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SMALL DOUBLING IN ORDERED GROUPS

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Abstract

We prove that if S is a finite subset of an ordered group that generates a nonabelian ordered group, then $|S^2| \geq 3|S| - 2$. This generalizes a classical result from the theory of set addition.

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1. Introduction

The structure theory of set addition, or Freiman-type theory, is an area founded by the first named author some time ago, and which concerns the structure of subsets of groups having so-called small 'doubling', see [F]. This area is very popular, see [B, C, GR, GT, HLS, R, S, T], and this paper contributes to the current programme of trying to understand what happens when we move from an abelian to a nonabelian setting.

First we mention the following theorem, which is a classical result in the theory of set addition.

THEOREM 1.1. *Let S be a finite subset of an ordered group. Then*

$$|S^2| \geq 2|S| - 1.$$

PROOF. Let $S = \{x_1, x_2, \ldots, x_k\}$, with $x_1 < x_2 < \cdots < x_k$. Then:

$$x_1^2 < x_1x_2 < x_2^2 < x_2x_3 < x_3^2 < \cdots < x_{k-1}^2 < x_{k-1}x_k < x_k^2$$

and each of these elements belongs to S^2. Hence $|S^2| \geq 2k - 1 = 2|S| - 1$, as required. □

This result is the best possible, as can be seen by considering geometric progressions. However, the critical examples (geometric progressions) are abelian in character and the main result of this paper is the following theorem, which is a strengthening of Theorem 1.1 if the group generated by S is nonabelian.

THEOREM 1.2. *Let S be a finite subset of an ordered group, which generates a nonabelian subgroup. Then*

$$|S^2| \geq 3|S| - 2.$$

Theorem 1.2 can be restated in the following Freiman-type equivalent form.

THEOREM 1.3. *Let S be a finite subset of an ordered group and suppose that*

$$|S^2| \leq 3|S| - 3.$$

Then S generates an abelian subgroup.

This result is the best possible, so that there is an ordered group G and a subset S generating a nonabelian group with $|S^2| = 3|S| - 2$.

We prove Theorem 1.3 in Section 3. Under a bit stronger assumption, we obtain the following extension of Freiman's theorem 1.9 in [F].

COROLLARY 1.4. *Let S be a finite subset of an ordered group G and suppose that*

$$t = |S^2| \leq 3|S| - 4.$$

Then there exist $x_1, g \in G$, such that $g > 1$, $gx_1 = x_1g$ and S is a subset of the geometric progression

$$\{x_1, x_1g, x_1g^2, \ldots, x_1g^{t-|S|}\}.$$

Finally we mention the following interesting result concerning ordered groups, which is proved in Section 2.

COROLLARY 1.5. *Let S be a finite subset of an ordered group G. Then*

$$N_G(S) = C_G(S).$$

Since the class of ordered groups contains the class of *torsion-free nilpotent groups*, our results hold in particular for finite subsets of torsion-free nilpotent groups.

We conclude this section with the following basic definition.

DEFINITION 1.6. If S, T are subsets of a group G, then we denote

$$ST = \{st : s \in S, t \in T\} \quad \text{and} \quad S^2 = \{s_1s_2 : s_1, s_2 \in S\}.$$

If $S = \{s\}$, then we denote ST by sT and if $T = \{t\}$, then we write St instead of $S\{t\}$. If G is an additive group, then we denote

$$2S = \{s_1 + s_2 : s_1, s_2 \in S\}.$$

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2. Finite subsets of ordered groups

We begin this section with the definitions of ordered groups and of orderable groups. We recall some properties of these groups that we shall use in this paper, and we mention some interesting examples of orderable groups.

In the second part of this section we investigate finite subsets in ordered groups.

Definition 2.1. Let G be a group and suppose that a total order relation $<$ is defined on the set G. We say that $(G, <)$ is an *ordered group* if, for all $a, b, x, y \in G$, the inequality $a < b$ implies that $xay < xby$.

A group G is *orderable* if there exists a total order relation $<$ on the set G, such that $(G, <)$ is an ordered group.

The following properties of ordered groups follow easily from the definition (we apply the notation of the definition and denote by 1 the unit element of G).

- If $a < b$ and n is a positive integer, then $a^n < b^n$ and $a^{-n} > b^{-n}$.
- If $a < 1$, then $x^{-1}ax < 1$.
- G is torsion-free.
- If $a, x \in G$ and $a = x^{-1}a^{-1}x$, then $a = 1$.

