(J)_n$. Since $(J)_n$ is locally nilpotent, $D^m=0$ for sufficiently large m. Therefore $D=0$ and $BA=I_n$.

REMARK. Hypothesis (ii) could be replaced by the weaker condition (ii′): $J=\bigcap_{\alpha\in A} J_\alpha$ is locally residually nilpotent.

The equivalence of conjecture (C_1) and (C_2) follows from

THEOREM 1. *Let G be a group. Then RG is n-finite for every commutative ring R if and only if kG is n-finite for every field k.*

Proof. Assume kG is n-finite for all fields k. If R is an integral domain with field of quotients k then RG is a subring of kG and, thus, RG is n-finite.

Now let R be any commutative ring, $\{P_\alpha\}_{\alpha\in A}$ the family of prime ideals of R and $\mathcal{N}=\bigcap_{\alpha\in A} P_\alpha$ the nil radical of R (cf. [6, p. 151]). Since $\mathcal{N}$ is nil and R is commutative, $\mathcal{N}$ is locally nilpotent. Let $J_\alpha=P_\alpha G$, $\alpha\in A$. Then J_α is an ideal of RG and $RG/J_\alpha=RG/P_\alpha G\cong(R/P_\alpha)G$. Since R/P_α is an integral domain, RG/J_α is n-finite. Moreover, $J=\bigcap_{\alpha\in A} J_\alpha=\bigcap_{\alpha\in A} P_\alpha G=\mathcal{N}G$. Since $\mathcal{N}$ is locally nilpotent so is $\mathcal{N}G$. The family of ideals $\{J_\alpha\}_{\alpha\in A}$ thus satisfies hypothesis of Lemma 1. Therefore RG is n-finite.

The opposite implication is obvious.

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§3. **$\mathscr{F}$-groups.** A group G is an $\mathscr{F}$-group if RG is strongly finite for all strongly finite rings R. It is easily seen that $(RG)_n \cong (R)_n G$. It thus follows that G is an $\mathscr{F}$-group if and only if RG is 1-finite for all strongly finite rings R. It is clear that the identity group $\{1\}$ is an $\mathscr{F}$-group, any group isomorphic to an $\mathscr{F}$-group is an $\mathscr{F}$-group and any subgroup of an $\mathscr{F}$-group is an $\mathscr{F}$-group. Since any finite set of elements of RG involve only finitely many elements of G it follows that G is an $\mathscr{F}$-group if and only if every finitely generated subgroup of G is an $\mathscr{F}$-group. In other words, G is an $\mathscr{F}$-group if and only if it is locally an $\mathscr{F}$-group.

LEMMA 1. *Let $G_1, G_2, \ldots, G_k$ be $\mathscr{F}$-groups. Then $G_1 \times G_2 \times \cdots \times G_k$ is an $\mathscr{F}$-group.*

Proof. It suffices to prove the result for $k=2$; the lemma then follows by induction on k. Let G, H be $\mathscr{F}$-groups and R a strongly finite ring. Then RG is strongly finite and, hence, $(RG)H$ is strongly finite. The mapping $\varphi\colon (RG)H \to R(G\times H)$ defined by

$$\sum_{h\in H}\Big(\sum_{g\in G} r(g,h)g\Big)h \to \sum_{(g,h)\in G\times H} r(g,h)(g,h)$$

is easily verified to be a ring isomorphism. Thus $R(G\times H)$ is strongly finite and, therefore, $G\times H$ is an $\mathscr{F}$-group.

THEOREM 2. *Let $\{N_\alpha\}_{\alpha\in A}$ be a family of normal subgroups of G such that, for each $\alpha\in A$, G/N_α is an $\mathscr{F}$-group. Let $N=\bigcap_{\alpha\in A} N_\alpha$. Then G/N is an $\mathscr{F}$-group.*

Proof. By passing to quotients if necessary, we may assume $N=\{1\}$. Let $N_1, \ldots, N_k \in \{N_\alpha\}$. The mapping $G \to G/N_1 \times \cdots \times G/N_k$ given by $g \to (gN_1, \ldots, gN_k)$ is a homomorphism of G into the $\mathscr{F}$-group $G/N_1\times\cdots\times G/N_k$ with kernel $\bigcap_{i=1}^k N_i$. Thus $G/\bigcap_{i=1}^k N_i$ is isomorphic to a subgroup of an $\mathscr{F}$-group and is itself an $\mathscr{F}$-group. Hence we may assume that the set $\{N_\alpha\}$ is closed under finite intersections. Consequently, given finitely many elements $x_1, x_2, \ldots, x_n \in G$ there exists $N_\beta \in \{N_\alpha\}$ such that $x_i \notin N_\beta$, $i=1, 2, \ldots, n$.

Let R be a strongly finite ring. Then $R(G/N_\alpha) \cong RG/I_G(N_\alpha)$ is strongly finite for each $\alpha\in A$. Let $\hat{R}$ denote the complete direct product of the rings $RG/I_G(N_\alpha)$, $\alpha \in A$. Then $\hat{R}$ is strongly finite. Let $\alpha\colon RG \to \hat{R}$ be the canonical homomorphism. The kernel of α is $J=\bigcap_{\alpha\in A} I_G(N_\alpha)$. Suppose $r\in J$, $r\neq 0$. Then $r=\sum_{i=1}^n r(g_i)g_i$ where the g_i, $i=1,2,\ldots,n$, are distinct elements of G and $r(g_i)\neq 0$, $i=1,2,\ldots n$. Let N_β be such that $g_ig_j^{-1}\notin N_\beta$, $i,j=1,2,\ldots,n$, $i\neq j$. Then $g_iN_\beta\neq g_jN_\beta$ if $i\neq j$. Thus the element $\bar{r}=\sum_{i=1}^n r(g_i)g_iN_\beta\neq 0$ in $R(G/N_\beta)$ and so its image $\sum_{i=1}^n r(g_i)g_i+I_G(N_\beta)$ under the natural isomorphism of $R(G/N_\beta)$ onto $RG/I_G(N_\beta)$ is not zero, that is, $r=\sum_{i=1}^n r(g_i)g_i\notin I_G(N_\beta)$, a contradiction. Hence $J=(0)$ and $\alpha\colon RG\to\hat{R}$ is one–one. Since RG is isomorphic to a subring of the strongly finite ring $\hat{R}$, RG is itself strongly finite and, therefore, G is an $\mathscr{F}$-group.

