87.76 Two more proofs of Lester's theorem

In this note we use areal coordinates ([1]) to prove the recent theorem for a scalene triangle that the circumcentre $O$, the nine-point centre $N$, and the two Fermat points $P$ and $Q$ are concyclic ([2]). The first proof makes use of the point of intersection $M$ of the Euler line $OM$ and the Fermat line $PQ$, and invokes the power lemma for concyclic points; the second proof arises from the surdic conjugacy of the areal coordinates for $P$ and $Q$.

We define the actual areal coordinates of a general point $W$ with respect to the triangle of reference $ABC$ by the triple $[x, y, z]$, where

$$x = \frac{\Delta WBC}{\Delta ABC}, \quad y = \frac{\Delta WCA}{\Delta ABC}, \quad z = \frac{\Delta WAB}{\Delta ABC},$$

so that $x + y + z = 1$. Relative areal coordinates $(\lambda x, \lambda y, \lambda z) (\lambda \neq 0)$ are sometimes preferred, with square/round brackets for actual:relative areal coordinates, respectively.

The alternative expression given in [1] for the distance $WW'$ between the points $W$ and $W'$ with actual areal coordinates $[x, y, z]$ and $[x', y', z']$ is

$$WW'^2 = \frac{(b^2 + c^2 - a^2)(x-x')^2}{2} + \frac{(c^2 + a^2 - b^2)(y-y')^2}{2} + \frac{(a^2 + b^2 - c^2)(z-z')^2}{2},$$

which is easily seen to be equivalent to

$$\frac{WW'^2}{2\Delta} = \cot A(x - x')^2 + \cot B(y - y')^2 + \cot C(z - z')^2 \quad (1)$$

where $\Delta$ denotes $\Delta ABC$. We shall frequently use the notation $[x_w, y_w, z_w]$ for the actual areal coordinates of a general point $W$.

It will be convenient to denote $\cot A$, $\cot B$, $\cot C$ (which occur frequently) by $\alpha$, $\beta$, $\gamma$, respectively. We shall often use the cotangent identity $\beta \gamma + \gamma \alpha + \alpha \beta = 1$, and the abbreviations $\alpha + \beta + \gamma = \Sigma$, $\alpha \beta \gamma = \Pi$.

The actual areal coordinates $[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$ and $[\beta \gamma, \gamma \alpha, \alpha \beta]$ for the centroid $G$ and orthocentre $H$ are easily found, and may be used to provide relative areal coordinates for all points on the Euler line in the form $(\lambda + \mu \beta \gamma, \lambda + \mu \gamma \alpha, \lambda + \mu \alpha \beta) (3\lambda + \mu \neq 0)$. In particular, we obtain the relative areal coordinates $(1 + \beta \gamma, 1 + \gamma \alpha, 1 + \alpha \beta)$ and $(1 - \beta \gamma, 1 - \gamma \alpha, 1 - \alpha \beta)$ for $N$ and $O$, respectively, since $O, G, N, H$ is the order of points on the Euler line with $3GN = NH$ and $3OG = OH$.

Now the midpoint $M$ (say) of $GH$ has relative areal coordinates $(1 + 3\beta \gamma, 1 + 3\gamma \alpha, 1 + 3\alpha \beta)$. Furthermore, it is not especially difficult to obtain the relative areal coordinates

$$\left(\sqrt{3}(\beta + \gamma) \pm (1 + 3\beta \gamma), \sqrt{3}(\gamma + \alpha) \pm (1 + 3\gamma \alpha), \sqrt{3}(\alpha + \beta) \pm (1 + 3\alpha \beta)\right)$$

for $P/Q$, whence the actual areal coordinates for $P$ and $Q$ are given by

$$x_P = \frac{\sqrt{3}(\beta + \gamma) + (1 + 3\beta \gamma)}{2\sqrt{3}(\Sigma + \sqrt{3})}, \quad x_Q = \frac{\sqrt{3}(\beta + \gamma) - (1 + 3\beta \gamma)}{2\sqrt{3}(\Sigma - \sqrt{3})},$$

with similar expressions for $y_P, z_P$ and $y_Q, z_Q$. 

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Note that
\[2\sqrt{3}(\Sigma + \sqrt{3})[x_p, y_p, z_p] - 2\sqrt{3}(\Sigma - \sqrt{3})[x_Q, y_Q, z_Q] = 2(1 + 3\beta y, 1 + 3\gamma a, 1 + 3\alpha \beta).
\]
This shows that the intersection of the Fermat line \(PQ\) and the Euler line is \(M\); and since \(\Sigma - \sqrt{3} > 0\) ([3, 2.38]), it also shows that the three points lie in the order \(Q, P, M\) on the Fermat line.

