Canad. J. Math. Vol. **58** (3), 2006 pp. 548–579

Hausdorff and Quasi-Hausdorff Matrices on Spaces of Analytic Functions

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Abstract. We consider Hausdorff and quasi-Hausdorff matrices as operators on classical spaces of analytic functions such as the Hardy and the Bergman spaces, the Dirichlet space, the Bloch spaces and BMOA. When the generating sequence of the matrix is the moment sequence of a measure μ, we find the conditions on μ which are equivalent to the boundedness of the matrix on the various spaces.

1 Introduction

1.1 Hausdorff and Quasi-Hausdorff Matrices

Let Δ be the forward difference operator, defined on scalar sequences $\{\mu_n\}_0^{+\infty}$ by $\Delta\mu_n = \mu_n - \mu_{n+1}$, and its iterates $\Delta^0 = \Delta, \Delta^k = \Delta \circ \Delta^{k-1}$ for $k = 1, 2, \dots$. The *Hausdorff matrix* $H = H(\mu_n)$, with generating sequence $\{\mu_n\}_0^{+\infty}$, is the infinite lower-triangular matrix with entries $c_{n,k} = \binom{n}{k}\Delta^{n-k}\mu_k$, $0 \le k \le n$.

An important special case occurs when $\{\mu_n\}_0^{+\infty}$ is the moment sequence of a measure. That is, $\mu_n = \int_{(0,1]} t^n \, d\mu(t)$, where μ is a finite positive Borel measure on $(0, 1]$. These matrices are denoted by H_μ and their entries are easily found to be

$$c_{n,k} = \binom{n}{k} \int_0^1 t^k (1-t)^{n-k} \, d\mu(t), \quad 0 \le k \le n.$$

They had been originally studied in connection with summability of series and later on as operators on sequence spaces and on spaces of functions. See [3,8,9,11,12]. The study of Hausdorff matrices H_μ as transformations on spaces of analytic functions such as the Hardy spaces H^p, $1 \le p \le +\infty$, was introduced for the first time in [5].

In general, let $\mathbb{X}$ be a Banach space of analytic functions on the unit disc $\mathbf{D}$. We consider a Hausdorff matrix $H_\mu = (c_{n,k})$ and for each function $f(z) = \sum_{n=0}^{+\infty} a_n z^n \in \mathbb{X}$, we consider the formal power series

$$H_\mu(f)(z) = \sum_{n=0}^{+\infty} \Big(\sum_{k=0}^{n} c_{n,k} a_k \Big) z^n$$

and also the transpose matrix $A_\mu = H_\mu^*$ and the corresponding formal power series

$$A_\mu(f)(z) = \sum_{k=0}^{+\infty} \Big(\sum_{n=k}^{+\infty} c_{n,k} a_n \Big) z^k.$$

Received by the editors March 2, 2004; revised March 3, 2005.
Research financed by the research program "Pythagoras 2" (75% European funds and 25% Greek national funds).
AMS subject classification: Primary: 47B38; secondary: 46E15, 40G05, 42A20.

The matrices A_μ are called *quasi-Hausdorff matrices*.

In this work we address the problem of finding, for various classical spaces $\mathbb{X}$, the exact necessary and sufficient conditions on the measure μ so that for every $f \in \mathbb{X}$, the series defining $H_\mu(f)$ and $A_\mu(f)$ converge in $\mathbf{D}$, the resulting functions belong to $\mathbb{X}$ and the operators H_μ and A_μ are bounded on $\mathbb{X}$.

The spaces $\mathbb{X}$ we shall consider are the Hardy spaces, the Bergman spaces A^p, $1 \leq p \leq +\infty$, the disc algebra A_0, the Dirichlet space D, the spaces BMOA and VMOA, the Bloch-space B and the little-Bloch-space B_0. For all the facts about these spaces see [4, 7, 14].

In the rest of this work the symbol C stands for an absolute positive constant, while $C(k, l, \dots)$ stands for a positive constant depending only on the parameters $k, l, \dots$. These constants may not be the same on their various occurrences, even in the same set of equalities and/or inequalities. The symbol $a \asymp b$ means that $\frac{a}{b}$ is bounded from above and from below by two positive absolute constants.

1.2 The Associated Integral Operators

For $t \in (0, 1]$ we consider the two families of transformations

$$\phi_t(z) = \frac{tz}{(t-1)z+1}, \quad \psi_t(z) = tz + 1 - t, \quad z \in \mathbf{D},$$

of the unit disc into itself and the family of weight functions

$$w_t(z) = \frac{1}{(t-1)z+1}, \quad z \in \mathbf{D}.$$

If μ is a finite positive Borel measure on $(0, 1]$, then we define

$$S_\mu(f)(z) = \int_{(0,1]} w_t(z) f(\phi_t(z))\, d\mu(t), \quad z \in \mathbf{D}.$$

The integral is finite, since, by the lemma of Schwartz and $\phi_t(0) = 0$, we have $|w_t(z)||f(\phi_t(z))| \leq \frac{1}{1-|z|} \sup_{|\zeta| \leq |z|} |f(\zeta)|$.

We also define

$$T_\mu(f)(z) = \int_{(0,1]} f(\psi_t(z))\, d\mu(t)$$

for those analytic functions f and points z for which the integral is defined.

The following result is proved in [5], but only under extra conditions on μ.

Lemma 1.1 *Let μ be a finite positive Borel measure on $(0, 1]$ and f be analytic in $\mathbf{D}$. Then the power series $H_\mu(f)(z)$ converges in $\mathbf{D}$ and $H_\mu(f)(z) = S_\mu(f)(z)$ for every $z \in \mathbf{D}$.*

https://doi.org/10.4153/CJM-2006-023-5 Published online by Cambridge University Press

Proof The absolute convergence of $H_\mu(f)(z)$ is proved by

$$\begin{aligned}\sum_{n=0}^{+\infty}\Big|\sum_{k=0}^{n} c_{n,k}\, a_k\Big|\,|z|^n &\leq \int_{(0,1]}\sum_{k=0}^{+\infty}|a_k|\Big(\sum_{n=k}^{+\infty}\binom{n}{k}(1-t)^{n-k}|z|^{n-k}\Big)\,t^k|z|^k\,d\mu(t)\\ &= \int_{(0,1]}\frac{1}{1-(1-t)|z|}\sum_{k=0}^{+\infty}|a_k|\Big(\frac{t|z|}{1-(1-t)|z|}\Big)^k\,d\mu(t)\\ &\leq \frac{1}{1-|z|}\,\mu(0,1]\sum_{k=0}^{+\infty}|a_k||z|^k < +\infty.\end{aligned}$$

The same calculation, without absolute values, gives $H_\mu(f)(z) = S_\mu(f)(z)$. ∎

Unlike the case of H_μ, the coefficients $b_k = \sum_{n=k}^{+\infty} c_{n,k}a_n$ of the power series $A_\mu(f)(z)$ may not converge. For the sake of completeness we state the following trivial lemma, known from [5].

Lemma 1.2 *Let μ be a finite positive Borel measure on $(0,1]$. Then, for each polynomial f, the function $A_\mu(f)$ is also a polynomial and $A_\mu(f)(z) = T_\mu(f)(z)$ for every $z \in \mathbf{D}$.*

1.3 Previous Results and the Structure of This Paper

The operators H_μ and S_μ are identical (Lemma 1.1) and this we denote in the whole work by $H_\mu \equiv S_\mu$. On the other hand, the operators A_μ and T_μ are not *a priori* identical outside the linear space of polynomials (Lemma 1.2). The easiest of the two is T_μ and its boundedness is studied first. One then needs an extra argument to pass to A_μ and this becomes involved in certain cases, like $H^\infty \equiv A^\infty$, BMOA and B, where polynomials are not dense.

Section 2: In [5] a condition (depending on p) on μ was proved to be sufficient for the boundedness of $H_\mu \equiv T_\mu\colon H^p \to H^p$ in all cases $1 \leq p < +\infty$ and the same condition was also proved necessary in case $p = 1$.

Independently, [13] gives the same sufficient condition for the boundedness of $H_\mu \equiv T_\mu\colon H^p \to H^p$ when $2 \leq p < +\infty$ and a weaker condition when $1 < p < 2$.

In the present parer we prove (Theorem 2.4) that the condition in [5] and in [13] (but, there, only when $2 \leq p < +\infty$) is also necessary and we cover the full range $1 \leq p \leq +\infty$. We also give (Proposition 2.1) another proof for the sufficiency of the condition, entirely different from the previous proofs in [5, 13].

Regarding the boundedness of A_μ and of T_μ on H^p, [5] gave the condition on μ which is necessary and sufficient in the case of T_μ and for $1 \leq p < +\infty$ and [13] gave the necessary and sufficient condition in the case of A_μ and for $1 < p < +\infty$.

Here we give (Proposition 2.2), in a different way, the necessary and sufficient condition for the boundedness of T_μ in the range $1 \leq p \leq +\infty$ and (Theorem 2.3) the necessary and sufficient condition for the boundedness of A_μ in the range $1 \leq p < +\infty$. We also prove the equality of the two operators on H^p when $1 \leq p < +\infty$.

Another result (Theorem 2.4) is that $H_\mu\colon H^p \to H^p$ and $A_\mu\colon H^{p'} \to H^{p'}$ are adjoint when $1 < p < +\infty$ and p' is the exponent conjugate to p.

The boundedness of T_μ and H_μ on A_0 is treated in Proposition 2.5 and Theorem 2.6. The more difficult cases of the disc algebra A_0 and, especially, of H^∞ for the operator A_μ are covered by Theorems 2.7 and 2.8.

Section 3: In [13] there is a sufficient condition for the boundedness of H_μ on the Bergman spaces A^p in the restricted range $4 \le p < +\infty$ and we give (Theorem 3.6) the necessary and sufficient condition for the full range $1 \le p \le +\infty$. In order to do this, it seemed technicaly necessary to introduce and study in detail (Proposition 3.4) the adjoint S_μ^* of $H_\mu \equiv S_\mu$.

In [13] a sufficient condition for the boundedness of A_μ on A^p is given when $1 < p < +\infty$. Here we give (Proposition 3.1) the necessary and sufficient conditions for the boundedness of T_μ when $1 \le p \le +\infty$ and (Theorem 3.2) for the boundedness of A_μ when $1 \le p < +\infty$. Observe that the case of $p = +\infty$ for A_μ is already treated in the previous section.

Sections 4, 5, 6: We prove the necessary and sufficient conditions for the boundedness of H_μ and of A_μ on the Dirichlet spaces, BMOA, VMOA, the Bloch and the little-Bloch spaces. For all these there were no previous results in the literature.

Section 7: Finally, we state, but without proof, our results concerning the Lipschitz classes and a few open problems that might be interesting.

In all cases we give exact estimates, and in some instances the exact values, of the norms of the operators on the various spaces.

2 The Hardy Spaces H^p, $1 \le p \le +\infty$, and the Disc Algebra

In the following proposition we find sufficient conditions on μ for the boundedness of S_μ on the H^p spaces, giving a more direct proof than the one in [5].

Proposition 2.1 *Let μ be a finite positive Borel measure on $(0,1]$ with*

(i) $\int_{(0,1]} t^{\frac{1}{p}-1}\, d\mu(t) < +\infty$, *for* $1 < p \le +\infty$,

(ii) $\int_{(0,1]} \log\frac{1}{t}\, d\mu(t)$ *for* $p = 1$.

Then $H_\mu \equiv S_\mu : H^p \to H^p$ *is a bounded operator and*

(iii) $\|H_\mu\|_{H^p \to H^p} \le C \max\left(\frac{1}{p-1}, 1\right) \int_{(0,1]} t^{\frac{1}{p}-1}\, d\mu(t) < +\infty$, *for* $1 < p \le +\infty$,

(iv) $\|H_\mu\|_{H^1 \to H^1} \le C \int_{(0,1]}(1 + \log\frac{1}{t})\, d\mu(t)$ *for* $p = 1$.

Proof Let $f \in H^p$, $1 \le p < +\infty$. Using the generalized Minkowski inequality,

$$\|S_\mu(f)\|_{H^p} \le \int_{(0,1]} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |w_t(e^{i\theta})|^p |f(\phi_t(e^{i\theta}))|^p \, d\theta \right\}^{\frac{1}{p}} d\mu(t). \tag{2.1}$$

We fix a $t \in (0,1]$ and work with the inner integral

$$A(t) = \int_{-\pi}^{\pi} \frac{1}{|1-(1-t)e^{i\theta}|^p} \left| f\Big(\frac{te^{i\theta}}{1-(1-t)e^{i\theta}} \Big) \right|^p \frac{d\theta}{2\pi}.$$

https://doi.org/10.4153/CJM-2006-023-5 Published online by Cambridge University Press

We define $e^{i\phi}$ to be the radial projection of $\frac{te^{i\theta}}{1-(1-t)e^{i\theta}}$ on the boundary $\partial\mathbf{D}$ of the unit disc. This means $e^{i\phi} = \frac{te^{i\theta}}{1-(1-t)e^{i\theta}} \frac{|1-(1-t)e^{i\theta}|}{t}$ and, either by trivial calculations or geometrically, one can see that,

$$\Big|\frac{d\phi}{d\theta}\Big| \geq \frac{C}{t}, \quad 0 < t \leq 1. \tag{2.2}$$

If $Nf(e^{i\phi}) = \sup_{0\leq r<1} |f(re^{i\phi})|$ is the radial maximal function, then the above estimate (2.2) gives for $\frac{1}{2} \leq t \leq 1$,

$$A(t) \leq C^p \int_{-\pi}^{\pi} |Nf(e^{i\phi})|^p \frac{d\phi}{2\pi} \leq C^p \|f\|^p_{H^p}. \tag{2.3}$$

Now, let $0 < t < \frac{1}{2}$ and write $A(t) = \int_{0<|\theta|\leq t} + \int_{t<|\theta|\leq\pi} = A_1(t) + A_2(t)$. For the first integral, using (2.2), we find

$$A_1(t) \leq \frac{1}{t^p} \int_{0<|\theta|\leq t} |Nf(e^{i\phi})|^p \Big|\frac{d\theta}{d\phi}\Big| \frac{d\phi}{2\pi} \leq \frac{C^p}{t^{p-1}} \|f\|^p_{H^p}.$$

In $A_2(t)$ we have $\big|\frac{te^{i\theta}}{1-(1-t)e^{i\theta}}\big| \leq C < 1$, implying $\big|f\big(\frac{te^{i\theta}}{1-(1-t)e^{i\theta}}\big)\big| \leq C\|f\|_{H^p}$. Hence,

$$A_2(t) \leq C^p \int_{t<|\theta|\leq\pi} \frac{1}{\theta^p}\, d\theta \|f\|^p_{H^p} \leq \begin{cases} \frac{C^p}{p-1}\frac{1}{t^{p-1}}\|f\|^p_{H^p}, & \text{if } 1 < p < +\infty, \\ C\log\frac{1}{t}\|f\|_{H^1} & \text{if } p = 1, \end{cases}$$

and, finally, in case $0 < t < \frac{1}{2}$,

$$A(t) \leq \begin{cases} \frac{C^p}{p-1}\frac{1}{t^{p-1}}\|f\|^p_{H^p}, & \text{if } 1 < p < +\infty, \\ C(1+\log\frac{1}{t})\|f\|_{H^1}, & \text{if } p = 1. \end{cases}$$

Together with (2.1) and (2.3), we get the announced estimates for $p \in [1, +\infty)$. In case $p = +\infty$ the estimate is immediate, since

$$|S_\mu(f)(z)| \leq \int_{(0,1]} |w_t(z)|\,|f(\phi_t(z))|\, d\mu(t) \leq \int_{(0,1]} \frac{1}{t}\, d\mu(t) \|f\|_{H^\infty}. \qquad \blacksquare$$

The following result is known from [5], where, in fact, the equality $\|T_\mu\|_{H^p\to H^p} = \int_{(0,1]} t^{-\frac{1}{p}}\, d\mu(t)$ is proved for $p \in [1, +\infty)$ through the use of composition operators. Here we present an alternative proof.

