A NOTE ON ARC-PRESERVING FUNCTIONS FOR MANIFOLDS[1]

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Hall and Puckett [2] have shown that an arc-preserving function defined on a locally connected continuum having no local separating points is a homeomorphism if its total image is not an arc or point. This note shows that their results can be extended to non-compact manifolds.

A function $f: X \to Y$, X and Y topological spaces, is arc-preserving if the image of every arc is an arc or point. And f is locally a homeomorphism if for each $x \in X$ $\exists$ a neighbourhood U of x $\ni$ $f|U$ is a homeomorphism. An n-manifold (n-manifold with boundary) is a connected separable metric space each point of which has a neighbourhood homeomorphic to $E^n(I^n)$.

THEOREM 1. Let f be a function whose domain is a n-manifold M_1 and whose range is in a n-manifold M_2. If f is arc-preserving and $\dim (f(M_1)) > 1$, then f is a homeomorphism.

Proof: As M_1 is a manifold cover it with a countable collection, $\{C_i\}$, of closed n-cells with bicollared boundaries. Now there exists a n-cell C' in $\{C_i\}$ such that $\dim (f(C')) > 1$. For suppose not, as f preserves arcs $f(C_i)$ must be connected and is hence either a point or an arcwise connected one dimensional set. We may suppose that $f(C_i)$ is a subset of an arc, for if not, $f|C_i$ is a homeomorphism

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[1] This result is from the author's doctoral dissertation, Virginia Polytechnic Institute, 1966, directed by Professor P.H. Doyle.

https://doi.org/10.4153/CMB-1967-059-2 Published online by Cambridge University Press

by 4.1 [2]. Thus $f(M_1) = f(\bigcup_{i=1}^{\infty} C_i) = \bigcup_{i=1}^{\infty} f(C_i)$ can be expressed as a countable union of closed arcs or points as each $f(C_i)$ is either a point or expressible as a countable union of arcs. However, this is not possible by the Sum Theorem for Dimension. Therefore, at least one C_i, say C', is such that $\dim (f(C')) > 1$. Then $f|C'$ is a homeomorphism by 4.1 [2]. Suppose now that f is not locally a homeomorphism at $p \in M_1$. Then $p \cup C'$ lies in a n-cell C'' with bicollared boundary by Lemma 1 [1]. Thus $f|C''$ is a homeomorphism by 4.1 [2]. Consequently f is locally a homeomorphism. Suppose f is not 1:1. Let p and q be contained in M_1 and such that $f(p) = f(q)$. Then $p \cup q$ lies interior to a closed n-cell, $\mathrm{cl}\, C''' \supset C''$ in M. Therefore $f|\mathrm{cl}\, C''$ is a homeomorphism. Thus $f(p) \neq f(q)$ and so f is 1:1. To see that f is open and continuous observe that, as f is locally a homeomorphism, it preserves local dimension and $f(M_1)$ is locally Euclidean by Brouwer's Invariance of Domain. Thus by Invariance of Domain f and f^{-1} are open. Therefore f is a homeomorphism.

An analogous theorem for manifolds with boundaries does not hold. But it does follow that:

COROLLARY 1. Let f be a function whose domain is an n-manifold M^n with boundary and whose range is an n-manifold. If f preserves arcs and if $\dim (f(M^n)) > 1$, then f is continuous, 1:1 and $f|\mathrm{int}(M^n)$ is a homeomorphism.

REFERENCES

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