The first proof

We first note that \(MN.MO = \frac{3}{2}NO^2\). Now whereas at least one other proof of Lester’s theorem uses Ptolemy’s theorem ([2]), we shall show that \(MP.MQ = MN.MO\), so that \(O, N, P, Q\) are concyclic by the power lemma. To do this we need only find the three distances \(NO, MP, MQ\). Then using the actual areal coordinates \([x_N, y_N, z_N], [x_O, y_O, z_O]\), we have

\[
\frac{NO^2}{2\Delta} = \alpha(x_N - x_0)^2 + \beta(y_N - y_0)^2 + \gamma(z_N - z_0)^2
\]

\[
= \alpha\left(-\frac{1}{4} + \frac{3}{2}\beta y\right)^2 + \beta\left(-\frac{1}{4} + \frac{3}{2}\gamma a\right)^2 + \gamma\left(-\frac{1}{4} + \frac{3}{2}\alpha \beta\right)^2.
\]

The abbreviations \(\alpha + \beta + \gamma = \Sigma\), \(\alpha \beta \gamma = \Pi\), and the cotangent identity, lead to the formulae

\[
\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{\Pi}, \quad \alpha^2 + \beta^2 + \gamma^2 = \Sigma^2 - 2, \quad \alpha^3 + \beta^3 + \gamma^3 = \Sigma^3 - 3\Sigma + 3\Pi.
\]

Hence

\[
\frac{8NO^2}{\Delta} = \alpha\left(-1 + \frac{3\Pi}{\alpha}\right)^2 + \beta\left(-1 + \frac{3\Pi}{\beta}\right)^2 + \gamma\left(-1 + \frac{3\Pi}{\gamma}\right)^2
\]

\[
= \alpha + \beta + \gamma - 18\Pi + 9\Pi^2\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)
\]

\[
= \Sigma - 18\Pi + 9\Pi = \Sigma - 9\Pi,
\]

that is, \(NO^2 = \left(\frac{3}{2}\right)(\Sigma - 9\Pi)\) so that \(MN.MO = \frac{3}{2}NO^2 = \left(\frac{3}{2}\right)(\Sigma - 9\Pi).

We next encounter some serious algebra on the Fermat line. Thus

\[
\frac{MP^2}{2\Delta} = \alpha(x_M - x_P)^2 + \beta(y_M - y_P)^2 + \gamma(z_M - z_P)^2
\]

\[
= \alpha\left\{\frac{1 + 3\Pi}{6} - \frac{1 + \sqrt{3}(\Sigma - \alpha)}{2\sqrt{3}\Sigma + 6}\right\}^2 + \ldots,
\]

leading to

\[
\frac{216(\Sigma + \sqrt{3})^2MP^2}{\Delta} = \alpha\left\{\left(\frac{1 + 3\Pi}{6}\right) - 6\left(1 + \sqrt{3}(\Sigma - \alpha) + \frac{3\Pi}{\alpha}\right)\right\}^2 + \ldots
\]

\[
= 12\alpha\left\{\Sigma\left(1 + \frac{3\Pi}{\alpha}\right) - 3(\Sigma - \alpha)\right\}^2 + \ldots
\]
Thus after squaring and collecting terms we have
\[
\frac{18 \left(\Sigma + \sqrt{3}\right)^2 M P^2}{\Delta} = a \left\{ -2 \Sigma + \frac{3 \Pi \Sigma}{a} + 3 \alpha \right\}^2 + \ldots
\]
\[
= a \left\{ 4 \Sigma^2 + \frac{9 \Pi^2 \Sigma^2}{a^2} + 9 \alpha^2 + 18 \Pi \Sigma - 12 \Sigma \alpha - \frac{12 \Pi \Sigma^2}{a} \right\} + \ldots
\]
\[
= \left\{ 4 \Sigma^2 a + \frac{9 \Pi^2 \Sigma^2}{a} + 9 \alpha^3 + 18 \Pi \Sigma a - 12 \Sigma a^2 - 12 \Pi \Sigma^2 \right\} + \ldots
\]
\[
= 4 \Sigma^3 + 9 \Pi \Sigma^2 + 9 \left( \Sigma^3 - 3 \Sigma + 3 \Pi \right) + 18 \Pi \Sigma^2 - 12 \Sigma (\Sigma^2 - 2) - 36 \Pi \Sigma^2
\]
\[
= \Sigma^3 - 9 \Pi \Sigma^2 - 3 \Sigma + 27 \Pi = (\Sigma^3 - 3)(\Sigma - 9 \Pi),
\]
whence
\[
M P^2 = \frac{\Delta \left( \Sigma - \sqrt{3} \right)}{18 \left( \Sigma + \sqrt{3} \right)} (\Sigma - 9 \Pi).
\]
Now since \(M P^2\) and \(MQ^2\) are conjugate surds, we also have
\[
M Q^2 = \frac{\Delta \left( \Sigma + \sqrt{3} \right)}{18 \left( \Sigma - \sqrt{3} \right)} (\Sigma - 9 \Pi),
\]
so after multiplying and taking square roots we obtain \(MP.MQ = + \left( \frac{\Delta}{18} \right) (\Sigma - 9 \Pi)\), since \(\Sigma - 9 \Pi = \frac{8 \Pi^2}{\Delta} > 0\). Hence we have the power equation \(MP.MQ = MN.MO\) as required to complete the first proof.

