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NOTE ON THE SINGULAR SUBMODULE

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1. **Introduction.** One very interesting and important problem in ring theory is the determination of the position of the singular ideal of a ring with respect to the various radicals (Jacobson, prime, Wedderburn, etc.) of the ring. A summary of the known results can be found in Faith [3, p. 47 ff.] and Lambek [5, p. 102 ff.]. Here we use a new technique to obtain extensions of these results as well as some new ones.

Throughout we adopt the Bourbaki [2] conventions for rings and modules: all rings have 1, all modules are unital, and all ring homomorphisms preserve the 1.

2. **The main result.** Let ${}_AM_B$ be a bimodule. For $b \in B$ define $l(b) = (m \in M \mid mb = 0)$, an A-submodule of M. And for $m \in M$ define $l(m) = (a \in A \mid am = 0)$, a left ideal of A.

Now define $Z(B) = Z_M(B) = (b \in B \mid l(b) \nabla M)$ where ∇ denotes essential extension (=large submodule), and $Z(M) = Z_A(M) = (m \in M \mid l(m) \nabla A)$. It is easy to verify that $Z(B)$ is a two-sided ideal of B and that $Z(M)$ is an $A-B$ submodule of M. In fact $Z_A(\ )$ defined in the category of $A-B$ bimodules is a subfunctor of the identity functor, usually called the singular submodule of Johnson [4].

Note also that $Z(B)$ is invariant with respect to every ring homomorphism $\psi: B \to C$ such that ${}_AM_C$ is also a bimodule; i.e. $\psi Z(B) \subseteq Z(C)$. Hence $\psi Z(B) \subseteq Z(\psi B) \subseteq Z(C)$.

The proofs of the following two lemmas are straightforward and hence omitted. (Lemma 1 is needed for Lemma 2.)

LEMMA 1. *For any $m \in M$ and $b \in B$,*

$$Am \cap l(b) \simeq l(mb)/l(m) \quad \text{as } A\text{-modules.}$$

LEMMA 2. *With the same notation consider the following three conditions:*

(i) $b \in Z(B)$

(ii) $l(m) \neq l(mb)$

(iii) $m \neq 0$.

Then any two conditions imply the third, and hence in the presence of any one the other two are equivalent.

MAIN THEOREM. *Suppose A has maximum condition on left annihilator ideals and let $x_1, x_2, \ldots$ be a sequence of elements of $Z(B)$. Define $b_n = x_1 x_2 \ldots x_n$. Then:*

(i) $M = Ul(b_n)$

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(ii) If *M has maximum condition on annihilator A-submodules then there exists an integer N with* $M = l(b_N)$

(iii) *If M is also B-faithful* (*e.g.* $B = \text{End}\,_A M$ *or M is B-free*) *then* $b_N = 0$.

Proof. (i) For $m \in M$, $l(mb_n)$ is an ascending chain of left annihilator ideals which becomes stationary with $l(mb_n) = l(mb_n x_{n+1})$ say. By Lemma 2 $mb_n = 0$ since $x_{n+1} \in Z(B)$. Hence $m \in l(b_n)$.

(ii) and (iii) are now clear.

COROLLARY. *Under all of the above conditions $Z(B)$ is T-nilpotent in the sense of Bass* (1), *and hence* $Z(B) \subseteq \text{rad}\, B =$ *prime radical of B. Therefore B semiprime* $\Rightarrow B$ *neat in the sense of Bourbaki* [2].

Proof. It is easy to verify that every T-nilpotent ideal is contained in the prime radical, using the equivalent definition given by Lambek [5, p. 55].

COROLLARY. *If the maximum length of chains of left annihilator ideals of A is N* (*e.g. if A is an artinian ring of length N*) *then* $M = l(b_N)$ *and* $b_N = 0$ *if M is B-faithful. In this case* $(Z(B))^N = 0$, *i.e.* $Z(B)$ *is nilpotent and hence contained in the Wedderburn radical* (= *sum of all nilpotent ideals*).

3. **Applications.** (1) Let $_A M$ be a quasi-injective module and $B = \text{End}\,_A M$. Then $Z(B) = \text{Rad}\, B$ (= the Jacobson radical). Hence if A has maximum condition on left annihilator ideals and M has maximum condition on annihilator A-submodules then $Z(B) = \text{Rad}\, B = \text{rad}\, B$.

(2) If $M = B$ then condition (iii) of the theorem holds always. Thus if $M = B = AG$, the group ring over a finite group G, and A has maximum condition on left annihilator ideals then $Z_{AG}(AG) \subseteq \text{rad}\, AG$.

(3) If $A = M = B$ then $Z(B)$ is the (left) singular ideal. Thus if B has maximum condition on left annihilator ideals then $Z(B)$ is T-nilpotent and hence contained in rad B.

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