MULTIALGEBRAS, UNIVERSAL ALGEBRAS AND IDENTITIES

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Abstract

In this paper we determine the smallest equivalence relation on a multialgebra for which the factor multialgebra is a universal algebra satisfying a given identity. We also establish an important property for the factor multialgebra (of a multialgebra) modulo this relation.


Keywords and phrases: polynomial function, factor multialgebra, identity, hypergroup, derived subhypergroup.

1. Introduction

The starting point of this paper can be found in [7] where Freni presents the smallest equivalence on a (semi)hypergroup for which the factor (semi)hypergroup is a commutative (semi)group.

Multialgebras (also called hyperstructures) are particular cases of relational systems which are generalizations of universal algebras. They have been studied for more than 60 years and have been used in different areas of mathematics (algebra, geometry, graph theory) as well as in applied sciences (see [5]).

It follows from [7] (and also from [6] and [13]) that, among the equivalence relations of a multialgebra, of great importance are those equivalence relations for which the factor multialgebra is a universal algebra. For a multialgebra \( \mathbb{A} \), the class of these relations is an algebraic closure system. It follows that we can always obtain a smaller such equivalence, which contains a relation \( R \) on \( A \). Using this, some of the results of [7] can be established in the general case of multialgebras. More precisely, we determine the smallest equivalence relation \( \alpha^{*}_{qr} \) on a multialgebra that has the property that the factor multialgebra is a universal algebra satisfying a given identity \( q = r \).
In [8], Grätzer proved that any multialgebra \( \mathfrak{A} \) is obtained as a factor of a universal algebra \( \mathfrak{B} \) by an appropriate equivalence relation \( \rho \subseteq B \times B \). For a multialgebra \( \mathfrak{B}/\rho \), we consider the universal algebra \( (\mathfrak{B}/\rho)/\alpha \mathfrak{A} \) and we prove, in Theorem 5.3, that this algebra is isomorphic to the factor algebra of \( \mathfrak{B} \) modulo the smallest congruence relation \( \theta \) of \( \mathfrak{B} \) which has the property that \( \rho \subseteq \theta \) and \( \alpha \mathfrak{A} = \mathfrak{B} \) is satisfied on \( \mathfrak{B}/\theta \).

In the last section we give an application to hypergroups, which are factor of a group modulo an equivalence relation determined by a subgroup.

While studying some properties of the factor multialgebra of a multialgebra we have found an answer to the first part of Problem 4 from [8]: What are the factor multialgebras of a group, abelian group, lattice, ring and so on? Characterize these with a suitable axiom system. In the third section of this paper, we prove that an \( n \)-ary identity \( q = r \) on an algebra \( \mathfrak{B} \) gives the weak identity \( q \cap r \neq \emptyset \) on the multialgebra \( \mathfrak{B}/\rho \). Yet, as mentioned at the end of [8], there exist multialgebras with one binary associative multioperation, which are not factor multialgebras of a semigroup. So, a multialgebra that satisfies a set of given weak (or strong) identities does not have to be a factor multialgebra of a universal algebra satisfying the corresponding identities. This means that our answer does not cover the second part of this problem.

### 2. Preliminaries

Let \( \mathbb{N} \) be the set of the nonnegative integers, let \( \tau = (n_\gamma)_{\gamma < o(\tau)} \) be a sequence over \( \mathbb{N} \), where \( o(\tau) \) is an ordinal, let \( f_\gamma \) be a symbol of an \( n_\gamma \)-ary (multi)operation for any \( \gamma < o(\tau) \), and let \( \mathfrak{P}^{(n)}(\tau) = (P^{(n)}(\tau), (f_\gamma)_{\gamma < o(\tau)}) \) be the algebra of \( n \)-ary terms (of type \( \tau \)).

Let \( A \) be a set and \( P^*(A) \) the family of nonempty subsets of \( A \). Let \( \mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)}) \) be a multialgebra, where for any \( \gamma < o(\tau) \), \( f_\gamma : A^{n_\gamma} \to P^*(A) \) is the multioperation of arity \( n_\gamma \) that corresponds to the symbol \( f_\gamma \). If the multialgebra \( \mathfrak{A} \) has no nullary multioperations, then we allow the support set \( A \) to be empty. Of course, any universal algebra is a multialgebra (we identify a one element set with its element).

If for any \( \gamma < o(\tau) \) and for any \( A_0, \ldots, A_{n_\gamma - 1} \in P^*(A) \), we define

\[
f_\gamma(A_0, \ldots, A_{n_\gamma - 1}) = \bigcup \{ f_\gamma(a_0, \ldots, a_{n_\gamma - 1}) \mid a_i \in A_i, \; i \in \{0, \ldots, n_\gamma - 1\} \},
\]

then we obtain a universal algebra on \( P^*(A) \) (see [12]). We denote this algebra by \( P^*(\mathfrak{A}) \) and consider the algebra \( P^{(n)}(P^*(\mathfrak{A})) \) of the \( n \)-ary term functions on \( P^*(\mathfrak{A}) \) (\( n \in \mathbb{N} \)). Clearly, any term from \( p \in P^{(n)}(\tau) \) induces a term function from \( P^{(n)}(P^*(\mathfrak{A})) \) ([9, Corollary 8.1]). We denote this term function by \( p \) (or by \( (p)_{P^*(\mathfrak{A})} \) when necessary).
Denote by $\mathcal{P}^{(n)}_{P^{(A)}(\mathcal{P}^{(A)})}$ the algebra of the $n$-ary polynomial functions of the universal algebra $\mathcal{P}^{(A)}(\mathcal{A})$ (see [3]) and by $\mathcal{P}^{(n)}_{A}(\mathcal{P}^{(A)})$ its subalgebra generated by

$$\{c_{a}^{n} \mid a \in A\} \cup \{e_{i}^{n} \mid i \in \{0, \ldots, n - 1\}\},$$

where $c_{a}^{n}, e_{i}^{n} : P^{*}(A)^{n} \rightarrow P^{*}(A)$ are defined by

$$c_{a}^{n}(A_{0}, \ldots, A_{n-1}) = \{a\} \quad \text{and} \quad e_{i}^{n}(A_{0}, \ldots, A_{n-1}) = A_{i}.$$

**REMARK 1.** For a multialgebra $\mathcal{A}$, $P^{(n)}(\mathcal{P}^{*}(\mathcal{A}))$ is a subalgebra of $\mathcal{P}^{(n)}_{A}(\mathcal{P}^{(A)})$.

