ADDENDUM TO "CHANGING THE ORDER OF INTEGRATION"

E. R. LOVE

(Received 22 April 1971)

Communicated by B. Mond

Dr. Roy O. Davies, of the University of Leicester, has sent me a neat proof of Theorem 3, p. 430, of my paper "Changing the Order of Integration" [1]. Perhaps other readers would appreciate this improvement, as I do; I therefore present it below, with his agreement, together with a consequent improved proof of Theorem 2. The counter-example, which is the main part of that paper, is unaffected.

Actually Davies proves a different theorem, from which Theorem 3 follows easily. His theorem is a two-dimensional version of the consistency theorem mentioned on p. 422; it is as follows.

THEOREM. *If $f(x, y)$ is L-integrable on the quadrant $H \times H$, with integral I, and*

$$\int_0^\infty dx \int_0^\infty f(x, y)dy$$

exists as a repeated improper R-integral, then its value is equal to I.

PROOF. By the consistency theorem,

$$\int_H f(x, y)dy = \int_0^\infty f(x, y)dy \quad \text{for almost all } x \in H; \tag{39}$$

for the Lebesgue integral on the left exists for almost all $x \in H$ by Fubini's theorem, and the improper Riemann integral on the right exists for all $x \in H$ by data.

By Fubini's theorem and (39),

$$I = \int_H dx \int_H f(x, y)dy = \int_H dx \int_0^\infty f(x, y)dy. \tag{41}$$

By the consistency theorem again,

$$\int_H dx \int_0^\infty f(x, y)dy = \int_0^\infty dx \int_0^\infty f(x, y)dy, \tag{42}$$

the left side existing by (41) and the right by data.

https://doi.org/10.1017/S1446788700010879 Published online by Cambridge University Press

The stated theorem now follows from (41) and (42).

Theorem 3, p. 430, follows directly from this and Fubini's theorem if we restate it in the equivalent form:

THEOREM 3′. *If $f(x, y)$ is L-integrable on the quadrant $H \times H$, and the repeated improper R-integrals* (33) *both exist, then they are equal.*

From this we obtain a simpler proof of Theorem 2, p. 428. This theorem is as follows.

THEOREM 2. *If $f(x, y)$ is measurable on the quadrant $H \times H$, and the repeated improper R-integrals* (33) *both exist and one is absolutely convergent, then they are equal.*

The proof now proposed starts, as before, with the first three paragraphs of the proof of Theorem 1, p. 422. This leads to (8) which may be written

$$\int_H dx \int_H |f(x,y)|\, dy = \int_0^\infty dx \int_0^\infty |f(x,y)|\, dy < \infty . \tag{34}$$

Consequently f is L-integrable on $H \times H$. Then Theorem 3′ applies and completes the proof of Theorem 2.

Reference

[1] E. R. Love, 'Changing the order of integration,' Austral. Math. Soc. 11 (1970), 421–432.

University of Melbourne
Parkville, Victoria 3052
Australia