Proposition 2.2 *Let $1 \leq p \leq +\infty$ and μ be a finite positive Borel measure on $(0, 1]$. Then $T_\mu\colon H^p \to H^p$ defines a bounded operator if and only if*

$$\int_{(0,1]} t^{-\frac{1}{p}}\, d\mu(t) < +\infty.$$

Also

$$\int_{(0,1]} t^{-\frac{1}{p}}\, d\mu(t) \leq \|T_\mu\|_{H^p\to H^p} \leq C \int_{(0,1]} t^{-\frac{1}{p}}\, d\mu(t).$$

Proof It is easy to prove that there exists a fixed $\alpha > 1$ such that for every $\theta \in [-\pi, \pi]$, every $r \in [0,1]$ and every $t \in (0,1)$, the point $tre^{i\theta} + 1 - t$ is contained in the kite-shaped region $\Gamma_\alpha(e^{it\theta}) = \left\{ z \in \mathbf{D} : \frac{|e^{it\theta} - z|}{1-|z|} \leq \alpha \right\}$.

Now, it is implied that $|f(tre^{i\theta} + 1 - t)| \leq N_\alpha f(e^{it\theta})$, where $N_\alpha f(\zeta) = \sup_{z \in \Gamma_\alpha(\zeta)} |f(z)|$ is the well-known non-tangential maximal function.

Assuming that $\int_{(0,1]} t^{-\frac{1}{p}} \, d\mu(t) < +\infty$, and using Minkowski's inequality,

$$\begin{aligned} \|T_\mu(f)\|_{H^p} &\leq \left\{ \int_{-\pi}^{\pi} \left\{ \int_{(0,1]} N_\alpha f(e^{it\theta}) \, d\mu(t) \right\}^p \frac{d\theta}{2\pi} \right\}^{\frac{1}{p}} \\ &\leq \int_{(0,1]} \left\{ \int_{-t\pi}^{t\pi} N_\alpha f(e^{i\theta})^p \, \frac{d\theta}{2\pi} \right\}^{\frac{1}{p}} t^{-\frac{1}{p}} \, d\mu(t) \leq C_\alpha \int_{(0,1]} t^{-\frac{1}{p}} \, d\mu(t) \|f\|_{H^p}. \end{aligned}$$

Assuming now that T_μ is bounded on H^p and considering the functions $f_\lambda(z) = \frac{1}{(1-z)^\lambda}$, $0 < \lambda < \frac{1}{p}$, it is clear that $T_\mu(f_\lambda) = \int_{(0,1]} \frac{1}{t^\lambda} \, d\mu(t) f_\lambda$. This implies $\int_{(0,1]} \frac{1}{t^\lambda} \, d\mu(t) \leq \|T_\mu\|_{H^p \to H^p}$ for all $\lambda < \frac{1}{p}$, finishing the proof. ■

Now we shall see that under the same conditions A_μ defines a bounded operator on Hardy spaces.

Theorem 2.3 *Let $1 \leq p < +\infty$ and μ be a finite positive Borel measure on $(0,1]$. Then $A_\mu \colon H^p \to H^p$ defines a bounded operator if and only if*

$$\|A_\mu\|_{H^p \to H^p} = \int_{(0,1]} t^{-\frac{1}{p}} \, d\mu(t) < +\infty.$$

Moreover, under this condition, $A_\mu(f) = T_\mu(f)$ for every $f \in H^p$.

Proof Let $1 < p < +\infty$ and $\int_{(0,1]} t^{-\frac{1}{p}} \, d\mu(t) < +\infty$. If $f(z) = \sum_{n=0}^{+\infty} a_n z^n \in H^p$, then $\|s_N - f\|_{H^p} \to 0$, where $s_N(z) = \sum_{n=0}^{N} a_n z^n$ are the partial sums of the Taylor series of f. From Lemma 1.2 and Proposition 2.2 we get immediately that $A_\mu(s_N) = T_\mu(s_N) \to T_\mu(f)$ in H^p. Using series representation, this means that

$$\sum_{k=0}^{+\infty} \Big(\sum_{n=k}^{N} c_{n,k} a_n \Big) z^k \to T_\mu(f)(z) = \sum_{k=0}^{+\infty} b_k z^k$$

in H^p. Thus, for each k, we get $\sum_{n=k}^{+\infty} c_{n,k} \, a_n = b_k$. Therefore, the series $A_\mu(f)(z)$ is identical to the function $T_\mu(f)(z)$ and, combining this with Proposition 2.2, we conclude that for each $p \in (1, +\infty)$, $A_\mu \colon H^p \to H^p$ defines a bounded operator and

$$\|A_\mu\|_{H^p} \leq C \int_{(0,1]} t^{-\frac{1}{p}} \, d\mu(t).$$

Let $p = 1$ and $\int_{(0,1]} \frac{1}{t}\, d\mu(t) < +\infty$. Let also $f(z) = \sum_{n=0}^{+\infty} a_n z^n \in H^1$ and consider the $(C,1)$ means $\sigma_N(z) = \sum_{n=0}^{N} \left(1 - \frac{n}{N+1}\right) a_n z^n$ of the Taylor series of f. Since $\|\sigma_N - f\|_{H^1} \to 0$, we get $A_\mu(\sigma_N) = T_\mu(\sigma_N) \to T_\mu(f)$ in H^1. This means

$$\sum_{k=0}^{+\infty} \Big(\sum_{n=k}^{N} \Big(1 - \frac{n}{N+1}\Big) c_{n,k} a_n \Big) z^k \to T_\mu(f)(z) = \sum_{k=0}^{+\infty} b_k z^k$$

in H^1, implying that for each k, the series $\sum_{n=k}^{+\infty} c_{n,k} a_n$ is $(C,1)$ summable to b_k. In order to get $\sum_{n=k}^{+\infty} c_{n,k} a_n = b_k$, it is enough to show Tauber's condition: $c_{n,k} a_n = O(\frac{1}{n})$. Since $|a_n| \le \|f\|_{H^1}$, it is enough to show $c_{n,k} = O(\frac{1}{n})$. Now

$$\begin{aligned} c_{n,k} &= \binom{n}{k} \int_{(0,1]} \int_{(0,t]} \left(s^{k+1} (1-s)^{n-k} \right)' ds \frac{d\mu(t)}{t} \\ &\le \binom{n}{k} \int_{(0,1]} \frac{d\mu(t)}{t} \int_{(0,1]} \left| (s^{k+1}(1-s)^{n-k})' \right| ds \\ &\le 2 \frac{(k+1)^{k+1}}{k!} \frac{n^k}{(n+1)^{k+1}} \int_{(0,1]} \frac{d\mu(t)}{t} = O\Big(\frac{1}{n}\Big). \end{aligned}$$

After having proved that $b_k = \sum_{n=k}^{+\infty} c_{n,k} a_n$, the rest of the argument is the same as in the case $1 < p < +\infty$.

For the necessity, we assume that A_μ is bounded and consider the functions $f_\lambda(z) = \frac{1}{(1-z)^\lambda} = \sum_{n=0}^{+\infty} (-1)^n \binom{-\lambda}{n} z^n$, for $0 < \lambda < \frac{1}{p}$. Since $(-1)^n \binom{-\lambda}{n} \ge 0$, we find that

$$\begin{aligned} \sum_{n=k}^{+\infty} c_{n,k} (-1)^n \binom{-\lambda}{n} &= \int_{(0,1]} t^k (-1)^k \binom{-\lambda}{k} \sum_{n=0}^{+\infty} \binom{-\lambda-k}{n} (-1)^n (1-t)^n \, d\mu(t) \\ &= (-1)^k \binom{-\lambda}{k} \int_{(0,1]} t^{-\lambda}\, d\mu(t). \end{aligned}$$

This implies that $\int_{(0,1]} t^{-\lambda}\, d\mu(t) < +\infty$ for every $\lambda, 0 < \lambda < \frac{1}{p}$, and

$$A_\mu(f_\lambda)(z) = \sum_{k=0}^{+\infty} \Big(\sum_{n=k}^{+\infty} c_{n,k} (-1)^n \binom{-\lambda}{n} \Big) z^k = \int_{(0,1]} t^{-\lambda}\, d\mu(t) f_\lambda(z).$$

Therefore $\int_{(0,1]} t^{-\lambda}\, d\mu(t) \le \|A_\mu\|_{H^p \to H^p}$, for every $\lambda \in (0, \frac{1}{p})$, finishing the proof. For the exact value of the norm, see the remark before Proposition 2.2. ■

The proof of the case $p = 1$ in the following theorem is in [5], and we include it for the sake of completeness.

Theorem 2.4 *Let $1 \le p \le +\infty$, $p' = \frac{p}{p-1}$ and μ be a finite positive Borel measure on $(0,1]$. Then H_μ is bounded on H^p if and only if*

$$\|H_\mu\|_{H^p \to H^p} = \int_{(0,1]} t^{\frac{1}{p}-1}\, d\mu(t) < +\infty, \quad \text{if } 1 < p \le +\infty$$

and

$$\|H_\mu\|_{H^1 \to H^1} \asymp \int_{(0,1]} \Big(1 + \log\frac{1}{t}\Big)\, d\mu(t) < +\infty.$$

If $1 < p < +\infty$, then, under the above conditions, $H_\mu \colon H^p \to H^p$ and $A_\mu \colon H^{p'} \to H^{p'}$ are adjoint.

Proof Proposition 2.1 proves the sufficiency part.

If $p = +\infty$ and H_μ is bounded on H^∞, then $H_\mu(1) \in H^\infty$ and, hence,

$$\int_{(0,1]} \frac{1}{t}\, d\mu(t) = \lim_{x \to 1-} \int_{(0,1]} \frac{1}{1-(1-t)x}\, d\mu(t) \le \|H_\mu\|_{H^\infty \to H^\infty}.$$

Also, if $\int_{(0,1]} \frac{1}{t}\, d\mu(t) < +\infty$, then

$$|H_\mu(f)(z)| \le \int_{(0,1]} \frac{1}{|1-(1-t)z|}\, d\mu(t) \|f\|_{H^\infty} \le \int_{(0,1]} \frac{1}{t}\, d\mu(t) \|f\|_{H^\infty}.$$

Now, let $p = 1$ and H_μ be bounded on H^1. Using Hardy's inequality, we get

$$\begin{aligned}\int_{(0,1]} \Big(1 + \log\frac{1}{t}\Big)\, d\mu(t) &\le C \int_{(0,1]} \frac{1}{1-t} \log\frac{1}{t}\, d\mu(t) \\ &= C \sum_{n=0}^{+\infty} \frac{1}{n+1} \int_{(0,1]} (1-t)^n\, d\mu(t) \\ &\le C \|H_\mu(1)\|_{H^1} \le C \|H_\mu\|_{H^1 \to H^1}.\end{aligned}$$

Let $1 < p < +\infty$, assume that H_μ is bounded on H^p and let $H'_\mu \colon H^{p'} \to H^{p'}$ be the bounded adjoint of H_μ. We claim that for all $f \in H^p$ and all polynomials g,

$$\int_0^{2\pi} H_\mu(f)(e^{i\theta}) \overline{g(e^{i\theta})}\, \frac{d\theta}{2\pi} = \int_0^{2\pi} f(e^{i\theta}) \overline{A_\mu(g)(e^{i\theta})}\, \frac{d\theta}{2\pi}.$$

This is trivial to prove when we replace $e^{i\theta}$ by $re^{i\theta}$; we subsequently let $r \to 1-$, bearing in mind that both f and $H_\mu(f)$ are in H^p. This identity implies that $H'_\mu(g) = A_\mu(g)$ for all polynomials g and, in view of the density of polynomials in $H^{p'}$ and of Theorem 2.3, the proof will be complete, if we prove that $\int_{(0,1]} t^{\frac{1}{p}-1}\, d\mu(t) < +\infty$.