**REMARK 2.** Let $\mathcal{A}$ be a universal algebra and let $P^{(n)}_{A}(\mathcal{A})$ be the set of the $n$-ary polynomial functions of $\mathcal{A}$. For any polynomial function $p \in P^{(n)}_{A}(\mathcal{P}^{*}(\mathcal{A}))$, the map

$$A^{n} \rightarrow A, (a_{0}, \ldots, a_{n-1}) \mapsto p(a_{0}, \ldots, a_{n-1})$$

defines a polynomial function from $P^{(n)}_{A}(\mathcal{A})$. Moreover, any polynomial function from $P^{(n)}_{A}(\mathcal{A})$ can be obtained in this way from a polynomial function from $P^{(n)}_{A}(\mathcal{P}^{*}(\mathcal{A}))$.

**REMARK 3.** It is known that if $\gamma < o(\tau)$ and $A_{0}, \ldots, A_{n_{\gamma} - 1}, B_{0}, \ldots, B_{n_{\gamma} - 1}$ are nonempty subsets of $A$ such that $A_{0} \subseteq B_{0}, \ldots, A_{n_{\gamma} - 1} \subseteq B_{n_{\gamma} - 1}$, then

$$f_{\gamma}(A_{0}, \ldots, A_{n_{\gamma} - 1}) \subseteq f_{\gamma}(B_{0}, \ldots, B_{n_{\gamma} - 1}).$$

It easily follows that if $n \in \mathbb{N}$, $p \in P^{(n)}_{P^{*}(A)}(\mathcal{P}^{*}(\mathcal{A}))$, and the nonempty subsets $A_{0}, \ldots, A_{n-1}, B_{0}, \ldots, B_{n-1}$ of $A$ are such that $A_{0} \subseteq B_{0}, \ldots, A_{n-1} \subseteq B_{n-1}$, then

$$p(A_{0}, \ldots, A_{n-1}) \subseteq p(B_{0}, \ldots, B_{n-1}).$$

A map $h : A \rightarrow B$ between the multialgebras $\mathcal{A}$ and $\mathcal{B}$ of the same type $\tau$ is called a homomorphism if for any $\gamma < o(\tau)$ and for all $a_{0}, \ldots, a_{n_{\gamma} - 1} \in A$, we have

$$h(f_{\gamma}(a_{0}, \ldots, a_{n_{\gamma} - 1})) \subseteq f_{\gamma}(h(a_{0}), \ldots, h(a_{n_{\gamma} - 1})).$$

A bijective map $h$ is a *multialgebra isomorphism* if both $h$ and $h^{-1}$ are multialgebra homomorphisms. It follows from [12] that the multialgebra isomorphisms can be characterized as being the bijective homomorphisms $h$ for which the inclusion (1) is an equality.

**REMARK 4.** By the construction of a polynomial (symbol) it follows that for a homomorphism $h : A \rightarrow B$, if $n \in \mathbb{N}$, $p \in P^{(n)}(\tau)$, and $a_{0}, \ldots, a_{n-1} \in A$, then

$$h(p(a_{0}, \ldots, a_{n-1})) \subseteq p(h(a_{0}), \ldots, h(a_{n-1})).$$
Let \( \rho \) be an equivalence relation on \( A \) and \( A/\rho = \{ \rho(x) \mid x \in A \} \) (where \( \rho(x) \) denotes the class of \( x \) modulo \( \rho \)). For a \( \gamma < o(\tau) \), the equalities

\[
\begin{align*}
&f_\gamma(\rho(a_0), \ldots, \rho(a_{n-1})) \\
&= \{ \rho(b) \mid b \in f_\gamma(b_0, \ldots, b_{n-1}), \ a_i b_i, \ i \in \{0, \ldots, n_\gamma - 1\} \}
\end{align*}
\]

define a multioperation \( f_\gamma \) on \( A/\rho \) (see [8]). One obtains a multialgebra \( \mathfrak{A}/\rho \) on \( A/\rho \) called the factor multialgebra of \( \mathfrak{A} \) modulo \( \rho \).

The definition of the multiopeations from \( \mathfrak{A}/\rho \) allows us to see the canonical map \( \pi_\rho \) from \( A \) onto \( A/\rho \) as a multialgebra homomorphism for any equivalence relation \( \rho \) on \( A \) (see [12]). Applying (2) for \( \pi_\rho \), we have

\[
\{ \rho(a) \mid a \in (p)_{\mathfrak{P}^*(\mathfrak{A})}(a_0, \ldots, a_{n-1}) \} \subseteq (p)_{\mathfrak{P}^*(\mathfrak{A}/\rho)}(\rho(a_0), \ldots, \rho(a_{n-1}))
\]

for any \( n \in \mathbb{N}, \ p \in \mathfrak{P}^{(n)}(\tau), \) and \( a_0, \ldots, a_{n-1} \in A \).

**Remark 5.** The inclusion (3) holds if we replace the \( n \)-ary term functions \((p)_{\mathfrak{P}^*(\mathfrak{A})}\) and \((p)_{\mathfrak{P}^*(\mathfrak{A}/\rho)}\) by the \( n \)-ary polynomial functions

\[
p \in \mathfrak{P}^{(n)}_A(\mathfrak{P}^*(\mathfrak{A})) \quad \text{and} \quad p' \in \mathfrak{P}^{(n)}_{A/\rho}(\mathfrak{P}^*(\mathfrak{A}/\rho)),
\]

respectively, where the polynomial function \( p' \) (which corresponds to \( p \)) is given as follows

(i) if \( p = c^n_a \), then \( p' = c^n_{\rho(a)} \);  
(ii) if \( p = e^n_i = (x_i)_{\mathfrak{P}^*(\mathfrak{A})} \), then \( p' = e^n_i = (x_i)_{\mathfrak{P}^*(\mathfrak{A}/\rho)} \);  
(iii) if \( p = f_\gamma(p_0, \ldots, p_{n_\gamma - 1}) \) and the polynomial functions which correspond to \( p_0, \ldots, p_{n_\gamma - 1} \in \mathfrak{P}^{(n)}_A(\mathfrak{P}^*(\mathfrak{A})) \) are \( p'_0, \ldots, p'_{n_\gamma - 1} \in \mathfrak{P}^{(n)}_{A/\rho}(\mathfrak{P}^*(\mathfrak{A}/\rho)) \) respectively, then \( p' = f_\gamma(p'_0, \ldots, p'_{n_\gamma - 1}) \).

