NEW CHARACTERIZATIONS OF POLYHEDRAL CONES

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1. Introduction. A pyramid clearly has all its projections closed, even when the line segments from vertex to base are extended to infinite half-lines. Not so a circular cone. For if the cone is on its side and supported by the (x, y) plane in such a way that its infinite half-line of support coincides with the positive x axis, then its horizontal projection on the (y, z) plane is the open upper half-plane $y > 0$, together with the single point $(0, 0)$. It is our purpose to show that the pyramid behaves better under projection precisely because of its *polyhedral* nature. And this principle can be reinterpreted to give a criterion for the positive extendibility of positive functionals defined on a subspace of a partially ordered vector space.

Throughout our discussion E will be a real finite-dimensional vector space and E' its dual space. A subset P of E stable under vector addition and multiplication by non-negative scalars is called a *convex cone.* In particular, every linear subspace is a convex cone. The smallest subspace containing P is $(-P) + P$, and the largest subspace contained in P is $(-P) \cap P$. Omitting parentheses, we shall write $-P + P$ and $-P \cap P$. It is customary to call $\dim(-P + P)$ the *dimension* of P, and $\dim(-P \cap P)$ the *lineality* of P. We define the *polar* P° of P to be the set of all functionals $f \in E'$ such that $f(x) \geqslant 0$ for all $x \in P$. P° is a closed convex cone, and in fact is the most general such cone, since the double polar $P^{\circ\circ}$ coincides with the closure of P. This fact authorizes us to use the notation $P^{\circ\circ}$ for the closure of P (provided that P is a convex cone). The elementary duality theory of closed convex cones can be summed up as follows:

(1) *Galois connection:* $(P + Q)^\circ = P^\circ \cap Q^\circ$ and $(P \cap Q)^\circ = (P^\circ + Q^\circ)^{\circ\circ}$.
(2) $\dim(-P + P) + \dim(-P^\circ + P^\circ) = \dim E$.
(3) If $P^\circ = E'$, then $P = E$.

The above statements for closed convex cones P and Q are easily rephrased for arbitrary convex cones. Proofs can be found in Fenchel **(2)**.

2. Cones with all projections closed. We shall call a cone *polyhedral* if it is the intersection of finitely many closed half-spaces. A theorem of Weyl **(3)** states that such a cone is also the convex hull of finitely many half-lines, and conversely. Thus P and P° are polyhedral together.

THEOREM. *If a closed convex cone P has all its 2-dimensional projections closed, then P is polyhedral. Conversely, a closed convex polyhedral cone P has its projections (of all dimensions) closed.*

Received February 22, 1956.

https://doi.org/10.4153/CJM-1957-001-5 Published online by Cambridge University Press

Proof. We first dispose of the converse. Let P be polyhedral and let T be a projector. Then P is precisely all non-negative combinations of finitely many fixed vectors $x_1, \ldots, x_m$ and TP is all non-negative combinations of $Tx_1, \ldots, Tx_m$. Hence TP is not only closed but even polyhedral.

Now assume that P and all its 2-dimensional projections are closed and make an induction on the dimension n of $-P + P$, which we can without loss of generality take equal to the dimension of the whole vector space E. Since there are only 6 linearly inequivalent closed convex cones of dimension $\leqslant 2$, and since all of these are polyhedral, we can begin with $n \geqslant 3$.

First suppose that P contains some line L, or equivalently that $\dim(-P \cap P) > 0$. If T projects P parallel to L into some hyperplane (L the null space of the projector T), then TP has lower dimension than P and has all its 2-dimensional projections closed; hence by induction TP is the smallest cone containing the images $Tx_1, \ldots, Tx_s$ of some $x_1, \ldots, x_s \in P$. Let x be any non-zero vector in L. Because $P = P + L$, then P is the complete inverse image of TP. Hence P is the smallest cone containing x, $-x$, $x_1, \ldots, x_s$.

Suppose now $\dim(-P \cap P) = 0$. The convex cone $-P^\circ + P^\circ$ is dense in E', hence is all of E', and P° contains n linearly independent functionals whose sum f is strictly positive at all non-zero points of P. Let H be the affine hyperplane $\{x : f(x) = 1\}$. P is the smallest cone containing $H \cap P$, and this compact convex set is the convex hull of its extreme points. We shall show that these are finite in number by showing that each point $x \in H \cap P$ has a neighbourhood containing no extreme point other than possibly x itself.