Let $0 < \lambda < \frac{1}{p'}$ and consider the functions $f_\lambda(z) = \frac{1}{(1-z)^\lambda} = \sum_{n=0}^{+\infty} \binom{n+\lambda-1}{n} z^n$. For the partial sums $s_{\lambda,N}$ of the Taylor series of f_λ we know that $\|s_{\lambda,N} - f_\lambda\|_{H^{p'}} \to 0$

and, since H'_μ is bounded, we find $A_\mu(s_{\lambda,N}) = H'_\mu(s_{\lambda,N}) \to H'_\mu(f_\lambda)$ in $H^{p'}$. Thus, for each $z = x \in [0,1)$ we get $A_\mu(s_{\lambda,N})(x) \to H'_\mu(f_\lambda)(x)$. Due to monotone convergence,

$$A_\mu(s_{\lambda,N})(x) \to \int_{(0,1]} \sum_{n=0}^{+\infty} \binom{n+\lambda-1}{n} (1-t+tx)^n \, d\mu(t) = \int_{(0,1]} \frac{1}{t^\lambda} f_\lambda(x)\, d\mu(t).$$

Therefore, $\int_{(0,1]} \frac{1}{t^\lambda}\, d\mu(t)\, f_\lambda(z) = H'_\mu(f_\lambda)(z)$ for every $z \in [0,1)$. By analytic continuation, this extends to all z in the unit disc, implying

$$\int_{(0,1]} \frac{1}{t^\lambda}\, d\mu(t) \le \|H'_\mu\|_{H^{p'} \to H^{p'}} = \|H_\mu\|_{H^p \to H^p}$$

for all $\lambda \in (0, \frac{1}{p'})$ and, finally, $\int_{(0,1]} t^{\frac{1}{p}-1}\, d\mu(t) \le \|H_\mu\|_{H^p \to H^p}$. ■

The last two results concern the behaviour of Hausdorff and quasi-Hausdorff matrices on the disc algebra A_0.

Proposition 2.5 *Let μ be a finite positive Borel measure on $(0,1]$. Then T_μ is bounded on A_0 and*

$$\|T_\mu\|_{A_0 \to A_0} = \mu(0,1].$$

Proof If $f \in A_0$ and z_1, z_2 in $\overline{\mathbf{D}}$, using $w = tz + 1 - t$, we find easily

$$|T_\mu(f)(z_1) - T_\mu(f)(z_2)| \le \mu(0,1] \max_{|w_1 - w_2| \le |z_1 - z_2|} |f(w_1) - f(w_2)|.$$

Therefore, $T_\mu(f)$ is in A_0.

The inequality $\|A_\mu\|_{A_0 \to A_0} \le \mu(0,1]$ is obvious and we get the opposite inequality, considering $T_\mu(1) = \mu(0,1]$. ■

Theorem 2.6 *Let μ be a finite positive Borel measure on $(0,1]$. Then $H_\mu \equiv S_\mu$ is bounded on A_0 if and only if $\int_{(0,1]} \frac{1}{t}\, d\mu(t) < +\infty$. Moreover,*

$$\|H_\mu\|_{A_0 \to A_0} = \int_{(0,1]} \frac{1}{t}\, d\mu(t).$$

Proof The necessity of the condition and the exact formula for the norm of the operator are proved in the same way as the case $p = +\infty$ of Theorem 2.4. Therefore, it is enough to prove the sufficiency of the condition. Hence, let $\int_{(0,1]} \frac{1}{t}\, d\mu(t) < +\infty$ and $f \in A_0$ and for any $\epsilon > 0$ find $\delta > 0$ so that $\int_{(0,\delta)} \frac{1}{t}\, d\mu(t) < \epsilon$. Then

$$\begin{aligned} |H_\mu(f)(z) - H_\mu(f)(z_0)| \le 2\epsilon \|f\|_{A_0} \\ + \int_{[\delta,1]} \Big| \frac{1}{1-(1-t)z} f\Big(\frac{tz}{1-(1-t)z}\Big) \\ - \frac{1}{1-(1-t)z_0} f\Big(\frac{tz_0}{1-(1-t)z_0}\Big) \Big| \, d\mu(t). \end{aligned}$$

Due to uniform convergence, the last term tends to 0 when $z \to z_0$. Hence, $\limsup_{z\to z_0} |H_\mu(f)(z) - H_\mu(f)(z_0)| \le 2\epsilon \|f\|_{A_0}$, implying that $H_\mu(f)$ is continuous at the arbitrary $z_0 \in \mathbf{D}$. ■

The behaviour of A_μ on the spaces A_0 and H^∞ remains open, and the last two results of this section exactly describe this behaviour.

Theorem 2.7 *Let μ be a finite positive Borel measure on $(0, 1]$. Then A_μ is bounded on A_0 if and only if $\sup_n \log n \int_{(0,1]} (1-t)^n \, d\mu(t) < +\infty$. In this case we have that $A_\mu \equiv T_\mu$ on A_0 and, hence, $\|A_\mu\|_{A_0 \to A_0} = \mu(0, 1]$.*

Proof (A) The maximum of $t^k(1-t)^{n-k}$ on $(0, 1]$ is $(\frac{k}{n})^k(1-\frac{k}{n})^{n-k}$ at $t = \frac{k}{n}$. Therefore,

$$\begin{aligned} c_{n,k} &= \binom{n}{k} \int_{(0,\frac{1}{\sqrt{n}})} t^k(1-t)^{n-k} \, d\mu(t) + \binom{n}{k} \int_{[\frac{1}{\sqrt{n}},1]} t^k(1-t)^{n-k} \, d\mu(t) \\ &\le C(k) \Big\{ \mu\Big(0, \frac{1}{\sqrt{n}}\Big) + n^k e^{-\frac{n-k}{\sqrt{n}}} \mu(0, 1] \Big\}. \end{aligned}$$

Hence, when $n \to +\infty$,

$$c_{n,k} \to 0. \tag{2.4}$$

Our next aim is to prove that $\{c_{n,k}\}$ is almost-decreasing. The meaning of this is expressed by (2.5) and (2.6) below. Clearly,

$$c_{n,k} - c_{n+1,k} \ge -\binom{n+1}{k} \int_{(0,\frac{k}{n+1})} t^k(1-t)^{n-k}\Big(\frac{k}{n+1} - t\Big) \, d\mu(t) = -r_{n,k}. \tag{2.5}$$

where

$$\begin{aligned} \sum_{n=k}^{+\infty} r_{n,k} &\le C(k) \sum_{n=k}^{+\infty} n^{k-1} \int_{(0,\frac{k}{k+1})} t^k \chi_{(0,\frac{k}{n+1})}(t) \, d\mu(t) \\ &= C(k) \int_{(0,\frac{k}{k+1})} t^k \sum_{n=k}^{+\infty} n^{k-1} \chi_{[k,\frac{k}{t}-1)}(n) \, d\mu(t) \\ &\le C(k)\mu\Big(0, \frac{k}{k+1}\Big) < +\infty. \end{aligned} \tag{2.6}$$

The next result is that $\{c_{n,k}\}$ is almost-convex, as expressed by (2.7) and (2.8).

$$\begin{aligned} &c_{n,k} - 2c_{n+1,k} + c_{n+2,k} = \\ &\quad \binom{n+2}{k} \int_{(0,1]} t^k(1-t)^{n-k} \Big\{ \Big(t - \frac{k}{n+2}\Big)^2 - \frac{k(n+2-k)}{(n+1)(n+2)^2} \Big\} \, d\mu(t). \end{aligned}$$

We set $t^{\pm}_{n,k} = \frac{k}{n+2} \pm \sqrt{\frac{k(n+2-k)}{(n+1)(n+2)^2}}$ and then $0 < t^{-}_{n,k} < t^{+}_{n,k} \leq \min\bigl(\frac{2k}{n+2}, 1\bigr)$. For all $t \in (t^{-}_{n,k}, t^{+}_{n,k})$ we have

$$\Bigl|\Bigl(t - \frac{k}{n+2}\Bigr)^2 - \frac{k(n+2-k)}{(n+1)(n+2)^2}\Bigr| \leq C\frac{k(n+2-k)}{n^3}.$$

Hence,

$$c_{n,k} - 2c_{n+1,k} + c_{n+2,k} \geq -C\binom{n}{k-1}\frac{1}{n}\int_{(t^{-}_{n,k}, t^{+}_{n,k})} t^k(1-t)^{n-k}\, d\mu(t) = -R_{n,k}. \tag{2.7}$$

where $R_{n,k} = 0$ when $k = 0$. This implies

$$\begin{aligned} \sum_{n=k}^{+\infty} nR_{n,k} &\leq C(k)\sum_{n=k}^{+\infty} n^{k-1}\int_{(0,1]} t^k \chi_{(0,t_{n,k})}(t)\ d\mu(t) \\ &\leq C(k)\int_{(0,1]} t^k \sum_{n=k}^{[\frac{2k}{t}]} n^{k-1}\, d\mu(t) \leq C(k)\mu(0,1] < +\infty. \end{aligned} \tag{2.8}$$

From (2.7),

$$\begin{aligned} c_{[\frac{m}{2}],k} - c_{m+1,k} &= \sum_{n=[\frac{m}{2}]}^{m} (c_{n,k} - c_{n+1,k}) \\ &\geq \sum_{n=[\frac{m}{2}]}^{m} (-R_{n,k} - \cdots - R_{m-1,k} + c_{m,k} - c_{m+1,k}) \\ &\geq -C\sum_{n=[\frac{m}{2}]}^{m} nR_{n,k} + Cm(c_{m,k} - c_{m+1,k}). \end{aligned} \tag{2.9}$$

From (2.8) and (2.9) we find $\limsup_{m\to+\infty} m(c_{m,k} - c_{m+1,k}) \leq 0$ and from (2.5),

$$m(c_{m,k} - c_{m+1,k}) \geq -C(k)m^k \int_{(0,\frac{k}{m+1})} t^k\, d\mu(t) \geq -C(k)\mu\Bigl(0, \frac{k}{m+1}\Bigr),$$

implying $\liminf_{m\to+\infty} m(c_{m,k} - c_{m+1,k}) \geq 0$. Therefore

$$n(c_{n,k} - c_{n+1,k}) \to 0. \tag{2.10}$$

Applying summation by parts together with (2.4) and (2.10),

$$\sum_{n=k}^{+\infty}(n+1)(c_{n,k} - 2c_{n+1,k} + c_{n+2,k}) = (k+1)c_{k,k} - kc_{k+1,k}. \tag{2.11}$$

which, together with (2.7) and (2.8), gives

$$(2.12)\quad \sum_{n=k}^{+\infty}(n+1)|c_{n,k}-2c_{n+1,k}+c_{n+2,k}| \le (k+1)c_{k,k}-kc_{k+1,k}+2\sum_{n=k}^{+\infty}(n+1)R_{n,k} < +\infty.$$

(B) Now let $f(z)=\sum_{n=0}^{+\infty}a_n z^n \in A_0$ and consider $s_n=a_0+\cdots+a_n$ and $\sigma_n=\frac{1}{n+1}(s_0+\cdots+s_n)$. After the usual summation by parts, we get

$$(2.13)\quad \sum_{n=k}^{N}c_{n,k}a_n=\sum_{n=k}^{N-2}(c_{n,k}-2c_{n+1,k}+c_{n+2,k})(n+1)\sigma_n + N(c_{N-1,k}-c_{N,k})\sigma_{N-1}-k(c_{k,k}-c_{k+1,k})\sigma_{k-1}+c_{N,k}s_N-c_{k,k}s_{k-1}.$$

Since $\{\sigma_n\}$ is bounded, from (2.10), (2.12) and (2.13) it is implied that the convergence of $\sum_{n=k}^{+\infty}c_{n,k}a_n$ is equivalent to the existence of $\lim_{n\to+\infty}c_{n,k}s_n$ in **C**.

Assume now that A_μ is bounded on A_0. Then, for every $f\in A_0$, the series $\sum_{n=0}^{+\infty}c_{n,0}a_n$ converges and, hence, the limit

$$\lim_{n\to+\infty}c_{n,0}s_n=\lim_{n\to+\infty}c_{n,0}\int_{-\pi}^{\pi}D_n(\theta)f(e^{i\theta})\,\frac{d\theta}{\pi}$$

exists in **C**, where

$$D_n(\theta)=\frac{1}{2}+\sum_{\nu=1}^{n}\cos\nu\theta=\frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}$$

is the Dirichlet kernel. From the Uniform Boundedness Principle we get that $\sup_n c_{n,0}\log n<+\infty$. Because $\lim_{n\to+\infty}c_{n,0}s_n=0$ for every polynomial f and because polynomials are dense in A_0, we conclude that $\lim_{n\to+\infty}c_{n,0}s_n=0$. for every $f\in A_0$.

Suppose that

$$(2.14)\quad \sup_n c_{n,k}\log n<+\infty$$

https://doi.org/10.4153/CJM-2006-023-5 Published online by Cambridge University Press

is true for some k. We shall prove it for $k+1$, implying that it is true for all k and hence, that

$$\lim_{n\to+\infty} c_{n,k}s_n = 0 \tag{2.15}$$

is true for all k. Now, from (2.9),

$$\begin{aligned} c_{n,k+1} &= \frac{n}{k+1}(c_{n-1,k} - c_{n,k}) + \frac{k}{k+1}c_{n,k} \\ &\le C(k) \sum_{m=[n/2]}^{+\infty} mR_{m,k} + C(k)(c_{[n/2],k} + c_{n,k}) \end{aligned}$$

and, performing more carefully the estimates that led to (2.8), we get

$$\begin{aligned} c_{n,k+1} &\le C(k) \sum_{m=[n/2]}^{+\infty} m^{k-1} \int_{(0,\frac{2k}{[n/2]})} t^k \chi_{(0,\frac{2k}{m})}(t)\,d\mu(t) + C(k)(c_{[n/2],k} + c_{n,k}) \\ &\le C(k) \int_{(0,\frac{8k}{n})} t^k \sum_{m=[n/2]}^{[\frac{2k}{t}]} m^{k-1}\,d\mu(t) + C(k)(c_{[n/2],k} + c_{n,k}) \\ &\le C(k)\mu\Big(0,\frac{8k}{n}\Big) + C(k)(c_{[n/2],k} + c_{n,k}) \le C(k)(c_{[n/2],k} + c_{n,k}). \end{aligned}$$

From this and (2.14), we get $\sup_n \log nc_{n,k+1} < +\infty$, and the proof of (2.14) and (2.15) is complete for all k. Now, (2.13) implies

$$\sum_{n=k}^{+\infty} c_{n,k}a_n = \sum_{n=k}^{+\infty}(c_{n,k}-2c_{n+1,k}+c_{n+2,k})(n+1)\sigma_n - k(c_{k,k}-c_{k+1,k})\sigma_{k-1} - c_{k,k}s_{k-1} \tag{2.16}$$

for every k.