Since the polynomial function \( p' \) is obtained by using the same steps as in the construction of \( p \), if we take into account (i) from above, and write \( p \) instead of \( p' \) then

\[
\{ \rho(a) \mid a \in p(a_0, \ldots, a_{n-1}) \} \subseteq p(\rho(a_0), \ldots, \rho(a_{n-1})).
\]

Let \( q, r \in \mathfrak{P}^{(n)}(\tau) \) and let \( \mathfrak{A} \) be a multialgebra of type \( \tau \). We say that the \( n \)-ary (strong) identity \( q = r \) is satisfied in the multialgebra \( \mathfrak{A} \) if

\[
q(a_0, \ldots, a_{n-1}) = r(a_0, \ldots, a_{n-1})
\]

for all \( a_0, \ldots, a_{n-1} \in A \), where \( q = (q)_{\mathfrak{P}^*(\mathfrak{A})} \) and \( r = (r)_{\mathfrak{P}^*(\mathfrak{A})} \). We also say that a weak identity \( q \cap r \neq \emptyset \) is satisfied in the multialgebra \( \mathfrak{A} \) if

\[
q(a_0, \ldots, a_{n-1}) \cap r(a_0, \ldots, a_{n-1}) \neq \emptyset
\]

for all \( a_0, \ldots, a_{n-1} \in A \).
REMARK 6. Many important particular multialgebras can be defined by using identities.

3. On the factor multialgebra of an algebra

THEOREM 3.1 ([8]). For any multialgebra $\mathcal{A}$ of type $\tau$, there exists a universal algebra $\mathcal{B}$ of type $\tau$ and an equivalence relation $\rho$ on $B$ such that $\mathcal{A} \cong \mathcal{B}/\rho$.

Since $\mathcal{B}$ is a universal algebra we can rewrite the multioperations from $\mathcal{B}/\rho$ as follows, for any $y < o(\tau)$ and any $b_0, \ldots, b_{n_y-1} \in B$,

$$f_y(\rho(b_0), \ldots, \rho(b_{n_y-1})) = \{\rho(c) \mid c = f_y(c_0, \ldots, c_{n_y-1}), b_i \rho c_i, i \in \{0, \ldots, n_y - 1\}\}.$$

REMARK 7. If we consider $\rho$ as in Remark 5 and if $b_0, \ldots, b_{n-1} \in B$, then we have

$$p(\rho(b_0), \ldots, \rho(b_{n-1})) \geq \{\rho(c) \mid c = p(c_0, \ldots, c_{n-1}), b_i \rho c_i, i \in \{0, \ldots, n - 1\}\}.$$

REMARK 8. It follows immediately that if $n \in \mathbb{N}$, $q, r \in \mathcal{P}(n)(\tau)$, and $q = r$ is satisfied on the universal algebra $\mathcal{B}$, then for any $b_0, \ldots, b_{n-1} \in B$ the class of

$$q(b_0, \ldots, b_{n-1}) = r(b_0, \ldots, b_{n-1})$$

modulo $\rho$ is in $q(\rho(b_0), \ldots, \rho(b_{n-1})) \cap r(\rho(b_0), \ldots, \rho(b_{n-1}))$, thus the weak identity $q \cap r \neq \emptyset$ is satisfied on the multialgebra $\mathcal{B}/\rho$.

Next, we give an example which shows that, in general, the inclusion (4) is not an equality. It also follows that the above weak identity on the multialgebra $\mathcal{B}/\rho$ does not need to be strong.

EXAMPLE 1. Let $(\mathbb{Z}_5, +)$ be the cyclic group of order 5 and let us consider on $\mathbb{Z}_5$ the equivalence relation $\rho = \{0, 1\} \times \{0, 1\} \cup \{2\} \times \{2\} \cup \{3, 4\} \times \{3, 4\}$. Then $\rho(0) = \rho(1) = \{0, 1\}$, $\rho(2) = \{2\}$, and $\rho(3) = \rho(4) = \{3, 4\}$. Using the above considerations we obtain a multialgebra with one binary multioperation (that is, a hypergroupoid) on $\mathbb{Z}_5/\rho = \{\{0, 1\}, \{2\}, \{3, 4\}\}$ with the table

<table>
<thead>
<tr>
<th>+</th>
<th>{0, 1}</th>
<th>{2}</th>
<th>{3, 4}</th>
</tr>
</thead>
<tbody>
<tr>
<td>{0, 1}</td>
<td>{0, 1}, {2}</td>
<td>{2}, {3, 4}</td>
<td>{0, 1}, {3, 4}</td>
</tr>
<tr>
<td>{2}</td>
<td>{2}, {3, 4}</td>
<td>{3, 4}</td>
<td>{0, 1}</td>
</tr>
<tr>
<td>{3, 4}</td>
<td>{0, 1}, {3, 4}</td>
<td>{0, 1}</td>
<td>{2}, {3, 4}</td>
</tr>
</tbody>
</table>
The inclusion (4) is not always an equality since
\[
\{ \rho(c) \mid c = (b_0 + b_1) + b_2, b_0 = b_1 = 2, b_2 \in \{3, 4\} \} = \{ \rho(c) \mid c \in \{2, 3\} \} = \{ \rho(2), \rho(3) \}
\]
and
\[
(\rho(2) + \rho(2)) + \rho(3) = \rho(3) + \rho(3) = \{ \rho(0), \rho(2), \rho(3) \}.
\]
We also have \( \rho(2) + (\rho(2) + \rho(3)) = \rho(2) + \rho(0) = \{ \rho(2), \rho(3) \} \). Thus the associativity holds only in a weak manner for the hypergroupoid \( (\mathbb{Z}_5/\rho, +) \).