Let H_0 be the linear hyperplane through the origin parallel to H and let L be the line through x. Let T project the whole space E onto H_0 along L. The cone TP is polyhedral by induction, hence consists of all non-negative combinations of some $y_1, \ldots, y_r \in H_0$. We can without loss of generality take these as images under T of

$$y_1 + x, \ldots, y_r + x \in H \cap P.$$

(For if they are originally the images of $y_1 + c_1x, \ldots, y_r + c_rx \in P$, then we can replace $y_1, \ldots, y_r$ by $c_1^{-1}y_1, \ldots, c_r^{-1}y_r$.)

Furthermore, near 0, TP coincides with the convex hull of $0, y_1, \ldots, y_r$. We assert that near x the set $H \cap P$ coincides with the convex hull of x, $x_1, \ldots, x_r$. For clearly any $x' \in H \cap P$ near x has its image Tx' near 0, hence $Tx' = c_1y_1 + \ldots + c_ry_r$ with all c's non-negative and $0 \leqslant c_1 + \ldots + c_r < 1$. And since the restriction of T to H is a one-one affine mapping of H onto H_0, it follows that x' is the convex combination

$$(1 - c_1 - \ldots - c_r)x + c_1x_1 + \ldots + c_rx_r.$$

Now $H \cap P$ is seen to be a convex polyhedron near x, and near x the only possible extreme point of $H \cap P$ is x itself. Thus $H \cap P$ coincides near x with the convex hull of $x, x_1, \ldots, x_r$.

https://doi.org/10.4153/CJM-1957-001-5 Published online by Cambridge University Press

An application of the Heine-Borel theorem completes the proof. We cover the compact convex set $H \cap P$ with finitely many open sets, each containing at most one extreme point of $H \cap P$. The whole set $H \cap P$ is then the convex hull of finitely many points, P is the convex hull of finitely many half-lines, and the theorem is proved.

3. Positive extendibility of positive functionals. The main practical interest of our theorem lies in Corollary 1 below. De Leeuw **(1)** uses it, and a good deal more, in proving a convexity theorem for polynomials in several complex variables. And in fact it was a question of his about positive extendibility of functionals that led us to conjecture the theorem of the present paper.

Given a convex cone P that contains no line, we can make E an *ordered vector space* by defining $x \geqslant y$ to mean $x - y \in P$. Conversely the vectors $\geqslant 0$ in an ordered vector space form a convex cone P that contains no line. When such a cone is closed it is called an *order cone*. And it is easy to prove that each of the following properties characterizes order cones among all closed convex cones P:

(1) P possesses extreme half-lines (though certain authors define extreme half-lines in such a way that this particular characterization has trivial exceptions).

(2) P is the convex hull of its extreme half-lines.

(3) P lies strictly to one side of some hyperplane.

(4) P consists of all half-lines through some compact convex set that does not contain the origin.

Our purpose in this section, however, is to characterize polyhedral order cones among all order cones. The simplest kind of polyhedral ordering of a vector space is the coordinatewise ordering relative to a basis, a vector being non-negative if and only if all its coordinates are non-negative. A natural example of non-polyhedral (and non-lattice) ordering is given by the cone of positive-definite matrices in the real n^2-dimensional space of n-by-n complex hermitian matrices.

COROLLARY 1. *Let E be a polyhedrally ordered vector space, and let F be a subspace with the induced ordering. Then every positive functional f on F can be extended to a positive functional on E. Conversely, let E be a vector space ordered by a closed cone in such a way that the above positive extension property holds for all subspaces F and positive functionals f on F. Then the ordering of E is polyhedral.*

Proof. Let T_F be the natural mapping of E' onto F', with null space F°. The condition for positive extendibility of functionals can be written

$$T_F(P^\circ) = T_F((P \cap F)^\circ) = T_F((P^\circ + F^\circ)^{\circ\circ})$$

or equivalently

$$P^\circ + F^\circ = (P^\circ + F^\circ)^{\circ\circ} + F^\circ = (P^\circ + F^\circ)^{\circ\circ}.$$

Thus the condition asserts that all projections of P° are closed, and we know by our theorem that this happens exactly when P° is polyhedral. But P and P° are polyhedral together.

COROLLARY 2. *Let F be a subspace of E, let P be a polyhedral cone in E containing no line, and let K_F be a half-space of F containing $F \cap P$. Then $K_F = F \cap K_E$, for some half-space of E containing P. Conversely, let P be a closed convex cone containing no line, and suppose that some half-space K_E of E like the above can be found for every choice of F and K_F. Then P is polyhedral.*

Proof. We have simply restated Corollary 1 in a geometric language that avoids explicit mention of functionals.

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