From (2.7), we have that for every $\rho \in [0,1)$

$$\begin{aligned} \sum_{k=0}^{+\infty}\Big(\sum_{n=k}^{+\infty} nR_{n,k}\Big)\rho^k &\le C\sum_{k=1}^{+\infty}\sum_{n=k}^{+\infty}\binom{n}{k-1}\int_{(0,1]} t^k(1-t)^{n-k}\,d\mu(t)\rho^k \\ &= C\int_{(0,1]} \frac{\rho}{(1-\rho)(1-t\rho)}\,d\mu(t) \ < +\infty, \end{aligned}$$

and, from (2.12),

$$\begin{aligned}\sum_{k=0}^{+\infty}\sum_{n=k}^{+\infty}&(n+1)|c_{n,k}-2c_{n+1,k}+c_{n+2,k}|\rho^k\\&\le\sum_{k=0}^{+\infty}\big((k+1)c_{k,k}-kc_{k+1,k}\big)\rho^k+2\sum_{k=0}^{+\infty}\sum_{n=k}^{+\infty}(n+1)R_{n,k}\rho^k\\&=\int_{(0,1]}\frac{1-3\rho t+2\rho t^2}{(1-\rho t)^3}\,d\mu(t)+2\sum_{k=0}^{+\infty}\sum_{n=k}^{+\infty}(n+1)R_{n,k}\rho^k<+\infty.\end{aligned}$$

Therefore, from (2.16) and for any $z\in\mathbf{D}$,

$$\begin{aligned}\sum_{k=0}^{+\infty}\Big(\sum_{n=k}^{+\infty}c_{n,k}a_n\Big)z^k&=\sum_{n=0}^{+\infty}\Big(\sum_{k=0}^{n}(c_{n,k}-2c_{n+1,k}+c_{n+2,k})z^k\Big)(n+1)\sigma_n\\&\qquad-\sum_{k=1}^{+\infty}k(c_{k,k}-c_{k+1,k})\sigma_{k-1}z^k-\sum_{k=1}^{+\infty}c_{k,k}s_{k-1}z^k\\&=\sum_{n=0}^{+\infty}\Big(\sum_{k=0}^{n}c_{n,k}z^k-2\sum_{k=0}^{n+1}c_{n+1,k}z^k+\sum_{k=0}^{n+2}c_{n+2,k}z^k\Big)(n+1)\sigma_n\\&=\int_{(0,1]}\sum_{n=0}^{+\infty}a_n(1-t+tz)^n\,d\mu(t)=T_\mu(f)(z).\end{aligned}$$

■

Theorem 2.8 *Let μ be a finite positive Borel measure on $(0,1]$. Then A_μ is bounded on H^∞ if and only if $\lim_{n\to+\infty}\log n\int_{(0,1]}(1-t)^n\,d\mu(t)=0$. In this case $A_\mu\equiv T_\mu$ on H^∞ and $\|A_\mu\|_{H^\infty\to H^\infty}=\mu(0,1]$.*

Proof All results in part (A) of the proof of Theorem 2.7 remain unchanged, since the function space is not involved there. On the other hand, part (B) depends upon the validity of (2.15) for all k.

If we assume $\lim_{n\to+\infty}c_{n,0}\log n=\lim_{n\to+\infty}\log n\int_{(0,1]}(1-t)^n\,d\mu(t)=0$, then exactly as before, we can show by induction that $\lim_{n\to+\infty}c_{n,k}\log n=0$ for all k. Since $|s_n|\le C\log n\|f\|_{H^\infty}$ for all $f\in H^\infty$, we immediately get (2.15), and the sufficiency part of the theorem is proved.

Now, assume that A_μ is bounded on H^∞. Then, exactly as before, we see that $\lim_{n\to+\infty}c_{n,0}s_n$ exists in $\mathbf{C}$ for all $f\in H^\infty$, and the Uniform Boundedness Principle implies, as before, that $\sup_n c_{n,0}\log n<+\infty$. But the polynomials are not dense in H^∞ and, hence, we cannot easily get

$$\lim_{n\to+\infty}c_{n,0}\log n=0. \tag{2.17}$$

https://doi.org/10.4153/CJM-2006-023-5 Published online by Cambridge University Press

Therefore, the rest of the proof consists in proving that if $\lim_{n\to+\infty} c_{n,0}s_n$ exists in $\mathbf{C}$ for all $f \in H^\infty$, then $\lim_{n\to+\infty} c_{n,0}\log n = 0$.

Suppose that, on the contrary, there is a sequence $\{n_j\}$ so that

$$c_{n_j,0}\log n_j \to \rho \neq 0. \tag{2.18}$$

We say that ϕ is of *type* $\mathcal{C}$ if it is 2π-periodic, is in $C^\infty(\mathbf{R}\setminus 2\pi\mathbf{Z})$, is real and odd, is decreasing in $(0,\pi]$ and satisfies $\phi(0+) = \frac{\pi}{2}$ and $\phi(\pi) = 0$. Then (see [15])

(i) $\widehat{\phi}(n)$ is imaginary, $\widehat{\phi}(-n) = -\widehat{\phi}(n)$ for all n and hence, $s_N\phi(0) = 0$ for all N,
(ii) $|s_N\phi(\theta)| \leq C_0$ for some absolute C_0, for all N and all θ,
(iii) $s_N\phi(\theta) \to \phi(\theta)$ uniformly in $\delta \leq |\theta| \leq \pi$, for all δ, and
(iv) $-\frac{\widetilde{s}_N\phi(0)}{\log N} = \frac{2i}{\log N}\sum_{n=1}^{N}\widehat{\phi}(n) \to 1$.

We now construct a sequence of exponential polynomials $\{\psi_k\}$ as follows.

We first consider a function ϕ_1 of *type* $\mathcal{C}$ and a large enough N_1 so that $2N_1$ is in the sequence $\{n_j\}$ of (2.18) and so that $\left|\frac{2i}{\log N_1}\sum_{n=1}^{N_1}\widehat{\phi_1}(n) - 1\right| < 1$. From (ii), $|s_{N_1}\phi_1(\theta)| \leq C_0$ for all θ.

Let $\psi_1 = s_{N_1}\phi_1$ and suppose that $\psi_1,\dots,\psi_k$ have been constructed so that

$$\deg\psi_j = N_j, \quad j = 1,\dots,k \quad \text{where } 2N_j \text{ are all from } \{n_j\} \text{ and,} \tag{2.19}$$

$$N_{j+1} \geq 3N_j, \quad j = 1,\dots,k-1 \tag{2.20}$$

$$\psi_j(0) = 0, \quad j = 1,\dots,k \tag{2.21}$$

$$|\psi_j(\theta)| \leq \frac{C_0}{2^{j-1}}, \quad \frac{\pi}{2^{j-1}} \leq |\theta| \leq \pi, \quad j = 1,\dots,k \tag{2.22}$$

$$|\psi_1(\theta)| + \cdots + |\psi_k(\theta)| \leq C_0\Big(1 + \frac{1}{2} + \cdots + \frac{1}{2^{k-1}}\Big) \text{ for all } \theta \text{ and} \tag{2.23}$$

$$\Big|\frac{2i}{\log N_j}\sum_{n=1}^{N_j}\widehat{\psi_j}(n) - 1\Big| < \frac{1}{j}, \quad j = 1,\dots,k. \tag{2.24}$$

From (2.21) we have that for some $\delta_k \in (0, \frac{\pi}{2^k}]$,

$$|\psi_1(\theta)| + \cdots + |\psi_k(\theta)| \leq C_0\big(\frac{1}{2} + \cdots + \frac{1}{2^k}\big), \quad |\theta| \leq \delta_k. \tag{2.25}$$

We consider any ϕ_{k+1} of *type* $\mathcal{C}$ and supported in $[-\delta_k,\delta_k]$ and take large enough N_{k+1} so that $2N_{k+1}$ is in $\{n_j\}$ and so that

$$N_{k+1} > 3N_k, \quad \Big|\frac{2i}{\log N_{k+1}}\sum_{n=1}^{N_{k+1}}\widehat{\phi_{k+1}}(n) - 1\Big| < \frac{1}{k+1}$$

and due to (iii),

$$|s_{N_{k+1}}\phi_{k+1}(\theta)| \leq \frac{C_0}{2^k}, \quad \delta_k \leq |\theta| \leq \pi. \tag{2.26}$$

Now, if we define $\psi_{k+1} = s_{N_{k+1}}\phi_{k+1}$, then (2.19)–(2.22) and (2.24) are automatically satisfied for $j = k + 1$. Combining (2.22) and (2.26) for $\delta_k \le |\theta| \le \pi$ and (ii) and (2.25) for $|\theta| \le \delta_k$, we get (2.23) for $k + 1$. Therefore, we have inductively constructed $\{\psi_k\}$ satisfying (2.19)–(2.24) with $k = +\infty$.

Consider the series

$$\sum_{k=1}^{+\infty} e^{i2N_k\theta}\psi_k(\theta).$$

By (2.23), the series defines a bounded 2π-periodic function f. Due to (2.19) and (2.20), $f \in H^\infty$ and the frequency ranges of the summands do not overlap. From (2.22), we have that the series converges uniformly in $\delta \le |\theta| \le \pi$ for all δ, and from (2.23), that its partial sums are uniformly bounded everywhere. Therefore, the series is the Fourier series of f, and thus,

$$c_{3N_m,0}s_{3N_m}f(0) = c_{3N_m,0}\sum_{k=1}^{m}\psi_k(0) = 0,$$

while, from (2.18) and (2.24),

$$c_{2N_m,0}s_{2N_m}f(0) = c_{2N_m,0}\Big\{\sum_{k=1}^{m}\psi_k(0) - \sum_{n=1}^{N_m}\widehat{\psi_k}(n)\Big\} = -c_{2N_m,0}\sum_{n=1}^{N_m}\widehat{\psi_k}(n)$$

$$= -c_{2N_m,0}\log 2N_m\frac{\sum_{n=1}^{N_m}\widehat{\psi_k}(n)}{\log 2N_m} \to \frac{1}{2}i\rho \ne 0.$$

This is a contradiction to the existence of $\lim_{n\to+\infty} c_{n,0}s_n f(0)$ for all $f \in H^\infty$. ■

3 The Bergman Spaces A^p, $1 \le p \le +\infty$.

In this section we study Hausdorff matrices and quasi-Hausdorff matrices on Bergman spaces $A^p, 1 \le p \le +\infty$. We find the necessary and sufficient conditions in order for H_μ and A_μ to define bounded operators on these spaces.

Proposition 3.1 *Let $1 \le p \le +\infty$ and μ be a finite positive Borel measure on $(0,1]$. Then $T_\mu\colon A^p \to A^p$ is a bounded operator if and only if*

$$\|T_\mu\|_{A^p\to A^p} = \int_{(0,1]} t^{-\frac{2}{p}}\,d\mu(t) < +\infty.$$

Proof Let $1 \le p \le +\infty$, $\int_{(0,1]} t^{-\frac{2}{p}}\,d\mu(t) < +\infty$ and $f \in A^p$. By Minkowski's inequality and the change of variable $w = \psi_t(z) = tz + 1 - t$,

$$\|T_\mu(f)\|_{A^p} \le \int_{(0,1]} t^{-\frac{2}{p}}\,d\mu(t)\|f\|_{A^p}.$$

Now assume that T_μ is bounded and consider the functions $f_\lambda(z) = \frac{1}{(1-z)^\lambda}$, $0 < \lambda < \frac{2}{p}$. Since $f_\lambda \in A^p$ and $T_\mu(f_\lambda)(z) = \int_{(0,1]} \frac{1}{t^\lambda}\, d\mu(t) f_\lambda(z)$, it follows that $\int_{(0,1]} \frac{1}{t^\lambda}\, d\mu(t) \le \|T_\mu\|_{A^p \to A^p}$ for all $\lambda \in (0, \frac{2}{p})$. Taking the limit as $\lambda \to \frac{2}{p}-$, we get the result we want. ■

Since $A^\infty = H^\infty$, the case $p = +\infty$ in the next theorem has been covered by Theorem 2.8.

Theorem 3.2 *Let $1 \le p < +\infty$ and μ be a finite positive Borel measure on $(0, 1]$. Then the operator $A_\mu \colon A^p \to A^p$ is bounded if and only if*

$$\|A_\mu\|_{A^p \to A^p} = \int_{(0,1]} t^{-\frac{2}{p}}\, d\mu(t) < +\infty.$$

Proof The proof is the same as the proof of Theorem 2.3, using now (see [14]) that if $f(z) = \sum_{n=0}^{+\infty} a_n z^n \in A^p$, then $\sum_{n=0}^{N} a_n z^n \to f$ in A^p, if $1 < p < +\infty$, and $\sum_{n=0}^{N} \big(1 - \frac{n}{N+1}\big) a_n z^n \to f$ in A^1. ■

We continue towards finding sufficient conditions so that $S_\mu \equiv H_\mu$ is bounded on A^p. The next result is only a preliminary rough form of Theorem 3.6.

Proposition 3.3 *Let $1 \le p \le +\infty$ and μ be a finite positive Borel measure on $(0, 1]$ under the further conditions*

(i) $\sum_{n=0}^{+\infty} \frac{1}{n+1} \big| \int_{(0,1]} (1-t)^n\, d\mu(t) \big|^2 < +\infty$, *for* $p = 2$,

(ii) $\int_{(0,1]} t^{\frac{2}{p}-1}\, d\mu(t) < +\infty$, *for* $2 < p \le +\infty$.