The next lemma due to B. H. Neumann, see [N], will be very useful in what follows.

Lemma 2.2. *Let $(G, <)$ be an ordered group and let $a, b \in G$. If $[a^n, b] = 1$ for some integer $n \neq 0$, then $[a, b] = 1$.*

Proof. For each integer $m > 0$ we have the following identities:

$$[a^m, b] \equiv (a^{-(m-1)}[a, b]a^{m-1})(a^{-(m-2)}[a, b]a^{m-2}) \cdots (a^{-1}[a, b]a^1)(a^0[a, b]a^0)$$

and

$$[a^{-m}, b] \equiv \prod_{k=-m}^{-1} (a^{-k}[a, b]^{-1}a^k).$$

Suppose that $[a, b] > 1$ ($[a, b] < 1$). Since $[a^m, b]$ is a product of conjugates of $[a, b]$, each of which is > 1 (< 1), it follows that $[a^m, b] > 1$ ($[a^m, b] < 1$). Similarly, it follows that $[a^{-m}, b] < 1$ ($[a^{-m}, b] > 1$). Hence if $[a, b] \neq 1$, then $[a^n, b] \neq 1$ and the result follows. □

There are many examples of orderable groups. An *abelian group* is orderable if and only if it is torsion-free, by a theorem of F. W. Levi, see [L]. K. Iwasawa, see [I], A. I. Mal'cev, see [M], and B. H. Neumann, see [N], proved independently that the class of ordered groups contains the class of *torsion-free nilpotent groups*.

Other examples of solvable orderable groups can be obtained using the following theorem of Kargapolov, see [K].

Theorem 2.3. *A torsion-free group G has the property that every full order for any subgroup of G can be extended to some full order of G if and only if there exists a normal abelian subgroup A of G such that G/A is abelian and, for any $a \in A$ and $b \in G \setminus A$, there exist* positive *integers m, n, $m \neq n$, such that $(a^m)^b = a^n$.*

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More information concerning ordered groups may be found, for example, in [G] and in [BMR].

We now prove an important proposition concerning finite subsets in ordered groups.

Proposition 2.4. *Let* $(G,<)$ *be an ordered group and let* S *be a finite subset of* G *of size* k. *If* $y \in G \setminus C_G(S)$, *then*

$$|yS \cup Sy| \geq k+1.$$

In particular, there exist $x_i, x_j \in S$ *such that* $yx_i \notin Sy$ *and* $x_jy \notin yS$.

Proof. Suppose, to the contrary, that $yS = Sy$. Since $y \notin C_G(S)$, there exists $x_1 \in S$ such that

$$yx_1 \neq x_1y.$$

As $yS = Sy$, there exists $x_2 \in S$ such that $x_2 \neq x_1$ and $yx_1 = x_2y$. Suppose that there exist $x_1, x_2, \ldots, x_t \in S$ such that

$$\begin{aligned} yx_1 &= x_2y \\ yx_2 &= x_3y \\ &\vdots \\ yx_{t-1} &= x_ty, \end{aligned} \tag{2.1}$$

where $x_i = x_j$ if and only if $i = j$.

Since $yS = Sy$, there exists $x_{t+1} \in S$ such that

$$yx_t = x_{t+1}y.$$

We claim that $x_{t+1} \notin \{x_1, x_2, \ldots, x_t\}$. Indeed, if $x_{t+1} = x_u$ for some integer u, $1 \leq u \leq t$, then by (2.1)

$$x_t = y^{-1}x_{t+1}y = y^{-1}x_uy = y^{-2}x_{u+1}y^2 = \cdots = y^{-(t-u+1)}x_ty^{t-u+1}$$

and hence $[x_t, y^{t-u+1}] = 1$. It follows by Lemma 2.2 and (2.1) that $yx_t = x_ty = yx_{t-1}$. But then $x_t = x_{t-1}$, in contradiction to (2.1). This proves our claim. Since this procedure can be carried out indefinitely, we have reached a contradiction to the finiteness of S. Hence $yS \neq Sy$ and the proposition follows. □

From Proposition 2.4 we derive the above Corollary 1.5 (repeated below for convenience) as follows.

Corollary 1.5. *Let* S *be a finite subset of an ordered group* G. *Then*

$$N_G(S) = C_G(S).$$

Proof. If $y \in N_G(S)$, then $yS = Sy$ and it follows from Proposition 2.4 that $y \in C_G(S)$. The opposite containment is trivial. □

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3. The main results

In this section we prove our main results and some corollaries. First we prove Theorem 1.3 (repeated below for convenience).