**Remark 9.** Some identities, such as those which characterize the commutativity of an operation of a universal algebra, hold strongly on the factor multialgebra.

**Example 2.** Let \( \mathfrak{B} \) be a universal algebra of type \( \tau \). Let \( \gamma < o(\tau) \) and assume that for a permutation \( \sigma \) of the set \( \{0, \ldots, n_\gamma - 1\} \),
\[
f_\gamma(x_0, \ldots, x_{n_\gamma - 1}) = f_\gamma(x_{\sigma(0)}, \ldots, x_{\sigma(n_\gamma - 1)})
\]
is satisfied on \( \mathfrak{B} \).

Consider \( b_0, \ldots, b_{n_\gamma - 1} \in B \). If \( \rho(c) \in f_\gamma(\rho(b_0), \ldots, \rho(b_{n_\gamma - 1})) \), then there exist \( c_0, \ldots, c_{n_\gamma - 1} \in B \) with \( b_0 \rho c_0, \ldots, b_{n_\gamma - 1} \rho c_{n_\gamma - 1} \) such that \( c = f_\gamma(c_0, \ldots, c_{n_\gamma - 1}) \). However,
\[
b_{\sigma(0)} \rho c_{\sigma(0)}, \ldots, b_{\sigma(n_\gamma - 1)} \rho c_{\sigma(n_\gamma - 1)} \quad \text{and} \quad f_\gamma(c_0, \ldots, c_{n_\gamma - 1}) = f_\gamma(c_{\sigma(0)}, \ldots, c_{\sigma(n_\gamma - 1)}),
\]
thus \( \rho(c) \in f_\gamma(\rho(b_{\sigma(0)}), \ldots, \rho(b_{\sigma(n_\gamma - 1)}) \). Clearly, the identity
\[
f_\gamma(x_{\sigma^{-1}(0)}, \ldots, x_{\sigma^{-1}(n_\gamma - 1)}) = f_\gamma(x_0, \ldots, x_{n_\gamma - 1})
\]
also holds on the universal algebra \( \mathfrak{B} \), hence
\[
f_\gamma(\rho(b_{\sigma(0)}), \ldots, \rho(b_{\sigma(n_\gamma - 1)})) \subseteq f_\gamma(\rho(b_0), \ldots, \rho(b_{n_\gamma - 1})).
\]
It means that for any \( b_0, \ldots, b_{n_\gamma - 1} \in B \), we have
\[
f_\gamma(\rho(b_0), \ldots, \rho(b_{n_\gamma - 1})) = f_\gamma(\rho(b_{\sigma(0)}), \ldots, \rho(b_{\sigma(n_\gamma - 1)})),
\]
hence the identity \( f_\gamma(x_0, \ldots, x_{n_\gamma - 1}) = f_\gamma(x_{\sigma(0)}, \ldots, x_{\sigma(n_\gamma - 1)}) \) is satisfied on the multialgebra \( \mathfrak{B} / \rho \).

Now we describe the factor multialgebras of semigroups, groups, abelian groups, rings, and lattices.

**3.1. The case of semigroups** Consider a semigroup \( (S, \cdot) \) and an equivalence relation \( \rho \) on \( S \). According to Remark 8, the hypergroupoid \( (S/\rho, \cdot) \) satisfies the associativity in a weak manner. Such a hyperstructure is called \( H_\tau \)-semigroup (see [14]).
3.2. The case of groups Consider a group \((G, \cdot)\) and let \(\rho\) be an equivalence relation on \(G\). The existence and uniqueness of solutions of the equations \(ya = b\) and \(ax = b\) allow us to define on \(G\) the operations / and \(\backslash\) by

\[ b/a = \{y \in G \mid b = ya\} \quad \text{and} \quad a\backslash b = \{x \in G \mid b = ax\}. \]

So, the group \(G\) can be seen as a universal algebra \((G, \cdot, /, \backslash)\) satisfying the following identities

\[
\begin{align*}
(x_0 \cdot x_1) \cdot x_2 &= x_0 \cdot (x_1 \cdot x_2), & x_1 &= x_0 \cdot (x_0 \backslash x_1), & x_1 &= (x_1 / x_0) \cdot x_0, \\
x_1 &= x_0 \backslash (x_0 \cdot x_1), & x_1 &= (x_1 \cdot x_0) / x_0.
\end{align*}
\]

We obtain the multialgebra \((G/\rho, \cdot, /, \backslash)\), which satisfies the above identities in a weak manner. It follows that \((G/\rho, \cdot)\) is an \(H_1\)-semigroup satisfying

\[
\rho(a) \cdot G/\rho = G/\rho = G/\rho \cdot \rho(a), \quad \text{for any} \quad a \in G,
\]

that is, is an \(H_1\)-group (see [14]). In general, an \(H_1\)-group does not have an identity element. In our case the class \(\rho(1)\) of the identity element of \(G\) satisfies the condition

\[
\rho(a) \in \rho(a) \cdot \rho(1) \cap \rho(1) \cdot \rho(a), \quad \text{for any} \quad a \in G,
\]

hence the weak identities \(x_0 \cdot 1 \cap x_0 \neq \emptyset\) and \(1 \cdot x_0 \cap x_0 \neq \emptyset\) are satisfied on the \(H_1\)-group \(G/\rho\). Moreover, any class \(\rho(a) \in G/\rho\) has an inverse since

\[
\rho(1) \in \rho(a^{-1}) \cdot \rho(a) \cap \rho(a) \cdot \rho(a^{-1})
\]

\((a^{-1}\) denotes the inverse of \(a\) in \(G\)).

If the group \(G\) is abelian, then the \(H_1\)-group \(G/\rho\) is commutative (see Example 2).