Then $H_\mu \equiv S_\mu : A^p \to A^p$ is bounded. Also,

(iii) $\|H_\mu\|_{A^2 \to A^2} \le C\big\{ \sum_{n=0}^{+\infty} \frac{1}{n+1} \big| \int_{(0,1]} (1-t)^n\, d\mu(t) \big|^2 \big\}^{\frac{1}{2}}$, *for* $p = 2$,

(iv) $\|H_\mu\|_{A^p \to A^p} \le C \max\big(\frac{1}{\sqrt{p-2}}, 1\big) \int_{(0,1]} t^{\frac{2}{p}-1}\, d\mu(t)$, *for* $2 < p \le +\infty$,

(v) $\|H_\mu\|_{A^p \to A^p} \le \frac{C}{\sqrt{2-p}} \mu(0, 1]$, *for* $1 \le p < 2$.

Proof Let $1 \le p < +\infty$, $p \ne 2$ and $f \in A^p$. Then

$$\begin{aligned}\|S_\mu(f)\|_{A^p} &\le \int_{(0,1]} \Big\{ \iint_{\mathbf{D}} \frac{1}{|1-(1-t)z|^p} \Big| f\Big(\frac{tz}{1-(1-t)z}\Big)\Big|^p\, dm(z) \Big\}^{\frac{1}{p}}\, d\mu(t) \\ &= \int_{(0,\frac{1}{2}]} + \int_{(\frac{1}{2},1]} = K_1 + K_2.\end{aligned}$$

Using the change of variable $w = \phi_t(z) = \frac{tz}{1-(1-t)z}$, we get

$$\begin{aligned} I(t) &:= \iint_{\mathbf{D}} \frac{1}{|1-(1-t)z|^p} \Big| f\Big(\frac{tz}{1-(1-t)z}\Big)\Big|^p\, dm(z) \\ &= \frac{(1-t)^{p-4}}{t^{p-2}} \iint_{\phi_t(\mathbf{D})} \Big| w + \frac{t}{1-t}\Big|^{p-4} |f(w)|^p\, dm(w).\end{aligned}$$

The image of the unit disc, $\phi_t(\mathbf{D}) = \{w \in \mathbf{D} : |w - \frac{1-t}{2-t}| < \frac{1}{2-t}\}$, is an open disc with the interval $(-\frac{t}{2-t}, 1)$ as diameter. We separate $\phi_t(\mathbf{D})$ into

$$A_0 = \Big\{ w \in \phi_t(\mathbf{D}) : \Big| w + \frac{t}{1-t} \Big| \leq \frac{t}{1-t} \Big\}$$

and

$$A_j = \Big\{ w \in \phi_t(\mathbf{D}) : 2^{j-1}\frac{t}{1-t} \leq \Big| w + \frac{t}{1-t} \Big| \leq 2^j \frac{t}{1-t} \Big\},$$

for $1 \leq j \leq N$, where $2^{N-1}\frac{t}{1-t} < \big| 1 + \frac{t}{1-t} \big| \leq 2^N \frac{t}{1-t}$ and thus,

$$2^N \asymp \frac{1}{t}. \tag{3.1}$$

If $\frac{1}{2} < t < 1$, then the disc $\phi_t(\mathbf{D})$ is covered by $A_0 \cup A_1$. In this case, it is trivial to see that $\frac{t}{1-t} \asymp \big| w + \frac{t}{1-t} \big|$ in $\phi_t(\mathbf{D})$. Hence, $I(t) \leq C \, \|f\|_{A^p}^p$. In case $t = 1$, obviously, $I(t) = \|f\|_{A^p}^p$. Therefore,

$$K_2 \leq C \|f\|_{A^p}.$$

Now let $0 < t \leq \frac{1}{2}$. Trying to estimate $I(t)$, we get that if $N \geq 2$, then the sets $A_0, A_1, \dots, A_{N-2}$ are included in $|z| \leq \frac{1}{2}$. Using that $|f(w)| \leq C\|f\|_{A^p}$, if $|w| \leq \frac{1}{2}$, we get $\iint_{A_j} \leq C^p (2^j t)^{p-2} \|f\|_{A^p}^p$ for $j = 0, \dots, N-2$. Also, for $j = N-1, N$, we get $\iint_{A_j} \leq C(2^j t)^{p-4} \|f\|_{A^p}^p$. Hence,

$$I(t) \leq C^p \Big\{ \sum_{j=0}^{N-2} (2^{p-2})^j \Big\} \|f\|_{A^p}^p + \frac{C}{t^2} \Big\{ (2^{p-4})^{N-1} + (2^{p-4})^N \Big\} \|f\|_{A^p}^p.$$

If $1 \leq p < 2$, from (3.1), $I(t) \leq \big\{ \frac{C^p}{1-2^{p-2}} + Ct^{2-p} \big\} \|f\|_{A^p}^p \leq \frac{C^p}{2-p} \|f\|_{A^p}^p$ and

$$K_1 \leq \frac{C}{\sqrt{2-p}} \mu(0,1] \|f\|_{A^p}.$$

If $p > 2$, from (3.1), $I(t) \leq \frac{C^p}{2^{p-2}-1} \frac{1}{t^{p-2}} \|f\|_{A^p}^p$, implying

$$K_1 \leq C \max\Big(\frac{1}{\sqrt{p-2}}, 1 \Big) \int_{(0,1]} t^{\frac{2}{p}-1} \, d\mu(t) \|f\|_{A^p}.$$

Combining the estimates for K_1 and K_2, we conclude the case $p \in [1,2) \cup (2, +\infty)$.

If $p = 2$, then

$$\begin{aligned}
&\|S_\mu(f)\|_{A^2}\\
&\quad\leq \Big\{ \iint_{\mathbf{D}} \Big| \int_{(0,1]} \frac{1}{1-(1-t)z}\, d\mu(t) \Big|^2 dm(z) \Big\}^{\frac12} |f(0)|\\
&\qquad + \Big\{ \iint_{\mathbf{D}} \Big| \int_{(0,1]} \frac{1}{1-(1-t)z} \Big(f\Big(\frac{tz}{1-(1-t)z}\Big) - f(0)\Big)\, d\mu(t) \Big|^2 dm(z) \Big\}^{\frac12}\\
&\quad\leq \Big\{ \sum_{n=0}^{+\infty} \frac{1}{n+1} \Big| \int_{(0,1]} (1-t)^n\, d\mu(t) \Big|^2 \Big\}^{\frac12} \|f\|_{A^2}\\
&\qquad + \int_{(0,1]} \Big\{ \iint_{\mathbf{D}} \frac{1}{|1-(1-t)z|^2} \Big| f\Big(\frac{tz}{1-(1-t)z}\Big) - f(0)\Big|^2 dm(z) \Big\}^{\frac12} d\mu(t).
\end{aligned}$$

Now, we set $g(z) = \frac{f(z)-f(0)}{z}$, which implies that $\|g\|_{A^2} \leq C\|f\|_{A^2}$. Therefore, the second term is $\leq \int_{(0,1]} \big\{ \iint_{\phi_t(\mathbf{D})} |g(w)|^2\, dm(w) \big\}^{\frac12} d\mu(t) \leq C\mu(0,1]\|f\|_{A^2}$, from which we find

$$\|S_\mu(f)\|_{A^2} \leq C \Big\{ \sum_{n=0}^{+\infty} \frac{1}{n+1} \Big| \int_{(0,1]} (1-t)^n\, d\mu(t) \Big|^2 \Big\}^{\frac12} \|f\|_{A^2}.$$

Finally, the case $p = +\infty$ is obvious. ■

Now, we consider $S_\mu \equiv H_\mu \colon A^p \to A^p$ as a bounded operator and try to find the necessary conditions on μ. To do this we formally define the operator

$$\begin{aligned}
S_\mu^*(f)(z) &= \int_{(0,1]} \Big\{ \frac{1-t}{(tz+1-t)^2} \int_0^{tz+1-t} f(\zeta) d\zeta + \frac{tz}{tz+1-t} f(tz+1-t) \Big\}\, d\mu(t)\\
&= \int_{(0,1]} \frac{d}{dz} \Big(\frac{z}{tz+1-t} \int_0^{tz+1-t} f(\zeta) d\zeta \Big)\, d\mu(t).
\end{aligned}$$

Proposition 3.4 *Let $1 \leq p \leq +\infty$ and μ be a finite positive Borel measure on $(0,1]$. Then the operator $S_\mu^* \colon A^p \to A^p$ is bounded if and only if μ satisfies*

$$\|S_\mu^*\|_{A^p \to A^p} \asymp \Big\{ \int_0^1 \Big(\int_{(0,1]} \frac{1}{t+r}\, d\mu(t) \Big)^{p'} r\, dr \Big\}^{\frac{1}{p'}} + \int_{(0,1]} t^{1-\frac{2}{p}}\, d\mu(t) < +\infty.$$

In particular, if $2 < p \leq +\infty$, then S_μ^* is bounded on A^p for all finite positive μ.

Proof If $p = +\infty$, we see easily, by distinguishing the cases $|tz+1-t| > \frac12$ and $|tz+1-t| \leq \frac12$, that the absolute value of the integrand in the formula for $S_\mu^*(f)(z)$ is less than $C\,\|f\|_{A^\infty}$. Therefore, $\|S_\mu^*(f)\|_{A^\infty} \leq C\mu(0,1]\,\|f\|_{A^\infty}$ and thus,

$$\mu(0,1] \leq \|S_\mu^*\|_{A^\infty \to A^\infty} \leq C\mu(0,1],$$

where the estimate from below we get from $f = 1$.

Let $1 \le p < +\infty$ and $f \in A^p$. Then

$$\|S_\mu^*(f)\|_{A^p} \le \Big\{ \iint_{\mathbf{D}} \Big| \int_{(0,\frac{1}{4}]} \Big|^p dm(z) \Big\}^{\frac{1}{p}} + \Big\{ \iint_{\mathbf{D}} \Big| \int_{(\frac{1}{4},1]} \Big|^p dm(z) \Big\}^{\frac{1}{p}} = I_1 + I_2. \tag{3.2}$$

Now

$$I_2 \le \int_{(\frac{1}{4},1]} \Big\{ \iint_{\mathbf{D}} \Big| \frac{d}{dz} \Big(\frac{z}{tz+1-t} \int_0^{tz+1-t} f(\zeta)d\zeta \Big) \Big|^p dm(z) \Big\}^{\frac{1}{p}} d\mu(t),$$

and, denoting the inner integral by $J(t)$ and with $w = \psi_t(z) = tz + 1 - t$, we get

$$J(t) = t^{p-2} \iint_{\psi_t(\mathbf{D})} \Big| \frac{d}{dw} \Big(\frac{\frac{1}{t}w + 1 - \frac{1}{t}}{w} \int_0^w f(\zeta)d\zeta \Big) \Big|^p dm(w).$$

Since $\frac{1}{4} < t \le 1$, the disc $\psi_t(\mathbf{D})$ intersects the disc $\{|w| \le \frac{1}{2}\}$ and we consider the sets $A_0 = \big\{ w \in \psi_t(\mathbf{D}) : |w| \le \frac{1}{2} \big\}$ and $A_1 = \psi_t(\mathbf{D}) \setminus A_0$. Then

$$J(t) = t^{p-2} \iint_{A_0} + t^{p-2} \iint_{A_1} = J_0(t) + J_1(t).$$

If $|w| \le \frac{3}{4}$, then $|f(w)| \le C \|f\|_{A^p}$ and thus the integrand in $J_0(t)$ is $\le C \|f\|_{A^p}$, implying $J_0(t) \le C^p t^{p-2} \|f\|_{A^p}^p$. Also,

$$\begin{aligned} J_1(t) &\le C^p t^{p-2} \iint_{A_1} \Big(\Big| \int_0^w f(\zeta)d\zeta \Big| + |f(w)| \Big)^p dm(w) \\ &\le C^p t^{p-2} \iint_{A_1} |w|^p \Big(\int_0^1 |f(\lambda w)|^p \, d\lambda \Big) \, dm(w) + C^p t^{p-2} \|f\|_{A^p}^p \\ &\le C^p t^{p-2} \int_0^1 \frac{1}{\lambda^2} \Big(\iint_{\lambda A_1} |f(w)|^p \, dm(w) \Big) \, d\lambda + C^p t^{p-2} \|f\|_{A^p}^p, \end{aligned}$$

and, since $\iint_{\lambda A_1} |f(w)|^p dm(w) \le C^p \, \|f\|_{A^p}^p \pi \lambda^2$, we find $J_1(t) \le C^p t^{p-2} \|f\|_{A^p}^p$. Thus, $J(t) \le C^p t^{p-2} \|f\|_{A^p}^p$, for $\frac{1}{4} \le t \le 1$, and finally, $I_2 \le C \, \mu(\frac{1}{4}, 1] \|f\|_{A^p}$.

