original coaxal system, whose centre is K′, parallel to TT′; and draw RM perpendicular to TT′, and join MP, MK′. Then if x and y be the coordinates of W, with respect to KV, KP as axes, we have

$$x^2 = K'W^2 = r^2 K'R^2 = r^2(K'M^2 - RM^2).$$

But RM = PM, both being tangents from a point on the radical axis to circles of the coaxal system. Therefore

$$x^2 = r^2(K'M^2 - PM^2) = r^2(K'K^2 - PK^2) = (y^2 - K\Pi^2)KT^2/K\Pi^2.$$

Hence the equation of the locus of W is

$$y^2/K\Pi^2 - x^2/KT^2 = 1.$$

It will be noticed that the hyperbolas in this case are the conjugates of those associated with the coaxaloid system whose Π-points (real) are Π and Π′, and whose ratio is also KT/KΠ.

It may be remarked that the foregoing proof and that of Art. 17 implicitly contain the following theorem which I have not seen stated elsewhere: If K′W (Fig. 5) be a variable line of fixed direction, such that its extremity K′ moves on a fixed line (KK″) and such that K′W/K′Π, where Π is a fixed point, is constant, the locus of W is a hyperbola.

Systems of Conics connected with the Triangle.

By J. A. Third, M.A.

[The references in square brackets are to the articles of my paper on Systems of Circles analogous to Tucker Circles.]

I.

Systems of Six-point Conics.

1. It has been shown [4] that if two triangles ABC and PQR (Fig. 6) be in perspective with respect to any point S as centre of perspective, the sides of PQR cut the non-corresponding sides of ABC in three pairs of points which lie on a circle, provided that the angles made by the sides of PQR with those of ABC possess certain

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values dependent on the position of S. In the general case, when these angles do not conform to the condition referred to, it follows from Pascal's theorem that the six points determined by the sides of PQR on the sides of ABC, lie on a conic.

Hence it follows that if any straight line be taken meeting QR, RP, PQ in D, E, F respectively (Fig. 16), and a triangle P′Q′R′ be drawn in perspective with PQR with respect to S as centre and DEF as axis of perspective, the sides of P′Q′R′ meet the non-corresponding sides of ABC in three pairs of points which also lie on a conic; for P′Q′R′ is in perspective with ABC with respect to S as centre of perspective. By causing P′Q′R′ to vary so as to remain in perspective with PQR with respect to S as centre of perspective and DEF as axis of perspective, we obtain a system of six-point conics connected with ABC.

If we project this system so that one of the conics becomes a circle and the line DEF becomes the line at infinity, then the sides of the variable triangle P′Q′R′ become parallel to the corresponding sides of PQR, and consequently [3] the other conics of the system also become circles. That is, the system of conics may be projected into a coaxaloid system of circles.

We thus obtain the following propositions.

(i) The conics of the system intersect on DEF in two (real or imaginary) points, corresponding to the circular points at infinity of the system of circles.

(ii) The envelope of the system is a conic touching the sides of ABC at the points where these are met by AS, BS, CS [11, 12].

(iii) The poles of DEF with respect to the envelope and the conics of the system are collinear. This corresponds to the property of the collinearity of the centres of the coaxaloid system of circles and their envelope [8, 11]. Generally the poles of any line, with respect to the conics, and consequently their centres lie on a conic [26].

(iv) If from any point on DEF a line be drawn to the pole of DEF with respect to a variable conic of the system, the locus of the points where this line meets the conic is another conic; for parallel diameters of the corresponding coaxaloid system of circles have their extremities lying on a conic [16]. By varying the point on DEF, we obtain a system of conics associated with the original system,

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and having the same envelope [17]. The poles of DEF with respect to the associated conics coincide with the pole of DEF with respect to the envelope; for in the corresponding coaxaloid system of circles, the associated hyperbolas are concentric with the envelope [16].