THEOREM 1.3. *Let S be a finite subset of an ordered group G and suppose that*

$$|S^2| \leq 3k - 3. \tag{$*$}$$

Then S generates an abelian subgroup.

PROOF. Let $S = \{x_1, x_2, \ldots, x_k\}$, with $x_1 < x_2 < \cdots < x_k$.

If $k = 2$, then $|S^2| \leq 3$. As $x_1^2 < x_1x_2 < x_2^2$, it follows that $S^2 = \{x_1^2, x_1x_2, x_2^2\}$ and we must have $x_2x_1 = x_1x_2$, as required.

So assume that $k > 2$ and that all subsets X of G satisfying $2 \leq |X| < k$ and $|X^2| \leq 3|X| - 3$ generate an abelian subgroup. Assume, moreover, that $\langle S \rangle$ is *nonabelian*. Our aim is to reach a contradiction.

Let i be the *maximal* integer such that

$$A = \{x_1, x_2, \ldots, x_i\}$$

generates an *abelian* subgroup. Then

$$1 \leq i < k, \qquad x_{i+1} \notin C_G(A), \qquad x_{i+1} \notin \langle A \rangle \tag{3.1}$$

and there exists $x_j \in A$ such that

$$x_{i+1}x_j \neq x_jx_{i+1}. \tag{3.2}$$

Let x_j be the *maximal* such element of A. Then

$$x_a \in C_G(x_{i+1}) \qquad \text{for each } x_a \in A \text{ satisfying } x_a > x_j. \tag{3.3}$$

Moreover, it follows from (3.1) that

$$A^2 \cap (x_{i+1}A \cup Ax_{i+1}) = \emptyset. \tag{3.4}$$

Write

$$D = \{x_{i+1}, x_{i+2}, \ldots, x_k\}.$$

If $|D| = 1$, then $i = k - 1$ and the order in S implies that $x_k^2 \notin A^2 \cup (x_{i+1}A \cup Ax_{i+1})$. Thus, by (3.4), (3.2), Theorem 1.1 and Proposition 2.4, we get that

$$\begin{aligned} |S^2| &\geq |A^2| + |x_{i+1}A \cup Ax_{i+1}| + |\{x_k^2\}| \geq (2i - 1) + (i + 1) + 1 \\ &= 3i + 1 = 3(k - 1) + 1 = 3k - 2 \end{aligned}$$

in contradiction to ($*$).

So assume that $|D| \geq 2$. We claim that

$$|D^2| \leq 3|D| - 3.$$

First we notice that the order in S implies that

$$D^2 \cap (A^2 \cup x_{i+1}A \cup Ax_{i+1}) = \emptyset. \tag{3.5}$$

This observation, together with (3.4), ($*$), (3.2), Theorem 1.1 and Proposition 2.4, yields the following inequality:

$$\begin{aligned} |D^2| \le |S^2| - |A^2| - |x_{i+1}A \cup Ax_{i+1}| &\le (3k-3) - (2i-1) - (i+1) \\ &= 3(k-i) - 3 = 3|D| - 3. \end{aligned}$$

This proves our claim.

Since $2 \le |D| < k$, it follows by the inductive assumption that $\langle D\rangle$ is *abelian*. In particular,

$$\langle D\rangle \le C_G(x_{i+1}). \tag{3.6}$$

This implies, in view of (3.2), that

$$D^2 \cap (x_jD \cup Dx_j) = \emptyset. \tag{3.7}$$

We claim that

$$Ax_{i+1} \cap x_jD = \{x_jx_{i+1}\}. \tag{3.8}$$

Indeed, suppose that

$$x_ax_{i+1} = x_jx_d \quad \text{for some } x_a \in A \text{ and } x_d \in D. \tag{3.9}$$