Next, working with I_1, we get

$$\begin{aligned} I_1 \le & \Big\{ \iint_{\mathbf{D}} \Big| \int_{(0,\frac{1}{4}]} \int_0^{1-t} f(\zeta)d\zeta \, d\mu(t) \Big|^p dm(z) \Big\}^{\frac{1}{p}} \\ & + \Big\{ \iint_{\mathbf{D}} \Big| \int_{(0,\frac{1}{4}]} \frac{d}{dz} \Big(\Big(\frac{z}{tz+1-t} - z \Big) \int_0^{1-t} f(\zeta)d\zeta \Big) \, d\mu(t) \Big|^p dm(z) \Big\}^{\frac{1}{p}} \\ & + \Big\{ \iint_{\mathbf{D}} \Big| \int_{(0,\frac{1}{4}]} \frac{d}{dz} \Big(\frac{z}{tz+1-t} \int_{1-t}^{tz+1-t} f(\zeta)d\zeta \Big) \, d\mu(t) \Big|^p dm(z) \Big\}^{\frac{1}{p}} \\ = & \; I_{11} + I_{12} + I_{13}. \end{aligned} \tag{3.3}$$

To estimate I_{13}, after using Minkowski's inequality we set

$$
\begin{aligned}
W(t) &= \iint_{\mathbf{D}} \Big| \frac{d}{dz}\Big(\frac{z}{tz+1-t} \int_{1-t}^{tz+1-t} f(\zeta)d\zeta \Big) \Big|^{p} dm(z) \\
&= t^{p-2} \iint_{\psi_t(\mathbf{D})} \Big| \frac{d}{dw}\Big(\frac{\frac{1}{t}w+1-\frac{1}{t}}{w} \int_{1-t}^{w} f(\zeta)d\zeta \Big) \Big|^{p} dm(w) \\
&\le C^p t^{p-2} \iint_{\psi_t(\mathbf{D})} \Big| \frac{1-t}{t\,w^2} \int_{1-t}^{w} f(\zeta)d\zeta \Big|^{p} dm(w) \\
&\qquad + C^p t^{p-2} \iint_{\psi_t(\mathbf{D})} \Big| \frac{w-1+t}{tw} f(w) \Big|^{p} dm(w) \\
&\le C^p t^{p-2} \Big\{ \int_0^1 \Big(\iint_{\psi_t(\mathbf{D})} |f(\lambda w + (1-\lambda)(1-t))|^p \, dm(w) \Big)^{\frac{1}{p}} d\lambda \Big\}^{p} \\
&\qquad + C^p t^{p-2} \|f\|_{A^p}^p \\
&= C^p t^{p-2} \Big\{ \int_0^1 \Big(\iint_{D(1-t;\lambda t)} |f(z))|^p \, dm(z) \Big)^{\frac{1}{p}} \lambda^{-\frac{2}{p}} d\lambda \Big\}^{p} + C^p t^{p-2} \|f\|_{A^p}^p \\
&\le C^p \frac{t^{p-2}}{t^2} \Big\{ \int_0^{\frac{1}{2}} |D(1-t;\lambda t)|^{\frac{1}{p}} \lambda^{-\frac{2}{p}} d\lambda \Big\}^{p} \|f\|_{A^p}^p \\
&\qquad + C^p t^{p-2} \Big\{ \int_{\frac{1}{2}}^{1} \lambda^{-\frac{2}{p}} d\lambda \Big\}^{p} \|f\|_{A^p}^p + C^p t^{p-2} \|f\|_{A^p}^p \\
&\le C^p t^{p-2} \|f\|_{A^p}^p .
\end{aligned}
$$

Thus, $I_{13} \le C \int_{(0,1]} t^{1-\frac{2}{p}} d\mu(t) \|f\|_{A^p}$. To estimate I_{11} and I_{12} we use the well-known $f(\zeta) = \iint_{\mathbf{D}} \frac{f(w)}{(1-\overline{w}\zeta)^2} \, dm(w)$, which gives

$$
\int_0^{1-t} f(\zeta)d\zeta = \iint_{\mathbf{D}} f(w) \frac{1-t}{1-(1-t)\overline{w}} \, dm(w). \tag{3.4}
$$

After trivial calculations, $I_{12} \le C\mu(0,1]\|f\|_{A^p}$.

If $p' = \frac{p}{p-1}$, then from (3.4) we also have

$$
\begin{aligned}
I_{11} &\le \Big\{ \iint_{\mathbf{D}} \Big| \int_{(0,\frac{1}{4}]} \frac{1-t}{1-(1-t)w} \, d\mu(t) \Big|^{p'} dm(w) \Big\}^{\frac{1}{p'}} \|f\|_{A^p} \\
&\le C \Big\{ \iint_{\mathbf{D}} \Big(\int_{(0,1]} \frac{1}{t+|1-w|} \, d\mu(t) \Big)^{p'} dm(w) \Big\}^{\frac{1}{p'}} \|f\|_{A^p} \\
&\le C \Big\{ \int_0^1 \Big(\int_{(0,1]} \frac{1}{t+r} \, d\mu(t) \Big)^{p'} r\,dr \Big\}^{\frac{1}{p'}} \|f\|_{A^p} .
\end{aligned}
$$

Combining all estimates we get

$$\|S^*_\mu\|_{A^p\to A^p} \le C\Big[\Big\{\int_0^1\Big(\int_{(0,1]}\frac{1}{t+r}\,d\mu(t)\Big)^{p'} r\,dr\Big\}^{\frac{1}{p'}} + \int_{(0,1]} t^{1-\frac{2}{p}}\,d\mu(t)\Big]. \tag{3.5}$$

Assume, conversely, that $S^*_\mu\colon A^p \to A^p$ is bounded and consider first $p \ge 2$. We change (3.2) to $\|S^*_\mu(f)\|_{A^p} \ge I_1 - I_2$ and (3.3) to $I_1 \ge I_{11} - I_{12} - I_{13}$. Choosing $f = 1$, we see that $\mu(0,1] \le \|S^*_\mu\|_{A^p\to A^p}$. This, together with all other estimates, implies $I_{11} \le C\|S^*_\mu\|_{A^p\to A^p}\|f\|_{A^p}$. In view of (3.4) and duality, we now have

$$\Big\{\iint_{\mathbf{D}}\Big|\int_{(0,\frac14]}\frac{1-t}{1-(1-t)w}\,d\mu(t)\Big|^{p'} dm(w)\Big\}^{\frac{1}{p'}} \le C\|S^*_\mu\|_{A^p\to A^p}.$$

Restricting $\mathbf{D}$ to a Stolz-angle of opening, say $\frac{\pi}{2}$, with vertex at 1, we get

$$\Big\{\int_0^1\Big(\int_{(0,1]}\frac{1}{t+r}\,d\mu(t)\Big)^{p'} r\,dr\Big\}^{\frac{1}{p'}} \le C\|S^*_\mu\|_{A^p\to A^p}.$$

If $1 \le p < 2$, consider the functions $f_\lambda(z) = \frac{d}{dz}\frac{z}{(1-z)^\lambda}$, $0 < \lambda < \frac{2}{p} - 1$, which satisfy $S^*_\mu(f_\lambda) = \int_{(0,1]} t^{-\lambda}\,d\mu(t) f_\lambda$. This implies $\int_{(0,1]} t^{1-\frac{2}{p}}\,d\mu(t) \le \|S^*_\mu\|_{A^p\to A^p}$, and now the inequality $\|S^*_\mu(f)\|_{A^p} \ge I_{11} - I_{12} - I_{13} - I_2$, through the same argument as above gives

$$\Big\{\int_0^1\Big(\int_{(0,1]}\frac{1}{t+r}\,d\mu(t)\Big)^{p'} r\,dr\Big\}^{\frac{1}{p'}} + \int_{(0,1]} t^{1-\frac{2}{p}}\,d\mu(t) \le C\|S^*_\mu\|_{A^p\to A^p}.$$

The last claim in the statement of the theorem is an immediate consequence of Comment (ii), below. ∎

Comment The quantities in Proposition 3.3 and the "duals" of the quantities in Proposition 3.4 are, as expected, closely related.

(i) For $p = 2$, we have through $\big\|\int_{(0,1]}\frac{1}{1-(1-t)w}\,d\mu(t)\big\|_{A^2}$ that

$$\Big\{\sum_{n=0}^{+\infty}\frac{1}{n+1}\Big|\int_{(0,1]}(1-t)^n d\mu(t)\Big|^2\Big\}^{\frac12} \asymp \Big\{\int_0^1\Big(\int_{(0,1]}\frac{1}{t+r}\,d\mu(t)\Big)^2 r\,dr\Big\}^{\frac12} + \mu(0,1]. \tag{3.6}$$

(ii) For $1 \le p < 2$,

$$\begin{aligned}\mu(0,1] &\le C\Big[\Big\{\int_0^1\Big(\int_{(0,1]}\frac{1}{t+r}d\mu(t)\Big)^p r\,dr\Big\}^{\frac1p} + \int_{(0,1]} t^{1-\frac{2}{p'}}\,d\mu(t)\Big]\\ &\le C\max\Big(\frac{1}{\sqrt{2-p}}, 1\Big)\,\mu(0,1].\end{aligned} \tag{3.7}$$

https://doi.org/10.4153/CJM-2006-023-5 Published online by Cambridge University Press

(iii) Finally, for $p > 2$,

$$\int_{(0,1]} t^{\frac{2}{p}-1}\,d\mu(t) \le \Big\{\int_0^1 \Big(\int_{(0,1]} \frac{1}{t+r}d\mu(t)\Big)^p r\,dr\Big\}^{\frac{1}{p}} + \int_{(0,1]} t^{1-\frac{2}{p'}}\,d\mu(t) \tag{3.8}$$
$$\le C\max\Big(\frac{1}{\sqrt{p-2}},1\Big)\int_{(0,1]} t^{\frac{2}{p}-1}\,d\mu(t).$$

Lemma 3.5 *Let μ be a finite Borel measure on $(0,1]$. If f,g are polynomials, then*

$$\iint_{\mathbf{D}} S_\mu^*(f)(z)\overline{g(z)}\,dm(z) = \iint_{\mathbf{D}} f(z)\overline{S_\mu(g)(z)}\,dm(z).$$

Proof The proof is a matter of trivial calculations with $f(z) = z^k$, $g(z) = z^l$. ■

Now, we are in the position to state the following final form of Proposition 3.3.

Theorem 3.6 *Let $1 \le p \le +\infty$ and μ be a finite positive Borel measure on $(0,1]$. Then $H_\mu \equiv S_\mu\colon A^p \to A^p$ is bounded if and only if the following additional condition is satisfied:*

$$\|H_\mu\|_{A^p\to A^p} \asymp \Big\{\int_0^1 \Big(\int_{(0,1]} \frac{1}{t+r}\,d\mu(t)\Big)^p r\,dr\Big\}^{\frac{1}{p}} + \int_{(0,1]} t^{1-\frac{2}{p'}}\,d\mu(t) < +\infty.$$

Moreover, if $1 \le p < +\infty$ and μ satisfies the above condition, then the adjoint of $H_\mu \equiv S_\mu\colon A^p \to A^p$ is the operator $S_\mu^*\colon A^{p'} \to A^{p'}$, $p' = \frac{p}{p-1}$.

Proof (i) Let $p = +\infty$. Then, the condition becomes $\int_{(0,1]} \frac{1}{t}d\mu(t) < +\infty$ and Proposition 3.3 implies that H_μ is bounded on A^∞.

Conversely, from $H_\mu(1)(z) = \int_{(0,1]} \frac{1}{1-(1-t)z}\,d\mu(t)$, we get

$$\int_{(0,1]} \frac{1}{t}\,d\mu(t) = \lim_{x\to 1^-}\int_{(0,1]} \frac{1}{1-(1-t)x}\,d\mu(t) \le \|H_\mu\|_{A^\infty\to A^\infty}.$$

Now let $1 \le p < +\infty$. Then the condition in the statement of the theorem implies, through (3.6), (3.7), (3.8) and Proposition 3.3, that H_μ is bounded on A^p. Therefore, it only remains to prove the necessity of the condition and that S_μ^* is the adjoint of H_μ. We assume that $H_\mu \equiv S_\mu\colon A^p \to A^p$ is bounded and we denote $S'_\mu\colon A^{p'} \to A^{p'}$ its bounded adjoint. From Lemma 3.5, we have that $S_\mu^*(f) = S'_\mu(f)$ for every polynomial f.

(ii) Let $1 \le p < 2$. Lemma 3.5 implies that for each polynomial $g(z)$,

$$\Big|\iint_{\mathbf{D}} S_\mu^*(1)(z)\overline{g(z)}\,dm(z)\Big| \le \|S_\mu(g)\|_{A^p} \le \|S_\mu\|_{A^p\to A^p}\|g\|_{A^p},$$

and, since the polynomials are dense in A^p, $\mu(0,1] = \|S_\mu^*(1)\|_{A^{p'}} \le \|S_\mu\|_{A^p\to A^p}$. Hence, by (3.7) and Proposition 3.4, $S_\mu^*\colon A^{p'} \to A^{p'}$ is bounded.

In case $1 < p < 2$ the polynomials are dense in $A^{p'}$. Therefore, S_μ^* is the adjoint of H_μ, implying that the two operators have the same norm and this subcase is complete, due to Proposition 3.4.

Now let $p = 1$. To prove that $S'_\mu = S^*_\mu$, we consider the following subcases:

(α) Let $\delta \in (0,1]$ and $\mu \equiv 0$ in $(0,\delta)$. Then the measure μ satisfies the necessary and the sufficient condition for S^* to be bounded on A^2. Consider an $f \in A^\infty \subset A^2$ and the $(C,1)$ means $\sigma_N(f)$ of its Taylor series. Then $\sigma_N(f) \xrightarrow{w^*} f$ and therefore, $S'_\mu(\sigma_N(f)) \xrightarrow{w^*} S'_\mu(f)$ in A^∞. Also, $\sigma_N(f) \to f$, implying $S^*_\mu(\sigma_N(f)) \to S^*_\mu(f)$ in A^2. But, from Lemma 3.5, we get $S^*_\mu(\sigma_N(f)) = S'_\mu(\sigma_N(f))$, and thus, $S^*_\mu(f) = S'_\mu(f)$.

(β) For arbitrary $\delta \in (0,1]$ define the measures $\mu_\delta = \mu_{[\delta,1]}$ and $\nu_\delta = \mu_{(0,\delta)}$. From case ($\alpha$) we get $S'_{\mu_\delta}(f) = S^*_{\mu_\delta}(f)$ and then

$$\|S'_\mu(f) - S'_{\mu_\delta}(f)\|_{A^\infty} = \|S_{\nu_\delta}\|_{A^1\to A^1}\|f\|_{A^\infty} \le C\mu(0,\delta]\,\|f\|_{A^\infty}$$

and

$$\|S^*_\mu(f) - S^*_{\mu_\delta}(f)\|_{A^\infty} = \|S^*_{\nu_\delta}(f)\|_{A^\infty} \le C\mu(0,\delta]\|f\|_{A^\infty}.$$

Letting $\delta \to 0$, we find $S^*_\mu f = S'_\mu f$.