It may readily be proved that the associated conics are, in general, hyperbolas. In the general case DEF does not touch the conics of the original system, so that the line drawn from any point (G) on DEF through the pole (H) of DEF with respect to a variable conic of the system, meets this conic in two points (W, W′) which never coincide except in the cases noted below. Hence the conic on which W, W′ move is a hyperbola, W and W′ lying on opposite branches.*

Degenerate forms of the hyperbolas are obtained (1) when DEF touches the conics, for then H and consequently W and W′, for all positions of G and of the variable conic, coincide in the common point of contact, (2) when the conics touch each other at the points (J, J′) where they meet DEF (*see next article*), for then H is the same for all the conics, and consequently for any given position of G, and for all positions of the variable conic, W and W′ are collinear, and (3) when G coincides with either J or J′, in which case W and W′ are, for all positions of the variable conic, coincident at either J or J′.

(v) The six points in which a variable conic of the system meets the sides of ABC determine two triangles inscribed in ABC (corresponding to the triangles of constant species in the case of the circles [7]), which are such that the sides about any angle of either together with the lines joining the vertex of the angle to the two points where the conic meets DEF, form a pencil whose cross-ratio is a constant.

(vi) The conics of the system are related to any other triangle circumscribed to their envelope, in the same way that they are related to ABC [13].

2. When the original conic of a system such as has been described in Art. 1 is inscribed in ABC, the system may obviously

* If the system be regarded as the projection of a coaxaloid system of circles having imaginary Π-points, in which case it is not connected with any real triangle as a system of six-point conics, W and W′ lie on the same branch.

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be projected into a system of coaxaloid circles one of which is inscribed in a triangle, *i.e.*, into one of the four concentric systems [22]. Hence, in this case, the conics have double contact with each other (real or imaginary) on DEF, and the pole of DEF with respect to every conic of the system is the same.

3. When DEF is made to move off to infinity while ABC and PQR, and consequently the original conic of the system, remain unaltered, *i.e.*, when the sides of the variable triangle P′Q′R′ are drawn parallel to the sides of PQR, the conics of the system, having two common points at infinity (Art. 1, i), are similar and similarly placed. In this case the poles of DEF, with respect to the envelope and the conics of the system, become the centres of the envelope and the conics. Hence the centres of the conics and their envelope are collinear; and systems of parallel diameters of the conics have their extremities lying on a system of hyperbolas which have the same envelope as the original conics and are concentric with this envelope. This case is, therefore, very closely analogous to that of coaxaloid circles.

As an illustration of the theorem of the present article, the following may be given: If two transversals which intersect in K, meet the sides BC, CA, AB of a triangle in L, M, N and *l*, *m*, *n* respectively, and if S be the centre of perspective of ABC and the triangle determined by M*n*, N*l*, L*m*, and if variable parallels M′*n*′, N′*l*′, L′*m*′ be drawn to M*n*, N*l*, L*m* intersecting correspondingly on CS, AS, BS, and meeting BC, CA, AB in the pairs of points L′ and *l*′, M′ and *m*′, N′ and *n*′ respectively, then the six variable points L′, *l*′, M′, *m*′, N′, *n*′ determine a system of hyperbolas whose asymptotes are parallel to the two transversals, and whose centres lie on a line through K. In this case the line-pair formed by the transversals constitutes a hyperbola similar and similarly placed to the others of the system.

When the original conic of a system is inscribed in the triangle, and DEF is at infinity, the conics, besides being similar and similarly placed, are concentric (Art. 2). Thus, if the original conic is a parabola touching the sides of the triangle, and DEF be at infinity, the other conics are coaxial parabolas.

4. Since DEF can occupy any position in the plane, every conic

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which meets the sides of a triangle in six points, belongs to an infinite number of systems such as have been described in Art. 1. Every conic which meets the sides of a triangle in six points gives rise, just as in the case of the circle [2], to four S-points. Consequently, each of the above-mentioned systems has as S-point one or other of four points, and as envelope one or other of four conics touching the sides of the triangle at the points where these are met by the connectors of the opposite vertices with S. Also there are four systems for every position of DEF

Hence we obtain the following theorems.

(i) A conic which meets the sides of a triangle in six points has, in general, double contact (real or imaginary) with four conics inscribed in the triangle.

(ii) If a conic touches the sides BC, CA, AB of a triangle at X, Y, Z respectively, and if S be the point of concurrence of AX, BY, CZ, the sides of every triangle in perspective with ABC with respect to S as centre of perspective, meet the non-corresponding sides of ABC in pairs of points which lie on a conic having double contact (real or imaginary) with the original conic. This is evident from the fact that each of the secondary conics belongs to a system having double contact with the original conic.