The second proof, and the centre \(L\)

The centre \(L\) \([x, y, z]\) of the circle \(NOP\) may be found by means of the equations
\[
L O^2 = L N^2 = L P^2.
\]
We have the equations
\[
\frac{L O^2}{2 \Delta} = a (x - x_0)^2 + \beta (y - y_0)^2 + \gamma (z - z_0)^2
\]
\[
\frac{L N^2}{2 \Delta} = a (x - x_N)^2 + \beta (y - y_N)^2 + \gamma (z - z_N)^2.
\]
which after subtraction and simplification lead to
\[
\alpha x + \beta y + \gamma z = \frac{3 (\Sigma - \Pi)}{8}. \quad (2)
\]
From \(\frac{L O^2}{2 \Delta} = \frac{L P^2}{2 \Delta}\), using similar but more complicated calculations, we obtain
\[
\alpha^2 x + \beta^2 y + \gamma^2 z = \frac{\Sigma (\Sigma - \Pi)}{4} - \frac{1}{3}.
\]
We now have the simultaneous linear equations \(A x = d\), namely
\[
\begin{pmatrix}
1 & 1 & 1 \\
\alpha & \beta & \gamma \\
\alpha^2 & \beta^2 & \gamma^2
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} =
\begin{pmatrix}
1 \\
\frac{3 (\Sigma - \Pi)}{8} \\
\frac{\Sigma (\Sigma - \Pi)}{4} - \frac{1}{3}
\end{pmatrix}.
\]
Solving for the actual area coordinates \([x, y, z]\) of \(L\), we obtain
\[
24 \det \mathbf{A} \mathbf{x} = \left\{ 8 (1 - 3\beta \gamma) + 3 (\Sigma - 3\alpha) (\Sigma - \Pi) \right\} (\beta - \gamma),
\]
with similar expressions for \(y\) and \(z\), where \(\det \mathbf{A} = (\beta - \gamma)(\gamma - \alpha)(\alpha - \beta)\).

What we have found are expressions for the coordinates of the centre of the circle \(NOP\). Thus, since these expressions do not involve \(\sqrt{3}\), we shall obtain the same coordinates for the centre of the circle \(NOQ\) if we perform similar calculations. This provides another proof that the four points \(O, N, P, Q\) are concyclic.

Note that the formula for \(\det \mathbf{A}\) shows that the special case of an isosceles triangle is singular although limits do exist for the equilateral case.

The radius \(\rho\)

From (1) we have
\[
\frac{\rho^2}{2\Delta} = \frac{LO^2}{2\Delta} = \alpha (x - x_0)^2 + \beta (y - y_0)^2 + \gamma (z - z_0)^2
\]
where \([x, y, z]\) still denote the actual areal coordinates of \(L\). This simplifies (using (2)) to
\[
\frac{\rho^2}{2\Delta} = ax^2 + by^2 + cz^2 - \frac{\Sigma - \Pi}{8}.
\]

Now since \(\Sigma - \Pi = \csc A \csc B \csc C\) and \(\Delta = 2R^2 \sin A \sin B \sin C\), where \(R\) denotes the circumradius, we have
\[
\left( \frac{\rho}{R} \right)^2 = 4(\Sigma - \Pi)^{-1} (ax^2 + by^2 + cz^2) - \frac{1}{2}.
\]

The reader may wish to demonstrate by synthetic methods that the ratio \(\rho/R = \frac{1}{2}\) whenever an angle of the triangle has value \(\pi/3\) or \(2\pi/3\).

Finally, numerical considerations seem to elicit the conjecture that \(\rho/R\) attains minimum value \(\frac{1}{2}\sqrt{2}\) if the triangle of reference has angles \(\pi/7, 2\pi/7, 4\pi/7\), that is when \(\Sigma = -\frac{1}{\Pi} = \sqrt{7} = \cot \omega\) where \(\omega\) is the Brocard angle ([1]).

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References

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