(iii) Let $p = 2$. Lemma 3.5 implies that for every polynomial f, $\|S^*_\mu(f)\|_{A^2} \le \|S_\mu\|_{A^2\to A^2}\|f\|_{A^2}$. Considering the polynomial $f_N(z) = \sum_{n=0}^N \int_{(0,\frac14]}(1-t)^{n+1}\,d\mu(t)z^n$, we get from the end of the proof of Proposition 3.4 that

$$\begin{aligned}\Big\{\sum_{n=0}^N \frac{1}{n+1}\Big|\int_{(0,\frac14]}(1-t)^n d\mu(t)\Big|^2\Big\}&\\ &\le C\|S^*_\mu(f_N)\|_{A^2\to A^2} \le C\|S_\mu\|_{A^2\to A^2}\|f_N\|_{A^2}\\ &= C\|S_\mu\|_{A^2\to A^2}\Big\{\sum_{n=0}^N \frac{1}{n+1}\Big|\int_{(0,\frac14]}(1-t)^n\,d\mu(t)\Big|^2\Big\}^{\frac12}.\end{aligned}$$

Therefore, $\big\{\sum_{n=0}^N \frac{1}{n+1}\big|\int_{(0,\frac14]}(1-t)^n\,d\mu(t)\big|^2\big\}^{\frac12} \le C\|S_\mu\|_{A^2\to A^2}$ and, by Proposition 3.3, S^*_μ is bounded on A^2. From the density of polynomials in A^2, we conclude that S^*_μ is the adjoint of H_μ.

(iv) Finally, let $2 < p < +\infty$. Let $0 < \lambda < \frac{2}{p'} - 1$ and

$$f_\lambda(z) = \frac{d}{dz}\frac{z}{(1-z)^\lambda} = \sum_{n=0}^{+\infty}(n+1)\binom{n+\lambda-1}{n}z^n \in A^{p'}.$$

We take the partial sums $s_{\lambda,N}$ of the Taylor series of f_λ for which we know that $\|s_{\lambda,N} - f_\lambda\|_{A^{p'}} \to 0$. Since S'_μ is bounded, we find $S^*_\mu(s_{\lambda,N}) = S'_\mu(s_{\lambda,N}) \to S'_\mu(f_\lambda)$ in $A^{p'}$. Hence, for each $z = x \in [0,1)$ we get $S^*_\mu(s_{\lambda,N})(x) \to S'_\mu(f_\lambda)(x)$.

Due to monotone convergence,

$$S_\mu^*(s_{\lambda,N})(x) \to \int_{(0,1]} \Big\{ \sum_{n=0}^{+\infty} \binom{n+\lambda-1}{n} \Big((1-t+tx)^n + ntx(1-t+tx)^{n-1} \Big) \Big\} \, d\mu(t)$$
$$= \int_{(0,1]} \frac{1}{t^\lambda} f_\lambda(x) \, d\mu(t).$$

Therefore, $\int_{(0,1]} \frac{1}{t^\lambda} \, d\mu(t) f_\lambda(x) = S'_\mu(f_\lambda)(x)$ for all $x \in [0,1)$ and, by analytic continuation, $\int_{(0,1]} \frac{1}{t^\lambda} \, d\mu(t) f_\lambda = S'_\mu(f_\lambda)$. Hence, $\int_{(0,1]} \frac{1}{t^\lambda} \, d\mu(t) \le \|S'_\mu\|_{A^{p'} \to A^{p'}}$ for $0 < \lambda < \frac{2}{p'} - 1$, and thus, $\int_{(0,1]} t^{\frac{2}{p}-1} \, d\mu(t) \le \|S'_\mu\|_{A^p \to A^p}$.

Proposition 3.4 implies now that S_μ^* is bounded on $A^{p'}$, and, through the density of polynomials in $A^{p'}$, it is the adjoint of $H_\mu \equiv S_\mu$. ∎

Comment One can easily find the exact value of the norm $\|H_\mu\|_{A^p \to A^p}$ in the restricted range $4 \le p < +\infty$, using Minkowski's inequality:

$$\|H_\mu(f)\|_{A^p} \le \int_{(0,1]} \Big\{ \iint_{\mathbf{D}} \frac{1}{t^{p-2}} \Big| f\Big(\frac{tz}{1-(1-t)z} \Big) \Big|^p \frac{t^2}{|1-(1-t)z|^4} \, dm(z) \Big\}^{\frac{1}{p}} \, d\mu(t)$$
$$= \int_{(0,1]} \Big\{ \iint_{\phi_t(\mathbf{D})} |f(w)|^p \, dm(w) \Big\}^{\frac{1}{p}} t^{\frac{2}{p}-1} \, d\mu(t) \le \int_{(0,1]} t^{\frac{2}{p}-1} \, d\mu(t) \|f\|_{A^p}.$$

In view of the last inequality in the proof of Theorem 3.6, we get

$$\|H_\mu\|_{A^p \to A^p} = \int_{(0,1]} t^{\frac{2}{p}-1} \, d\mu(t).$$

4 BMOA **and** VMOA

Theorem 4.1 *Let μ be a finite positive Borel measure on $(0,1]$. Then T_μ is bounded on either* BMOA *or* VMOA *if and only if $\int_{(0,1]} \log \frac{1}{t} \, d\mu(t) < +\infty$, and in both cases its norm is $\asymp \int_{(0,1]} (1 + \log \frac{1}{t}) \, d\mu(t)$.*

Proof Let $\int_{(0,1]} \log \frac{1}{t} \, d\mu(t) < +\infty$ and $f \in$ BMOA. By the growth estimate on f,

$$\int_{(0,1]} |f(1-t+tz)| \, d\mu(t) \le C \int_{(0,1]} \log \frac{1}{1-|1-t+tz|} \, d\mu(t) \|f\|_{\mathrm{BMOA}}$$
$$\le C \int_{(0,1]} \log \frac{1}{t(1-|z|)} \, d\mu(t) \|f\|_{\mathrm{BMOA}} < +\infty$$

and hence, $T_\mu f(z)$ is well defined.

There exist f_1, f_2 analytic in $\mathbf{D}$ with $\Re(f_1), \Re(f_2)$ in $L^\infty, f = f_1 + if_2 + f(0)$, $f_1(0) = f_2(0) = 0$ and $\|f\|_{\mathrm{BMOA}} \asymp \|\Re(f_1)\|_{L^\infty} + \|\Re(f_2)\|_{L^\infty} + |f(0)|$. Obviously, $\Re(T_\mu f_j) = T_\mu(\Re(f_j))$, whence,

$$\begin{aligned}\|T_\mu f_j\|_{\mathrm{BMOA}} &\leq C|T_\mu f_j(0)| + C\|T_\mu(\Re(f_j))\|_{L^\infty}\\ &\leq C\int_{(0,1]} |f_j(1-t)|d\mu(t) + C\mu(0,1]\|\Re(f_j)\|_{L^\infty}\\ &\leq C\int_{(0,1]} \Big(1+\log\frac{1}{t}\Big)\, d\mu(t)\|f\|_{\mathrm{BMOA}}.\end{aligned}$$

Thus, $\|T_\mu f\|_{\mathrm{BMOA}} \leq C\int_{(0,1]}\big(1+\log\frac{1}{t}\big)\, d\mu(t)\|f\|_{\mathrm{BMOA}}$.

Now suppose that T_μ is bounded on BMOA and take $f(z) = \log\frac{1}{1-z}$. Then $T_\mu f(0) = \int_{(0,1]}\log\frac{1}{t}\, d\mu(t)$ is finite and $\int_{(0,1]}\log\frac{1}{t}\, d\mu(t) \leq C\|T_\mu\|_{\mathrm{BMOA}\to\mathrm{BMOA}}$. Similarly, taking $f(z) = 1$, we find $\mu(0,1] \leq C\|T_\mu\|_{\mathrm{BMOA}\to\mathrm{BMOA}}$.

To deal with the case of VMOA, assume $\int_{(0,1]}\log\frac{1}{t}\, d\mu(t) < +\infty$.

From Proposition 2.5, we know that T_μ maps A_0 into A_0 and, therefore, into VMOA. Since T_μ is bounded on BMOA and A_0 is dense in VMOA, we get that it is bounded on VMOA with no larger norm.

For the opposite, we take $f_\epsilon(z) = \log\frac{1}{1+\epsilon-z} \in$ VMOA, and then let $\epsilon \to 0$. ∎

Theorem 4.2 *Let μ be a finite positive Borel measure on* $(0,1]$. *Then $H_\mu \equiv S_\mu$ is bounded on either* BMOA *or* VMOA *if and only if $\int_{(0,1]}\frac{d\mu(t)}{t} < +\infty$. Moreover in both cases the norm of the operator is $\asymp \int_{(0,1]}\frac{d\mu(t)}{t}$.*

Proof Assume $\int_{(0,1]}\frac{d\mu(t)}{t} < +\infty$ and take $f \in$ BMOA. We define $g(z) = zf(z) \in$ BMOA and use the decomposition $g = g_1 + ig_2$ with g_j analytic in $\mathbf{D}$, $g_j(0) = 0$ and $\|g\|_{\mathrm{BMOA}} \asymp \|\Re(g_1)\|_{L^\infty} + \|\Re(g_2)\|_{L^\infty}$.

Since $|\frac{tz}{1-(1-t)z}| \leq |z|$, we see that $H_\mu(f)(z) = \frac{1}{z}\int_{(0,1]} g\big(\frac{tz}{1-(1-t)z}\big)\frac{d\mu(t)}{t}$ is well defined for every $z \in \mathbf{D}$ and as in the proof of Theorem 4.1,

$$\begin{aligned}\|H_\mu(f)\|_{\mathrm{BMOA}} &\leq C|H_\mu(f)(0)| + C\|zH_\mu(f)\|_{\mathrm{BMOA}}\\ &\leq C\mu(0,1]|f(0)| + C\int_{(0,1]}\frac{d\mu(t)}{t}\|g\|_{\mathrm{BMOA}} \leq C\int_{(0,1]}\frac{d\mu(t)}{t}\|f\|_{\mathrm{BMOA}}.\end{aligned}$$

Assume H_μ is bounded on BMOA, take $f(z) = \log\frac{1}{1-z}$ and $x \in [0,1)$. Then,

$$\begin{aligned}\int_{(0,1]}\frac{1}{1-(1-t)x}\log\frac{1-(1-t)x}{1-x}\, d\mu(t) &= H_\mu(f)(x)\\ &\leq C\log\frac{1}{1-x}\|H_\mu\|_{\mathrm{BMOA}\to\mathrm{BMOA}}.\end{aligned}$$

Restricting to $(\sqrt{1-x},1]$, we find $\int_{(\sqrt{1-x},1]} \frac{1}{t}\, d\mu(t) \le C\|H_\mu\|_{\mathrm{BMOA}\to\mathrm{BMOA}}$ and it only remains to let $x \to 1-$.

If $\int_{(0,1]} \frac{d\mu(t)}{t} < +\infty$, then from Theorem 2.6 H_μ maps A_0 into itself. Since A_0 is dense in VMOA and H_μ is bounded on BMOA, we have that it is bounded on VMOA with no increase in norm.

For the opposite, take $f_\epsilon(z) = \log \frac{1}{1+\epsilon-z} \in$ VMOA and let $\epsilon \to 0+$. ■

Theorem 4.3 *Let μ be a finite positive Borel measure on $(0,1]$. Then A_μ is bounded on* BMOA *if and only if $\int_{(0,1]} \log \frac{1}{t}\, d\mu(t) < +\infty$. In this case $A_\mu \equiv T_\mu$ on* BMOA. *Exactly the same are true for the space* VMOA.

Proof We employ the notation in the proof of Theorem 2.7. We assume $\int_{(0,1]} \log \frac{1}{t}\, d\mu(t) < +\infty$ and take $f \in$ BMOA. It is easy to see that

$$|s_n| \le C\|f\|_{\mathrm{BMOA}} \log n. \tag{4.1}$$

In fact, write $f = f_1 + i f_2 + f(0)$ with f_1, f_2 analytic in $\mathbf{D}$, $\Re(f_1), \Re(f_2) \in L^\infty$, $f_1(0) = f_2(0) = 0$ and $\|f\|_{\mathrm{BMOA}} \asymp |f(0)| + \|\Re(f_1)\|_{L^\infty} + \|\Re(f_2)\|_{L^\infty}$. Then,

$$\begin{aligned}
|s_n| &\le |f(0)| + \Big|\sum_{k=1}^{n} \widehat{f_1}(k)\Big| + \Big|\sum_{k=1}^{n} \widehat{f_2}(k)\Big| \\
&= |f(0)| + \Big|\sum_{k=1}^{n} \int_0^{2\pi} \Re(f_1)(\theta) e^{-ik\theta} \frac{d\theta}{2\pi}\Big| + \Big|\sum_{k=1}^{n} \int_0^{2\pi} \Re(f_2)(\theta) e^{-ik\theta} \frac{d\theta}{2\pi}\Big| \\
&\le |f(0)| + C(\|\Re(f_1)\|_{L^\infty} + \|\Re(f_2)\|_{L^\infty}) \log n \le C\|f\|_{\mathrm{BMOA}} \log n.
\end{aligned}$$

We get $c_{n,k} \log n \le C(k)\big\{ \int_{(0,\frac{1}{\sqrt{n}})} \log \frac{1}{t}\, d\mu(t) + \log n\, n^k e^{-\frac{n-k}{\sqrt{n}}} \mu(0,1]\big\} \to 0$ from the estimate that gave us (2.4), and this, together with (4.1), implies

$$c_{n,k} s_n \to 0. \tag{4.2}$$

From (2.5),

$$|c_{n,k} - c_{n+1,k}| \le c_{n,k} - c_{n+1,k} + 2C(k) n^{k-1} \int_{(0,\frac{k}{n+1})} t^k\, d\mu(t)$$