3.3. The case of rings In [14], a hyperstructure \((R, +, \cdot)\) is called \(H_1\)-ring if \((R, +)\) is an \(H_1\)-group, \((R, \cdot)\) is an \(H_1\)-semigroup and for any \(a, b, c \in R\) we have

\[
a(b + c) \cap (ab + ac) \neq \emptyset \quad \text{and} \quad (b + c)a \cap (ba + ca) \neq \emptyset.
\]

It is easy to see that the factor multialgebra of a ring is an \(H_1\)-ring (for which the first multioperation is commutative).

Remark 10. Since the absorption (which is required in the definition of a hyperlattice, see [11]) is not satisfied in a strong manner in the factor multialgebra, the factor multialgebra of a lattice is not necessarily a hyperlattice.
EXAMPLE 3. Consider the lattice \((\mathbb{N}, \wedge, \vee)\), where \(\mathbb{N} = \{0, 1, 2, \ldots\}\) is the set of the nonnegative integers, \(a \wedge b = \gcd(a, b)\), and \(a \vee b = \text{lcm}(a, b)\). Let us denote by \(\mathbb{P}\) the set \(\{2, 3, 5, \ldots\}\) of prime numbers and consider
\[
\rho = \mathbb{P} \times \mathbb{P} \cup \{(a, a) \mid a \in \mathbb{N} \setminus \mathbb{P}\}.
\]
Clearly, \(\rho\) is an equivalence relation on \(\mathbb{N}\) and we have
\[
\rho(2) \in \rho(2) \vee (\rho(2) \wedge \rho(6)) = \rho(2) \vee \{\rho(1), \rho(2)\} = \{\rho(2)\} \cup \{\rho(pq) \mid p, q \in \mathbb{P}, p \neq q\},
\]
so the absorption holds only in a weak manner.

4. A class of equivalence relations on a multialgebra

Let \(\rho\) be an equivalence relation on the set \(A\). We denote by \(\overline{\rho}\) the relation defined on \(P^*(A)\) as follows. If \(X, Y \in P^*(A)\), then
\[
X \overline{\rho} Y \iff x \rho y, \forall x \in X, \forall y \in Y (\iff X \times Y \subseteq \rho).
\]
It follows immediately that \(\overline{\rho}\) is symmetric and transitive. In general, \(\overline{\rho}\) is not reflexive. Indeed, let us take, for example, the equality relation on \(A\), denoted here by \(\delta_A\). The relation \(\delta_A\) is reflexive if and only if \(|A| = 1\).

**Proposition 4.1.** Let \(\mathfrak{A} = (A, (f\gamma)_{\gamma \leq o(\gamma)})\) be a multialgebra of type \(\tau\) and let \(\rho\) be an equivalence relation on \(A\). The following conditions are equivalent:

(a) \(\mathfrak{A}/\rho\) is a universal algebra;
(b) If \(\gamma < o(\tau)\), \(a, b, x_i \in A\), \(i \in \{0, \ldots, n_\gamma - 1\}\), with \(a \rho b\), then
\[
f_\gamma(x_0, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_{n_\gamma-1}) \overline{\rho} f_\gamma(x_0, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_{n_\gamma-1}),
\]
for all \(i \in \{0, \ldots, n_\gamma - 1\}\);
(c) If \(\gamma < o(\tau)\) and \(x_i, y_i \in A\) with \(x_i \rho y_i\) for any \(i \in \{0, \ldots, n_\gamma - 1\}\), then
\[
f_\gamma(x_0, \ldots, x_{n_\gamma-1}) \overline{\rho} f_\gamma(y_0, \ldots, y_{n_\gamma-1});
\]
(d) If \(n \in \mathbb{N}\), \(p \in P^A_\Lambda(P^*\mathfrak{A})\), and \(x_i, y_i \in A\) with \(x_i \rho y_i\) for any \(i \in \{0, \ldots, n-1\}\), then
\[
p(x_0, \ldots, x_{n-1}) \overline{\rho} p(y_0, \ldots, y_{n-1}).
\]

**Proof.** (a) if and only if (b) was proved in [2, Remark 12.a]. From the proof of (b) implies (a) it follows that (b) implies (c) and (c) implies (a). Since (d) implies (c) is obvious, it is enough to show that (c) implies (d) to complete the proof. This was proved in [2, Theorem 13] for the case when \(p \in P^A\mathfrak{A}^*(\mathfrak{A})\). The reasoning is almost the same, that is why we skip it here. \(\blacksquare\)
EXAMPLE 4. A hypergroupoid \((H, \cdot)\) is called semihypergroup if
\[(a \cdot b) \cdot c = a \cdot (b \cdot c), \quad \text{for any } a, b, c \in H.\]
An equivalence relation \(\rho\) on a semihypergroup \((H, \cdot)\) is called strongly regular if for any \(a, b, x \in H\) with \(a \rho b\) we have \(a x \rho b x\) and \(x a \rho x b\). It is clear that the strongly regular equivalences of a semihypergroup are those relations \(\rho\) for which \((H/\rho, \cdot)\) is a groupoid. Note that \((H/\rho, \cdot)\) is a semigroup (see [4, Theorem 31]).

REMARK 11. If the equivalence relation \(\rho\) satisfies one of the equivalent conditions in Proposition 4.1, then the operations in the factor multialgebra (which is a universal algebra) are defined as follows: if \(\gamma < o(\tau), a_0, \ldots, a_{n_\gamma - 1} \in A\), and \(b \in f_\gamma(a_0, \ldots, a_{n_\gamma - 1})\), then
\[f_\gamma(\rho\langle a_0 \rangle, \ldots, \rho\langle a_{n_\gamma - 1} \rangle) = \rho\langle b \rangle.\]

LEMMA 4.2. Let \(\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})\) be a multialgebra of type \(\tau\). The set \(E_{ua}(\mathfrak{A})\) of the equivalence relations \(\rho\) on \(A\) for which \(\mathfrak{A}/\rho\) is a universal algebra is an algebraic closure system on \(A \times A\).