The converse of the immediately preceding theorem may be thus stated: If a conic κ touch the sides BC, CA, AB of a triangle at X, Y, Z, and S be the point of concurrence of AX, BY, CZ, and if a variable six-point conic κ' have double contact with κ, then two of the eight variable triangles formed by joining the six points where κ' meets the sides of ABC from side to side in pairs, is in perspective with ABC with respect to S as centre of perspective.

This leads to the following two interesting particular cases.

(*a*) If ABC be a triangle circumscribed to a conic, and two variable tangents be drawn to the conic meeting BC, CA, AB, one in L, M, N and the other in l, m, n respectively, then the variable triangle determined by the connectors Mn, Nl, Lm (or mN, nL, lM) is in perspective with ABC with respect to a fixed point S as centre of perspective, S being the point of concurrence of the lines joining A, B, C to the points of contact of the conic with the opposite sides. In this case a variable line-pair is taken instead of the variable conic of the general case.

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(*b*) If ABC be a triangle circumscribed to a parabola, and a variable tangent be drawn to the parabola meeting BC, CA, AB in L, M, N respectively, and if through L, M, N parallels be drawn to CA, AB, BC (or AB, BC, CA) respectively, the variable triangle so determined is in perspective with ABC with respect to a fixed point S as centre of perspective, S being the point of concurrence of the lines joining A, B, C to the points of contact of the parabola with opposite sides. In this case, one of the variable tangents of (*a*) is the tangent at infinity of the parabola. The present case could be established independently of (*a*) by recollecting that the Simson lines of a point, with the line at infinity, form a coaxaloid system of circles [7], and have, therefore, a common S-point. (*a*) could then be deduced from (*b*) by projection.

II.

Systems of Six-tangent Conics.

5. By Brianchon's theorem if two triangles ABC and PQR be in perspective with respect to any line *s* as axis of perspective, the vertices of PQR connect with the non-corresponding vertices of ABC by means of six lines which touch a conic. Hence if any point U be taken in the plane, and a triangle P′Q′R′ be drawn in perspective with PQR with respect to *s* as axis and U as centre of perspective, the vertices of P′Q′R′ connect with the non-corresponding vertices of ABC by means of six lines which also touch a conic. By causing P′Q′R′ to vary so as to remain in perspective with PQR with respect to *s* as axis and U as centre of perspective, we obtain a system of six-tangent conics connected with ABC.

The general properties of such a system are most readily obtained by observing that it is the polar reciprocal of a six-point system such as is discussed in Art. 1, *s* and U of the former corresponding to S and DEF of the latter. We thus obtain the following propositions.

(i) The conics of the system have a pair of common tangents (real or imaginary) which meet in U.

(ii) The envelope of the system is a conic circumscribed to ABC,

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and touching at its vertices the lines connecting these with the points where the opposite sides meet *s*.

(iii) The polars of U with respect to the envelope and the conics of the system are concurrent; generally the polars of any point with respect to the conics envelop a conic.

(iv) The envelope of the tangents to a variable conic of the system from the point where any line through U meets the polar of U with respect to the conic, is a conic. By varying the line through U, we obtain a system of conics associated with the original system and having the same envelope.

(v) The six tangents from the vertices of ABC to a variable conic of the system determine two triangles circumscribed to ABC which are such that the extremities of any side of either together with the points of intersection of that side and the tangents to the conic from U, form a range whose cross-ratio is a constant.

(vi) The conics of the system are related to any other triangle inscribed in their envelope in the same way that they are related to ABC.

To these may be added:—

(vii) The poles of any line with respect to the conics, and consequently their centres, lie on a conic [25].

6. When one of the conics, say the original conic, of a system such as has been described in the preceding article, is circumscribed to ABC, the system is obviously the polar reciprocal of a system of the kind referred to in Art. 2. Hence, in this case, the conics touch their two common tangents at the same pair of points, and the polar of U with respect to every conic of the system is the same.

7. If U be a focus of one conic of a system, say the original conic, it is a focus of all the conics of the system. This is evident if we reciprocate a coaxaloid system of circles with respect to U as centre of reciprocation. In this case, since the centres of the circles reciprocate into the directrices corresponding to U of the conics, or by Art. 1, (iii), the directrices corresponding to the common focus are concurrent.