If $x_a > x_j$, then it follows by (3.9), (3.3) and (3.6) that $x_j \in \langle x_a, x_{i+1}, x_d\rangle \le C_G(x_{i+1})$, in contradiction to (3.2). On the other hand, if $x_a < x_j$, then it follows by (3.9) that $x_{i+1} > x_d$, which is impossible, since x_{i+1} is the smallest element in D. Thus $x_a = x_j$, $x_d = x_{i+1}$ and our claim follows. Since $|Ax_{i+1}| = |A| = i$ and $|x_jD| = |D| = k - i$, (3.8) implies that

$$|Ax_{i+1} \cup x_jD| = k - 1. \tag{3.10}$$

We also claim that

$$A^2 \cap (x_jD \cup Dx_j) = \emptyset. \tag{3.11}$$

Indeed, suppose that there exist $x_a, x_b \in A$ and $x_d \in D$ satisfying

$$x_ax_b = x_jx_d.$$

Since $x_b < x_d$, it follows that $x_a > x_j$. But $\langle A\rangle$ is abelian, so $x_ax_b = x_bx_a$ and similarly we get $x_b > x_j$. Thus, by (3.3) and (3.6), $x_j \in \langle x_a, x_b, x_d\rangle \le C_G(x_{i+1})$, in contradiction to (3.2). Hence $A^2 \cap x_jD = \emptyset$ and a similar proof yields $A^2 \cap Dx_j = \emptyset$. Thus our claim holds.

It follows by (3.4), (3.5), (3.7) and (3.11) that

$$|A^2 \cup D^2 \cup Ax_{i+1} \cup x_jD| = |A^2| + |D^2| + |Ax_{i+1} \cup x_jD|$$

and hence, by Theorem 1.1 and (3.10), we get

$$|A^2 \cup D^2 \cup Ax_{i+1} \cup x_jD| \ge (2i-1) + (2(k-i)-1) + (k-1) = 3k-3.$$

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Thus, by ($*$),

$$S^2 = A^2 \cup D^2 \cup Ax_{i+1} \cup x_j D. \tag{3.12}$$

Consider now the element $x_{i+1}x_j \in S^2$. By (3.5), $x_{i+1}x_j \notin D^2$ and by (3.4), $x_{i+1}x_j \notin A^2$.

Suppose, first, that $x_{i+1}x_j \in Ax_{i+1}$. Then

$$x_{i+1}x_j = x_a x_{i+1} \quad \text{for some } x_a \in A. \tag{3.13}$$

If $x_a > x_j$, then by (3.13) and (3.3) $x_j \in C_G(x_{i+1})$, in contradiction to (3.2). Again by (3.2) $x_a \neq x_j$. Hence $x_a < x_j$.

By (3.2) and Proposition 2.4, there exists $x_b \in A$ such that $x_{i+1}x_b \notin Ax_{i+1}$. Since $x_{i+1}x_j \in Ax_{i+1}$, we know that $x_b \neq x_j$ and if $x_b > x_j$, then (3.3) implies that $x_{i+1}x_b = x_b x_{i+1} \in Ax_{i+1}$, a contradiction. Hence $x_b < x_j$.

Since $x_{i+1}x_b \notin Ax_{i+1}$ and since, by (3.4) and (3.5), also $x_{i+1}x_b \notin A^2 \cup D^2$, it follows by (3.12) that $x_{i+1}x_b \in x_j D$ and there exists $x_d \in D$ such that $x_j x_d = x_{i+1}x_b$. Since $\langle A \rangle$ is abelian, it follows that $x_j x_d x_j = x_{i+1}x_b x_j = x_{i+1}x_j x_b$. As by (3.13) $x_{i+1}x_j = x_a x_{i+1}$, we get $x_j x_d x_j = x_a x_{i+1} x_b$ and $x_j > x_b$ implies that $x_j x_d < x_a x_{i+1}$. But $x_j > x_a$ and $x_d \geq x_{i+1}$, so $x_j x_d > x_a x_{i+1}$, a contradiction.

Suppose, finally, that $x_{i+1}x_j \in x_j D$. It follows that

$$x_{i+1}x_j = x_j x_d \quad \text{for some } x_d \in D.$$

By (3.2) and Proposition 2.4 there exists $x_f \in D$ such that $x_f x_j \notin x_j D$. Since $x_{i+1}x_j \in x_j D$, we must have $x_{i+1} < x_f$.

Now, $x_f x_j \notin x_j D$ and it follows from (3.7) and (3.11) that $x_f x_j \notin D^2 \cup A^2$. Hence by (3.12) we must have $x_f x_j \in Ax_{i+1}$. Thus

$$x_a x_{i+1} = x_f x_j \quad \text{for some } x_a \in A. \tag{3.14}$$

Since $x_f x_j \notin x_j D$, we must have $x_a \neq x_j$. If $x_a > x_j$, then it follows by (3.3) and (3.6) that $x_j \in \langle x_f, x_a, x_{i+1} \rangle \leq C_G(x_{i+1})$, in contradiction to (3.2). Hence $x_a < x_j$.