PROOF. Consider a family \((\rho_i \mid i \in I)\) of relations from \(E_{ua}(\mathfrak{A})\), let \(\gamma < o(\tau)\) and \(x_j, y_j \in A\) \((j \in \{0, \ldots, n_\gamma - 1\})\).
If we assume that \((x_j, y_j) \in \bigcap_{i \in I} \rho_i\) for all \(j \in \{0, \ldots, n_\gamma - 1\}\), it follows that \(x_j \rho_i y_j\) for any \(i \in I\) and any \(j \in \{0, \ldots, n_\gamma - 1\}\). Thus, for any \(i \in I\) we have
\[f_\gamma(x_0, \ldots, x_{n_\gamma - 1}) = f_\gamma(y_0, \ldots, y_{n_\gamma - 1}).\]
Therefore, \(x \rho_i y\) for any \(i \in I\), \(x \in f_\gamma(x_0, \ldots, x_{n_\gamma - 1})\), and \(y \in f_\gamma(y_0, \ldots, y_{n_\gamma - 1})\). It means that \((x, y) \in \bigcap_{i \in I} \rho_i\) for any \(x \in f_\gamma(x_0, \ldots, x_{n_\gamma - 1})\) and \(y \in f_\gamma(y_0, \ldots, y_{n_\gamma - 1})\) or, equivalently,
\[\big( f_\gamma(x_0, \ldots, x_{n_\gamma - 1}), f_\gamma(y_0, \ldots, y_{n_\gamma - 1}) \big) \in \bigcap_{i \in I} \rho_i.\]
Thus, we have proved that \(\bigcap_{i \in I} \rho_i \in E_{ua}(\mathfrak{A})\).
If \((x_j, y_j) \in \bigcup_{i \in I} \rho_i\) for all \(j \in \{0, \ldots, n_\gamma - 1\}\), then for each \(j \in \{0, \ldots, n_\gamma - 1\}\) there exists \(i_j \in I\) such that \((x_j, y_j) \in \rho_{i_j}\). Assuming the ordered set \((\rho_i \mid i \in I), \subseteq\) is directed and \(I \neq \emptyset\), we obtain an element \(m \in I\) such that \(\rho_{i_j} \subseteq \rho_m\) for all \(j \in \{0, \ldots, n_\gamma - 1\}\). Consequently, we have \(f_\gamma(x_0, \ldots, x_{n_\gamma - 1}) = f_\gamma(y_0, \ldots, y_{n_\gamma - 1})\). Hence \((x, y) \in \rho_m \subseteq \bigcup_{i \in I} \rho_i\) for any \(x \in f_\gamma(x_0, \ldots, x_{n_\gamma - 1})\) and \(y \in f_\gamma(y_0, \ldots, y_{n_\gamma - 1})\), which means that
\[\big( f_\gamma(x_0, \ldots, x_{n_\gamma - 1}), f_\gamma(y_0, \ldots, y_{n_\gamma - 1}) \big) \in \bigcup_{i \in I} \rho_i.\]
Thus \(\bigcup_{i \in I} \rho_i \in E_{ua}(\mathfrak{A})\). □
COROLLARY 4.3. Let $\mathfrak{A} = (A, (f_y)_{y \in \sigma(\tau)})$ be a multialgebra of type $\tau$. If $R \subseteq A \times A$, then the relation $\alpha(R) = \bigcap\{\rho \in E_{ua}(\mathfrak{A}) \mid R \subseteq \rho\}$ is the smallest equivalence relation on $A$ containing $R$ for which the factor multialgebra is a universal algebra.

REMARK 12. If the multialgebra $\mathfrak{A}$ is not a universal algebra, then the smallest element of $E_{ua}(\mathfrak{A})$ is not $\delta_A$.

For a multialgebra $\mathfrak{A}$ the smallest equivalence from $E_{ua}(\mathfrak{A})$ will be denoted by $\alpha^*$ (or $\alpha^*_\mathfrak{A}$ when necessary) and it will be called the fundamental relation of $\mathfrak{A}$. We recall that the fundamental relation $\alpha^*$ of the multialgebra $\mathfrak{A}$ is the transitive closure of the relation $\alpha$ defined by $x \alpha y$ if and only if $x, y \in p(a_0, \ldots, a_{n-1})$ for some $n \in \mathbb{N}$, $p \in P^{(n)}_A(\mathfrak{P}^*(\mathfrak{A}))$, and $a_0, \ldots, a_{n-1} \in A$ (see [10]).

Let $n \in \mathbb{N}$ and $q, r \in P^{(n)}(\tau)$. In [11, Proposition 3] we proved that if the weak identity $q \cap r \neq \emptyset$ is satisfied on the multialgebra $\mathfrak{A}$, then the identity $q = r$ is satisfied in the factor multialgebra $\mathfrak{A}/\alpha^*$ (which is a universal algebra). This happens because $\alpha^*$ contains the relation

$$R_{qr} = \left\{(x, y) \in A \times A \mid x \in q(a_0, \ldots, a_{n-1}), y \in r(a_0, \ldots, a_{n-1})\right\}.$$

REMARK 13. Even if the weak identity $q \cap r \neq \emptyset$ is not satisfied on the multialgebra $\mathfrak{A}$, we can obtain a factor multialgebra of $\mathfrak{A}$ that is a universal algebra satisfying the identity $q = r$ by taking the factor multialgebra determined by a relation from $E_{ua}(\mathfrak{A})$ which contains $R_{qr}$. Also, each relation from $E_{ua}(\mathfrak{A})$ that gives a factor multialgebra satisfying the identity $q = r$ must contain the relation $R_{qr}$. It means that the smallest relation from $E_{ua}(\mathfrak{A})$ for which the factor multialgebra is a universal algebra satisfying the identity $q = r$ is the relation $\alpha(R_{qr})$. From now on, we will denote the relation $\alpha(R_{qr})$ by $\alpha^*_{qr}$.

REMARK 14. It follows immediately that $\alpha^* = \alpha(\emptyset) = \alpha(\delta_A) = \alpha^*_{\mathfrak{A}_0\mathfrak{K}^0}$, and it is obvious that $\alpha^* \subseteq \alpha^*_{qr}$ for any $q, r \in P^{(n)}(\tau)$.

We give a characterization of the relation $\alpha^*_{qr}$ in terms of unary polynomial functions from $P^{(1)}_A(\mathfrak{P}^*(\mathfrak{A}))$.