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and, using (4.1),

$$\begin{aligned}
&\sum_{n=k}^{N-1} |c_{n,k} - c_{n+1,k}||s_n| \\
&\quad \le C(k)\|f\|_{\mathrm{BMOA}} \Big\{ \sum_{n=k}^{N-1} (c_{n,k} - c_{n+1,k}) \log n + \sum_{n=k}^{N-1} n^{k-1} \log n \int_{(0,\frac{k}{n+1})} t^k \, d\mu(t) \Big\} \\
&\quad \le C(k)\|f\|_{\mathrm{BMOA}} \Big\{ \sum_{n=k}^{+\infty} \log \frac{n}{n-1} c_{n,k} + c_{k,k} \log k \\
&\qquad\qquad + \int_{(0,1]} t^k \sum_{n=k}^{+\infty} n^{k-1} \log n \chi_{(0,\frac{k}{n+1})}(t) \, d\mu(t) \Big\} \\
&\quad \le C(k)\|f\|_{\mathrm{BMOA}} \Big\{ \sum_{n=k}^{+\infty} \frac{1}{n} \binom{n}{k} \int_{(0,1]} t^k (1-t)^{n-k} \, d\mu(t) + c_{k,k} \log k \\
&\qquad\qquad + \int_{(0,1]} t^k \sum_{n=k}^{[\frac{k}{t}]} n^{k-1} \log n \, d\mu(t) \Big\} \\
&\quad \le C(k)\|f\|_{\mathrm{BMOA}} \Big\{ \mu(0,1] + c_{k,k} \log k + \int_{(0,1]} \log \frac{1}{t} \, d\mu(t) \Big\} < +\infty.
\end{aligned}$$

Therefore, the series $\sum_{n=k}^{+\infty} (c_{n,k} - c_{n+1,k}) s_n$ coverges and (4.2) together with summation by parts gives

$$\sum_{n=k}^{+\infty} c_{n,k} a_n = \sum_{n=k}^{+\infty} (c_{n,k} - c_{n+1,k}) s_n - c_{k,k} s_{k-1}. \tag{4.3}$$

We shall need the estimates: $\sum_{n=1}^{+\infty} t^n \log n \le C \sum_{n=1}^{+\infty} t^n \sum_{k=1}^{n} \frac{1}{k} = \frac{C}{1-t} \log \frac{1}{1-t}$ and $\sum_{n=1}^{+\infty} n t^n \log n \le \frac{C}{1-t} + \frac{C}{1-t} \log \frac{1}{1-t}$. Taking any $\rho \in [0,1)$, we have from (4.1) and the first of these estimates,

$$\begin{aligned}
&\sum_{n=0}^{+\infty} \sum_{k=0}^{n} (c_{n,k} - c_{n+1,k}) \rho^k |s_n| \\
&\quad = \sum_{n=0}^{+\infty} \int_{(0,1]} \Big\{ t(1-t+t\rho)^n (1-\rho) + t^{n+1} \rho^{n+1} \Big\} \, d\mu(t) |s_n| \\
&\quad \le C\|f\|_{\mathrm{BMOA}} \int_{(0,1]} \Big\{ t(1-\rho) + t(1-\rho) \frac{1}{t(1-\rho)} \log \frac{1}{t(1-\rho)} \\
&\qquad + t\rho + t\rho \frac{1}{1-t\rho} \log \frac{1}{1-t\rho} \Big\} \, d\mu(t) < +\infty.
\end{aligned} \tag{4.4}$$

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From (2.5), (4.1) and the second of the estimates,

$$
\begin{aligned}
\text{(4.5)}\qquad \sum_{k=0}^{+\infty}\sum_{n=k}^{+\infty} r_{n,k}|s_n|\rho^k &\le \sum_{k=1}^{+\infty}\sum_{n=k}^{+\infty}\binom{n}{k-1}\int_{(0,1]} t^k(1-t)^{n-k}\,d\mu(t)|s_n|\rho^k\\
&= \int_{(0,1]}\frac{t\rho}{1-t}\sum_{n=1}^{+\infty}\big((1-t+t\rho)^n - t^n\rho^n\big)\,|s_n|d\mu(t)\\
&\le \int_{(0,1]} t\rho\sum_{n=1}^{+\infty} n(1-t+t\rho)^{n-1}|s_n|\,d\mu(t)\\
&\le C\|f\|_{\mathrm{BMOA}}\int_{(0,1]} t\rho\Big\{\frac{1}{t(1-\rho)}\\
&\qquad + \frac{1}{t(1-\rho)}\log\frac{1}{t(1-\rho)}\Big\}\,d\mu(t) < +\infty.
\end{aligned}
$$

Finally, from (4.4) and (4.5) we get

$$
\begin{aligned}
\sum_{k=0}^{+\infty}\sum_{n=k}^{+\infty}|c_{n,k}-c_{n+1,k}||s_n|\rho^k &\le \sum_{n=0}^{+\infty}\sum_{k=0}^{n}(c_{n,k}-c_{n+1,k})\rho^k|s_n| + 2\sum_{k=0}^{+\infty}\sum_{n=k}^{+\infty} r_{n,k}|s_n|\rho^k\\
&< +\infty.
\end{aligned}
$$

Therefore, using (4.3) and a change of the order of summation,

$$
\sum_{k=0}^{+\infty}\Big(\sum_{n=k}^{+\infty} c_{n,k}a_n\Big) z^k = \int_{(0,1]}\sum_{n=0}^{+\infty} a_n(1-t+tz)^n\,d\mu(t) = T_\mu(f)(z).
$$

To prove the necessity part of the theorem, consider $f(z) = \log\frac{1}{1-z}$ and observe that the first coefficient of $A_\mu(f)$ is $\sum_{n=1}^{+\infty}\frac{1}{n}\int_{(0,1]}(1-t)^n d\mu(t) = \int_{(0,1]}\log\frac{1}{t}\,d\mu(t)$.

Since we have proved that the condition $\int_{(0,1]}\log\frac{1}{t}\,d\mu(t) < +\infty$ implies $A_\mu \equiv T_\mu$ on BMOA, we can use Theorem 4.1 to prove that under the same condition A_μ is bounded on VMOA.

The necessity is proved by considering $f(z) = \log\frac{1}{1+\epsilon-z}$ and letting $\epsilon \to 0+$. ■

5 The Bloch and Little–Bloch Spaces

The proofs of the next two theorems, although they are mildly involved, do not present any new ideas and it seems better to omit them. They just use the standard growth estimates of functions in the Bloch space B:

$$
|f'(z)|(1-|z|) \le C\|f\|_B, \quad |f(z)| \le C\Big(\log\frac{1}{1-|z|} + 1\Big)\|f\|_B.
$$

Theorem 5.1 *Let μ be a finite positive Borel measure on $(0,1]$. Then $H_\mu \equiv S_\mu$ is bounded on B if and only if*

$$\|H_\mu\|_{B\to B} \asymp \int_{(0,1]} \frac{1}{t}\, d\mu(t) < +\infty.$$

The same condition is necessary and sufficient for H_μ to be bounded on B_0.

Theorem 5.2 *Let μ be a finite positive Borel measure on $(0,1]$. Then T_μ is bounded on both B and B_0 if and only if $\int_{(0,1]} \log\frac{1}{t}\, d\mu(t) < +\infty$. Moreover, in both cases, its norm is equivalent to $\int_{(0,1]} \big(1+\log\frac{1}{t}\big)\, d\mu(t)$.*

Theorem 5.3 *Let μ be a finite positive Borel measure on $(0,1]$. Then A_μ is bounded on B if and only if $\int_{(0,1]} \log\frac{1}{t}\, d\mu(t) < +\infty$. In this case: $A_\mu \equiv T_\mu$ on B. Exactly the same are true for the space B_0.*

Proof The proof is identical to the proof of the analogous result for the spaces BMOA and VMOA, provided we prove that for every $f(z) = \sum_{n=0}^{+\infty} a_n z^n \in B$:

$$|s_n| = |a_0 + \cdots + a_n| \le C\|f\|_B \log n.$$

It is true (see [2]) that for every $g(z) = \sum_{n=0}^{+\infty} b_n z^n$ analytic in $\mathbf{D}$

$$\Big|\lim_{r\to 1-} \sum_{n=0}^{+\infty} a_n b_n r^n\Big| \le 2\|f\|_B \|g\|_T,$$

where the last norm is defined by $\|g\|_T = |g(0)| + \frac{1}{2\pi}\int_0^1\int_0^{2\pi} \big|g'(re^{i\theta})\big|\, d\theta dr$.

Hence, it is enough to consider the function $g(z) = \sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}$ and prove that $\|g\|_T \le C\log n$. This is probably known, but since we have no reference for it, we present a quick proof.

$$\begin{aligned}\|g\|_T &\le 1 + \frac{1}{\pi}\int_0^1\int_0^{\pi} \frac{|1-r^n e^{in\theta}|}{|1-re^{i\theta}|^2}\, d\theta dr + \frac{1}{\pi}\int_0^1\int_0^{\pi} \frac{n|r^n e^{in\theta} - r^{n+1}e^{i(n+1)\theta}|}{|1-re^{i\theta}|^2}\, d\theta dr\\ &= 1 + A + B.\end{aligned}$$

Now, $B = \frac{n}{\pi}\int_0^1 r^n \int_0^{\pi} \frac{1}{|1-re^{i\theta}|}\, d\theta dr \le Cn\int_0^1 r^n\big(1+\log\frac{1}{1-r}\big)\, dr \le C\log n$. Also

$$\begin{aligned}A &= \frac{1}{\pi}\int_0^1 \sum_{k=0}^{[\frac{n}{2}]-1} \int_{k\frac{2\pi}{n}}^{(k+1)\frac{2\pi}{n}} \frac{|1-r^n e^{in\theta}|}{|1-re^{i\theta}|^2}\, d\theta dr + \frac{1}{\pi}\int_0^1\int_{[\frac{n}{2}]\frac{2\pi}{n}}^{\pi} \frac{|1-r^n e^{in\theta}|}{|1-re^{i\theta}|^2}\, d\theta dr\\ &\le Cn\int_0^1\int_0^{\frac{2\pi}{n}} \frac{1}{|1-re^{i\theta}|}\, d\theta dr + C\int_0^1\int_0^{\frac{2\pi}{n}} \sum_{k=1}^{[\frac{n}{2}]-1} \frac{1}{|1-re^{i(\theta+k\frac{2\pi}{n})}|^2}\, d\theta dr + \frac{C}{n}\\ &\le C\log n + C\int_0^1\int_0^{\frac{2\pi}{n}}\int_{\frac{2\pi}{n}}^{\pi} \frac{1}{|1-re^{it}|^2}\, dt\, d\theta dr\\ &\le C\log n + \frac{C}{n}\int_0^1\int_{\frac{2\pi}{n}}^{\pi} \frac{1}{(1-r)^2+t^2}\, dt dr \le C\log n,\end{aligned}$$

and the proof is finished. ∎

6 The Dirichlet Space

Through the form $\langle f, g\rangle_* = \sum_{n=0}^{+\infty} a_n\overline{b_n}$, defined for $f(z) = \sum_{n=0}^{+\infty} a_n z^n \in D$ and $g(z) = \sum_{n=0}^{+\infty} b_n z^n \in A^2$ a duality relation between the Dirichlet space D and the Bergman space A^2 is introduced. Clearly, $|\langle f, g\rangle_*| \le \|f\|_D \, \|g\|_{A^2}$.

If either H_μ is bounded on D or A_μ is bounded on A^2, then

$$\langle H_\mu(f), g\rangle_* = \langle f, A_\mu(g)\rangle_*,$$

where the necessary change in the order of summation is justified from

$$\sum_{n=0}^{+\infty}\Big(\sum_{k=0}^{n} c_{n,k}|a_k|\Big)\,|b_n| \le \|H_\mu\Big(\sum_{n=0}^{+\infty}|a_n|z^n\Big)\|_D\|g\|_{A^2} \le \|H_\mu\|_{D\to D}\|f\|_D\|g\|_{A^2}$$

in the first case and from

$$\sum_{k=0}^{+\infty}\Big(\sum_{n=k}^{+\infty} c_{n,k}|b_n|\Big)\,|a_k| \le \|A_\mu\Big(\sum_{n=0}^{+\infty}|b_n|z^n\Big)\|_{A^2}\|f\|_D \le \|A_\mu\|_{A^2\to A^2}\|g\|_{A^2}\|f\|_D$$

in the second case. In the same manner we see that the same equality holds if either H_μ is bounded on A^2 or A_μ is bounded on D. From these dualities and from Theorems 3.2 and 3.6 together with Comment (i) in Section 3, we get

Theorem 6.1 *Let μ be a finite positive Borel measure on $(0,1]$. Then,*

(1) *$H_\mu \equiv S_\mu$ is bounded on D if and only if*

$$\|H_\mu\|_{D\to D} = \int_{(0,1]} \frac{1}{t}\, d\mu(t) < +\infty.$$

(2) *$A_\mu \equiv T_\mu$ is bounded on D if and only if*

$$\|A_\mu\|_{D\to D} \asymp \Big\{\sum_{n=0}^{+\infty}\frac{1}{n+1}\Big|\int_{(0,1]}(1-t)^n\, d\mu(t)\Big|^2\Big\}^{\frac{1}{2}} < +\infty.$$

7 Some Final Comments

It might be interesting to explore the action of Hausdorff and quasi-Hausdorff matrices and their integral analogues on the spaces H^p, when $0 < p < 1$, and also their boundedness as operators : $H^p \to H^q$, when $p \ne q$. Besides some conjectures, we have no positive result in this direction.

For the Lipschitz spaces Λ_α, $0 < \alpha < 1$, we are able to prove that H_μ is bounded on Λ_α if and only if $\int_{(0,1]} \frac{1}{t^{\alpha+1}}\, d\mu(t) < +\infty$ and that A_μ is bounded on Λ_α for all μ. The proofs of these results contain no new ideas or techniques and hence we omit them.

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