Since $\langle D \rangle$ is abelian, it follows from (3.14) that

$$x_{i+1}x_a x_{i+1} = x_{i+1}x_f x_j = x_f x_{i+1} x_j.$$

Now $x_{i+1}x_j = x_j x_d$, so $x_{i+1}x_a x_{i+1} = x_f x_j x_d$. But $x_{i+1} < x_f$, so $x_a x_{i+1} > x_j x_d$. However, $x_a < x_j$ and $x_{i+1} \leq x_d$, so $x_a x_{i+1} < x_j x_d$, a contradiction.

We have shown that $x_{i+1}x_j \in S^2$ does not belong to $A^2 \cup D^2 \cup Ax_{i+1} \cup x_j D$, in contradiction to (3.12). It follows from this contradiction that $\langle S \rangle$ is abelian. □

The result of the previous theorem is the best possible. In fact, we exhibit in the following example an ordered group G and a finite subset S of G such that $\langle S \rangle$ is *nonabelian*, $|S| = k \geq 2$ and $|S^2| = 3k - 2$.

Example. Let $G = A \rtimes \langle b \rangle$ be a semidirect product of an abelian subgroup A, isomorphic to the additive rational group $(\mathbb{Q}, +)$, with an infinite cyclic group $\langle b \rangle$, such that

$$b^{-1}ab = a^2 \quad \text{for each } a \in A.$$

Then G is torsion-free and it is orderable by Theorem 2.3.

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Let $a \in A \setminus \{1\}$ and let $S = \{b, ba, ba^2, \ldots, ba^{k-1}\}$. Since $ab = ba^2$, it is easy to see that $S^2 = \{b^2, b^2a, b^2a^2, b^2a^3, \ldots, b^2a^{3k-3}\}$. Thus $\langle S\rangle$ is nonabelian and $|S^2| = 3k-2$.

Theorem 1.3 is clearly equivalent to Theorem 1.2 (repeated below for convenience).

Theorem 1.2. *Let S be a finite subset of an ordered group, which generates a nonabelian subgroup. Then*

$$|S^2| \geq 3|S| - 2.$$

In order to prove Corollary 1.4 we need the following proposition, which extends Freiman's theorem 1.9 in [F] from finite subsets of integers to finite subsets in ordered groups, generating abelian subgroups. Although this result is mentioned in [HLS], for the sake of completeness we have decided to report it with its proof.

Proposition 3.1. *Let S be a finite subset of an ordered group G and suppose that*

$$t = |S^2| \leq 3|S| - 4$$

and S generates an abelian group. Then there exist $x_1, g \in G$ such that $g > 1$, $gx_1 = x_1g$ and S is a subset of the geometric progression

$$\{x_1, x_1g, x_1g^2, \ldots, x_1g^{t-|S|}\}.$$

Proof. Let $S = \{x_1, x_2, \ldots, x_k\}$, with $x_1 < x_2 < \cdots < x_k$. Clearly we may assume that $G = \langle S\rangle$, an abelian group.

Write $y_i = x_1^{-1}x_i$ for $i \in \{1, \ldots, k\}$ and let $K = \{1, y_2, \ldots, y_k\}$. Then $1 < y_2 < y_3 < \cdots < y_k$, $S = x_1K$, $S^2 = x_1^2K^2$ and $|S^2| = |K^2|$, so it suffices to prove the theorem when $x_1 = 1$. So assume that $x_1 = 1$. We argue by induction on k.

Assume first that $k = 3$ and $S = \{1, x_2, x_3\}$. Then the elements $1, x_2, x_2^2, x_2x_3, x_3^2$ are all different, since $1 < x_2 < x_3$. But $|S^2| \leq 3 \times 3 - 4 = 5$, so $S^2 = \{1, x_2, x_2^2, x_2x_3, x_3^2\}$, and the only possibility for $x_3 \in S^2$ is $x_3 = x_2^2$. Hence $S = \{1, g, g^2\}$ with $g = x_2 > 1$ and $2 = t - k$, as required.

Suppose now that $k > 3$ and that the theorem holds for subsets X of G satisfying $3 \leq |X| < k$ and $|X^2| \leq 3|X| - 4$. Let $g = x_kx_{k-1}^{-1}$. Then $g > 1$, since $x_k > x_{k-1}$.