REMARK 15. By the construction of the polynomial functions from $P^{(1)}_A(\mathfrak{P}^*(\mathfrak{A}))$ it follows that the usual mapping composition of two elements from $P^{(1)}_A(\mathfrak{P}^*(\mathfrak{A}))$ is an element of $P^{(1)}_A(\mathfrak{P}^*(\mathfrak{A}))$.

Indeed, if $f, p \in P^{(1)}_A(\mathfrak{P}^*(\mathfrak{A}))$, then $f \circ p : P^*(A) \to P^*(A)$, and we have:

(i) if $a \in A$, $f = c^1_a$ and $X \in P^*(A)$ is arbitrary, then $(f \circ p)(X) = c^1_a(p(X)) = a = c^1_a(X)$. Thus $f \circ p = c^1_a \in P^{(1)}_A(\mathfrak{P}^*(\mathfrak{A}))$.
(ii) if \( f = e_0^1 \) and \( X \in P^*(A) \) is arbitrary, then \( (f \circ p)(X) = e_0^1(p(X)) = p(X) \). Thus \( f \circ p = p \).

(iii) if \( \gamma < o(\tau) \) and for \( f^0, \ldots, f^{n-1} \in P_A^{(1)}(\mathcal{P}^*(\mathfrak{A})) \) was proved that

\[
\begin{align*}
  f^0 \circ p &= p_0 \in P_A^{(1)}(\mathcal{P}^*(\mathfrak{A})), \\
  \ldots, \\
  f^{n-1} \circ p &= p_{n-1} \in P_A^{(1)}(\mathcal{P}^*(\mathfrak{A})),
\end{align*}
\]

then for \( f = f_\gamma(f^0, \ldots, f^{n-1}) \) we have

\[
(f \circ p)(X) = f(p(X)) = f_\gamma(f^0(p(X)), \ldots, f^{n-1}(p(X))) = f_\gamma(p_0(X), \ldots, p_{n-1}(X)) = f_\gamma(p_0, \ldots, p_{n-1})(X),
\]

and thus \( f \circ p = f_\gamma(p_0, \ldots, p_{n-1}) \in P_A^{(1)}(\mathcal{P}^*(\mathfrak{A})) \).

Now we can prove the main result of this section.

**Theorem 4.4.** Let \( n \in \mathbb{N} \), \( q, r \in P^{(n)}(\tau) \), let \( \mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)}) \) be a multialgebra of type \( \tau \), and let \( \alpha_{qr} \subseteq A \times A \) be the relation defined by

\[
\begin{align*}
  x \alpha_{qr} y &\iff \exists p \in P_A^{(1)}(\mathcal{P}^*(\mathfrak{A})), \exists a_0, \ldots, a_{n-1} \in A \text{ such that} \\
  x &\in p(q(a_0, \ldots, a_{n-1})), \ y \in p(r(a_0, \ldots, a_{n-1})) \text{ or} \\
  y &\in p(q(a_0, \ldots, a_{n-1})), \ x \in p(r(a_0, \ldots, a_{n-1})).
\end{align*}
\]

The relation \( \alpha^*_qr \) is the transitive closure of the relation \( \alpha_{qr} \).

**Proof.** Let \( \alpha^\prime qr \) be the transitive closure of the relation \( \alpha_{qr} \). We will show that

\( \alpha^*_qr = \alpha^\prime qr \) by proving the following statements:

(A) The relation \( \alpha_{qr} \) is an equivalence relation on \( A \) containing \( R_{qr} \).

(B) If \( f \in P_A^{(1)}(\mathcal{P}^*(\mathfrak{A})) \), and for the elements \( a, b \in A \) we have \( a\alpha_{qr}b \), then

\[
f(a)\alpha_{qr}f(b).
\]

(C) The factor multialgebra \( \mathfrak{A}/\alpha_{qr} \) is a universal algebra satisfying the identity \( q = r \).

(D) If \( \rho \in E_{ua}(\mathfrak{A}) \) such that the identity \( q = r \) holds on the universal algebra \( \mathfrak{A}/\rho \), then \( \alpha^\prime qr \subseteq \rho \).

**Proof of (A).** It is obvious that \( \alpha_{qr} \) is symmetric. Taking \( p = c_a^1 \) for any \( a \in A \), we obtain the reflexivity of \( \alpha_{qr} \). Thus \( \alpha^\prime qr \) is the smallest equivalence relation on \( A \) containing \( \alpha_{qr} \). The inclusion \( R_{qr} \subseteq \alpha^\prime qr \) follows by considering \( p = e_0^1 \).

**Proof of (B).** From \( a\alpha_{qr}b \) it follows that there exist \( m \in \mathbb{N}^* \) and \( z_0, \ldots, z_{m-1} \in A \) such that

\[
a = z_0\alpha_{qr}z_1\alpha_{qr} \ldots \alpha_{qr}z_{m-1} = b.
\]

Let us consider \( j \in \{0, \ldots, m - 2\} \). Since
there exist \( p \in P_A^{(1)}(\mathbb{F}^*(\mathbb{A})) \) and \( a_0, \ldots, a_{n-1} \in A \) such that
\[
\begin{align*}
z_j &\in p(q(a_0, \ldots, a_{n-1})), \\
z_{j+1} &\in p(r(a_0, \ldots, a_{n-1})), \\
z_{j+1} &\in p(q(a_0, \ldots, a_{n-1})), \\
z_j &\in p(r(a_0, \ldots, a_{n-1})).
\end{align*}
\]
However, \( p' = f \circ p \in P_A^{(1)}(\mathbb{F}^*(\mathbb{A})) \) and we have
\[
\begin{align*}
f(z_j) &\subseteq p'(q(a_0, \ldots, a_{n-1})), \\
f(z_{j+1}) &\subseteq p'(r(a_0, \ldots, a_{n-1})), \\
f(z_{j+1}) &\subseteq p'(q(a_0, \ldots, a_{n-1})), \\
f(z_j) &\subseteq p'(r(a_0, \ldots, a_{n-1})).
\end{align*}
\]
However, for any \( u_j \in f(z_j) \) and any \( u_{j+1} \in f(z_{j+1}) \) we have \( u_j \alpha_{qr} u_{j+1} \) and, consequently, \( u_0 \alpha_{qr} u_{m-1} \). Since \( u_0 \in f(a) \) and \( u_{m-1} \in f(b) \) are arbitrary, we obtain
\[
f(a) \alpha_{qr} f(b).
\]
**Proof of (C).** Consider \( \gamma < \alpha(\tau) \) and the arbitrary elements \( a, b, x_0, \ldots, x_{n-1} \in A \) such that \( a \alpha_{qr} b \). Applying (B) to the unary polynomial functions (from \( P_A^{(1)}(\mathbb{F}^*(\mathbb{A})) \))
\[
f_{\gamma}(e_0^1, c_{x_1}, \ldots, c_{x_{n-1}}), f_{\gamma}(c_{x_0}, e_0^1, c_{x_1}, \ldots, c_{x_{n-1}}), \ldots, f_{\gamma}(c_{x_0}, \ldots, c_{x_{n-1}}, e_0^1),
\]
it follows that \( \alpha_{qr} \) verifies (b) from Proposition 4.1, hence \( \mathbb{A}/\alpha_{qr} \) is a universal algebra. Remark 13 and (A) complete the proof of (C).