When U is taken so as to coincide with one of the Π-points,

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say Π, of the reciprocal system of coaxaloid circles (Fig. 9), the eccentricity of the conic into which the circle K′ reciprocates is

$$K'\Pi / \text{ radius of circle } K'$$
$$= K'\Pi/(K'\Pi \,.\, KT/K\Pi) = K\Pi/KT = \text{a constant.}$$

Therefore, in this case, the conics into which the circles reciprocate, have the same eccentricity, *i.e.*, they are similar. Also, since the centre of reciprocation U is a focus of the envelope of the circles, the envelope of the conics is, in general, a circle.

Hence we have the following theorem: If a system of similar conics have a common focus U and concurrent directrices, their envelope is, in general, a circle; and if a triangle be inscribed in this circle, and *s* be the line on which the tangents to the circle at the vertices meet the opposite sides, the tangents from the vertices to a variable conic of the system determine by their intersection two variable triangles which are always in perspective with themselves with respect to *s* and U as axis and centre of perspective.

In order to determine the centre and radius of the circle enveloping a system of similar conics having a common focus and concurrent directrices we may proceed as follows. Let U be the common focus, O the point of concurrence of the directrices, *d* the perpendicular through O to UO, and A, A the vertices, lying on UO, of the conic of the system, denoted by κ, whose directrix is *d*. Then since *d* is the reciprocal, with respect to U, of the centre of the auxiliary circle of the envelope of the reciprocal system of coaxaloid circles, κ is the reciprocal of this circle. Again, since O is the reciprocal of the line of centres of the reciprocal system of coaxaloid circles, and U (being the centre of reciprocation) the reciprocal of the line at infinity, UO is the reciprocal of the point at infinity on the line of centres of the coaxaloid circles, *i.e.*, of the point of concurrence of the line of centres and the two tangents common to the envelope of the coaxaloid circles and its auxiliary circle (the tangents at T, T′ in Fig. 9). Therefore A and A′, where κ meets UO, are the reciprocals of these tangents, and are, consequently, the common points of κ and the circle enveloping the conics. Therefore AA′ is a diameter of this circle, whose centre and radius, are, consequently, determined.

In the particular case when the conics are parabolas, one of the vertices A, A, say A′, is at infinity; the enveloping circle, con-

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sequently, degenerates into the line at infinity and the tangent at A to the parabola whose directrix is d. Otherwise: If a system of parabolas have a common focus and concurrent directrices, they have a common tangent (real) in addition to the line at infinity.

If one conic of a system, say the original conic, be circumscribed to the triangle ABC, and U be its focus, then the conics, besides having a common focus, have a common directrix. This result is obtained by reciprocating a concentric system of coaxaloid circles connected with a triangle, from any centre (U). Hence we have the theorem: If a conic be circumscribed to a triangle, U a focus, d the corresponding directrix, and s the line on which the tangents at the vertices meet the opposite sides, and if a system of conics be constructed having the same focus U and directrix d, the tangents from the vertices of the triangle to a variable conic of this system determine two variable triangles which remain in perspective with themselves with respect to s and U as axis and centre of perspective respectively.

In the particular case when the conic circumscribed to ABC is the circumcircle and U the circumcentre, then the common directrix d is the line at infinity, and hence the other conics are circles concentric with the circumcircle. The theorem involved in this case could also be obtained by elementary geometry.

8. By reciprocating the theorems of Art. 4 we could obtain an analogous set of theorems applying to six-tangent conics connected with the triangle. I shall limit myself to enunciating two of these.

(i) If a conic be circumscribed to a triangle ABC, and s be the line on which the tangents at the vertices meet the opposite sides, then the vertices of every triangle in perspective with ABC with respect to s as axis of perspective, connect with the non-corresponding vertices of ABC by pairs of lines which touch a conic having double contact (real or imaginary) with the original conic (Art. 4, ii).

(ii) If ABC be a triangle inscribed in a conic, and if two variable points be taken on the conic, connecting with A, B, C, the one by the lines L, M, N and the other by the lines l, m, n respectively, then the variable triangle determined by the points of intersection of the pairs M and n, N and l, L and m (or m and N, n and L, l and M) is in perspective with ABC with respect to a fixed line s as axis of perspective, s being the line on which the tangents to the conic at A, B, C meet the opposite sides.

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