Assume first that for each i, $1 \leq i \leq k-1$, we have $x_{i+1} = x_ig^{s_{i+1}}$, where s_{i+1} are positive integers. Then, as $x_1 = 1$, it follows that $x_{i+1} = g^{q_{i+1}}$, where q_{i+1} are integers and $0 < q_2 < q_3 < \cdots < q_k$. Let $D = \{0, q_2, \ldots, q_k\}$. Since $S = \{1, g^{q_2}, \ldots, g^{q_k}\}$, it follows that $|2D| = |S^2| \leq 3k - 4$. As q_{i+1} are integers, Freiman's theorem 1.9 in [F] implies that D is a subset of the arithmetic progression $\{0, q, 2q, \ldots, (t-k)q\}$ for some integer $q > 0$. Thus S is a subset of the set $\{1, g^q, g^{2q}, \ldots, g^{(t-k)q}\}$, where $g^q > 1$, as required.

Now assume that there exists an integer i, $1 \leq i \leq k-1$, such that for all positive integers l

$$x_{i+1} \neq x_ig^l,$$

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and let i be the *maximal* such integer. It follows by the definition of g that $i < k-1$. Moreover, the definition of i implies that for each integer s, $i < s \leq k-1$, there exists a positive integer t_s such that $x_k = x_s g^{t_s}$, but for $s = i$ such an integer does not exist.

Let $S' = S \setminus \{x_k\}$. Obviously $x_k^2, x_k x_{k-1} \in S^2 \setminus (S')^2$ because of the order in S. We also claim that $x_k x_i \in S^2 \setminus (S')^2$. In fact, if $x_k x_i \in (S')^2$, then $x_k x_i = x_u x_v = x_v x_u$ for some integers u, v, $1 \leq u, v \leq k-1$. Since $x_k > x_u$, we must have $i < v$ and similarly $i < u$. Therefore there exist positive integers t_u, t_v such that $x_k = x_u g^{t_u}$ and $x_k = x_v g^{t_v}$. Thus $x_k x_i = x_u x_v = x_k^2 g^{-(t_u+t_v)}$, yielding $x_k = x_i g^{t_u+t_v}$ with $t_u + t_v > 0$, in contradiction to the definition of i. This contradiction proves that $x_k x_i \in S^2 \setminus (S')^2$. Since $x_k x_i \notin \{x_k^2, x_k x_{k-1}\}$, it follows that

$$|(S')^2| \leq |S^2| - 3 \leq 3k - 4 - 3 = 3(k-1) - 4 = 3|S'| - 4.$$

By induction there exists $g' > 1$ such that each x_j, $1 < j \leq k-1$, satisfies $x_j = (g')^{q_j}$ for some positive integer q_j. In particular, if $x_w, x_j \in S'$ and $x_w x_j > 1$, then $x_w x_j = (g')^{q_{w,j}}$, where $q_{w,j}$ is a positive integer.

Recall that $x_k > 1$ and $x_k^2 \notin (S')^2$. We claim that if $x_k \neq (g')^h$ for all positive integers h, then each $x_b \in S'$ satisfies $x_k x_b \notin (S')^2$. Indeed, assume that this is not the case and $x_k x_b = (g')^z$ for some positive integer z. Then $x_k = (g')^l$ for some integer l and since $x_k, g' > 1$, l is positive. We have reached a contradiction to our assumption. This proves our claim, and it follows that $|S^2| - |(S')^2| \geq k$. Thus

$$|(S')^2| \leq |S^2| - k \leq 3k - 4 - k = 2(k-1) - 2 = 2|S'| - 2,$$

in contradiction to Theorem 1.2. Hence also $x_k = (g')^{q_k}$ for some positive integer q_k. It follows from the order in S and from $g' > 1$ that $0 < q_2 < q_3 < \cdots < q_k$. Again applying Freiman's theorem 1.9 in [F] to $D = \{0, q_2, \ldots, q_k\}$, it follows as above that S is as required. □

Corollary 1.4 (repeated below for convenience) follows immediately from Theorem 1.3 and Proposition 3.1.

Corollary 1.4. *Let S be a finite subset of an ordered group G and suppose that*

$$t = |S^2| \leq 3|S| - 4.$$

Then there exist $x_1, g \in G$, such that $g > 1$, $gx_1 = x_1 g$ and S is a subset of the geometric progression

$$\{x_1, x_1 g, x_1 g^2, \ldots, x_1 g^{t-|S|}\}.$$

Proof. By Theorem 1.3, $\langle S \rangle$ is abelian, and hence, by Proposition 3.1, it is a subset of a geometric progression, as stated. □

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