For any group property $\mathscr{E}$, a group G is said to be residually an $\mathscr{E}$-group if there

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exists a family $\{N_\alpha\}$ of normal subgroups of G such that, for each α, G/N_α is an $\mathscr{E}$-group and $\bigcap_\alpha N_\alpha=\{1\}$. Thus an equivalent form of Theorem 2 is

THEOREM 2′. *A group G is an $\mathscr{F}$-group if and only if it is residually an $\mathscr{F}$-group.*

COROLLARY. *If $\{G_\alpha\}_{\alpha\in A}$ is a family of $\mathscr{F}$-groups then the complete direct product $\hat{G}=\prod_{\alpha\in A} G_\alpha$ is an $\mathscr{F}$-group.*

Proof. For each $\alpha\in A$, set $N_\alpha=\{(g_\beta)_{\beta\in A}\mid g_\alpha=1\}$. Then $\hat{G}/N_\alpha\cong G_\alpha$ and $\bigcap_\alpha N_\alpha=\{1\}$. By Theorem 2, $\hat{G}$ is an $\mathscr{F}$-group.

THEOREM 3. *Let G be a group and H a subgroup of G of finite index. If H is an $\mathscr{F}$-group then G is an $\mathscr{F}$-group.*

Proof. Let $G=Hx_1\cup Hx_2\cup\cdots\cup Hx_n$ be a decomposition of G into distinct cosets of H. Let R be a strongly finite ring. Then RG is a free left RH-module having $x_1,\ldots,x_n$ as a free basis. Thus

$$\mathrm{Hom}_{RH}(RG, RG)\cong (RH)_n$$

and, since RH is strongly finite, it follows from IV, §1, that $\mathrm{Hom}_{RH}(RG, RG)$ is strongly finite. For each $\alpha\in RG$ the mapping $\bar{\alpha}$: $RG\to RG$ defined by $(\beta)\bar{\alpha}=\beta\alpha$ is an RH-homomorphism of RG into itself (as a module). The mapping $\alpha\to\bar{\alpha}$ is easily verified to be a ring homomorphism of RG into $\mathrm{Hom}_{RH}(RG, RG)$. If $\bar{\alpha}=0$ then $0=(1)\bar{\alpha}=1\alpha=\alpha$ and, hence, the homomorphism is one–one. Thus RG is isomorphic to a subring of a strongly finite ring and is itself strongly finite. Therefore G is an $\mathscr{F}$-group.

COROLLARY. *Any finite group is an $\mathscr{F}$-group. Therefore all locally finite groups and all residually finite groups are $\mathscr{F}$-groups.*

A result of K. Hirsch asserts that any finitely generated nilpotent group is residually finite (see [5, p. 80]). Consequently, we have

COROLLARY. *Any nilpotent group, locally nilpotent group or residually nilpotent group is an $\mathscr{F}$-group.*

Since abelian groups are nilpotent and free groups are residually nilpotent (see [5, p. 80]) this implies

COROLLARY. *Any abelian group is an $\mathscr{F}$-group. Any free group is an $\mathscr{F}$-group. Thus, any locally free group or residually free group is an $\mathscr{F}$-group.*

The most obvious next stage in the investigation is to examine whether or not solvable groups are $\mathscr{F}$-groups. We have made only the following short steps in this direction.

THEOREM 4. *Let G be a group, N a normal subgroup and assume*

(i) *G/N is abelian,*
(ii) *N is finite.*

Then G is an $\mathscr{F}$-group.

Proof. It is sufficient to assume G/N finitely generated. Then $G/N \cong G_1/N \times \cdots \times G_k/N$ where each factor G_i/N is cyclic. Each finite cyclic factor is an $\mathscr{F}$-group (by Theorem 3) and so the problem reduces to the special case: G/N infinite cyclic. Let xN be a generator of G/N. Then $y \to x^{-1}yx$ is an automorphism of N. Since N has a finite automorphism group, x^m centralizes N for some $m>0$. Thus $N^* = \langle x^m, N\rangle = \langle x^m\rangle \times N$ is an $\mathscr{F}$-group and $[G: N^*]=m$. By Theorem 3, G is an $\mathscr{F}$-group.

REMARK. Theorem 4 remains true if (ii) is replaced by (ii'): N is an $\mathscr{F}$-group and has a periodic automorphism group.

COROLLARY. *Let G be a group, N a normal subgroup and assume*

(i) *G/N is abelian,*
(ii) *N is finitely generated abelian.*

Then G is an $\mathscr{F}$-group.

Proof. It is not difficult to show that N has a family $\{H_\alpha\}$ of *characteristic* subgroups such that N/H_α is finite for each α and $\bigcap_\alpha H_\alpha = \{1\}$. The H_α are then normal in G and, by Theorem 4, G/H_α is an $\mathscr{F}$-group. By Theorem 2, G is an $\mathscr{F}$-group.

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