**Proof of (D).** Let \( \rho \) be a relation from \( E_{ua}(\mathbb{A}) \) such that the identity \( q = r \) is satisfied on the universal algebra \( \mathbb{A}/\rho \), let \( p \in P_A^{(1)}(\mathbb{F}^*(\mathbb{A})) \), \( a_0, \ldots, a_{n-1} \in A \), and
\[
x \in p(q(a_0, \ldots, a_{n-1})), \\
y \in p(r(a_0, \ldots, a_{n-1})).
\]
We will show by induction over the steps of the construction of a polynomial function from \( P_A^{(1)}(\mathbb{F}^*(\mathbb{A})) \) that \( x \mathcal{R} y \).

If \( a \in A \) and \( p = c_a^1 \), then \( x = y = a \) and \( x \mathcal{R} y \).

If \( p = e_0^1 \), then \( x \in q(a_0, \ldots, a_{n-1}), y \in r(a_0, \ldots, a_{n-1}) \) and using Remark 13 we have \( (x, y) \in R_{qr} \subseteq \rho \).

Assume that the statement is true for \( p_0, \ldots, p_{n_\gamma-1} \) \((\gamma < \alpha(\tau))\) and consider \( p = f_{\gamma}(p_0, \ldots, p_{n_\gamma-1}) \). If
\[
x \in p(q(a_0, \ldots, a_{n-1})) = f_{\gamma}(p_0, \ldots, p_{n_\gamma-1})(q(a_0, \ldots, a_{n-1}))
\]
and
\[
y \in p(r(a_0, \ldots, a_{n-1})) = f_{\gamma}(p_0, \ldots, p_{n_\gamma-1})(r(a_0, \ldots, a_{n-1}))
\]
then there exist \( x_i \in p_i(q(a_0, \ldots, a_{n-1})), y_i \in p_i(r(a_0, \ldots, a_{n-1})) \), \( i \in \{0, \ldots, n_\gamma - 1\} \), such that \( x \in f_{\gamma}(x_0, \ldots, x_{n-1}) \) and \( y \in f_{\gamma}(y_0, \ldots, y_{n-1}) \). Since \( x_i \mathcal{R} y_i \) for all \( i \in \{0, \ldots, n_\gamma - 1\} \), using Proposition 4.1, it results \( x \mathcal{R} y \). Analogously, if we take \( x \in p(r(a_0, \ldots, a_{n-1})) \) and \( y \in p(q(a_0, \ldots, a_{n-1})) \), then \( x \mathcal{R} y \).

It follows that \( \alpha_{qr} \subseteq \rho \), thus \( \alpha_{qr}' \subseteq \rho \).
EXAMPLE 5. Let \((H, \cdot)\) be a semihypergroup. The smallest strongly regular equivalence on \(H\) such that the factor semihypergroup is a commutative semigroup was determined in [7]. This relation, denoted by \(\gamma^*\), is the transitive closure of the relation \(\gamma = \bigcup_{n \in \mathbb{N}} \gamma_n\), where \(\gamma_1 = \delta_H\) and, for any \(n > 1\), \(\gamma_n\) is defined by

\[ xy_ny \Leftrightarrow \exists (z_1, \ldots, z_n) \in H^n, \exists \sigma \in S_n : x \in \prod_{i=1}^{n} z_{\sigma(i)}, y \in \prod_{i=1}^{n} z_{\sigma(i)} \]  

\((S_n\) denotes the set of the permutations of the set \([1, \ldots, n]\)). Since the set of cycles \((1, 2), (2, 3), \ldots, (n-1, n)\) generates the group \(S_n\), it follows that \(\gamma^*\) is the transitive closure of the relation \(\gamma' = \bigcup_{n \in \mathbb{N}} \gamma'_n\) where \(\gamma'_1 = \delta_H\) and for \(n > 1\),

\[ xy'_n y \text{ if and only if there exist } (z_1, \ldots, z_n) \in H^n, \text{ and } i \in \{1, \ldots, n-1\} \text{ such that } x \in z_1 \cdots z_{i-1}z_{i+1} z_{i+2} \cdots z_n, \text{ and } y \in z_1 \cdots z_{i-1}z_{i+1} z_{i+2} \cdots z_n. \]

Clearly, \(\gamma' = \alpha_{qr}\) with \(q = x_0x_1\) and \(r = x_1x_0\).

From [7] it follows that if \((H, \cdot)\) is a hypergroup, then the relation \(\gamma\) is transitive and \(\gamma^* = \gamma\) is the smallest equivalence relation on \(H\) such that \(H/\gamma^*\) is a commutative group.

REMARK 16. If \((H, \cdot)\) is a hypergroup and \(\rho\) is a strongly regular equivalence on \(H\), then \(H/\rho\) is a group (see [4, Theorem 31]). If \(q, r\) are two \(n\)-ary terms, then the smallest equivalence relation on \(H\) such that the factor hypergroupoid is a semigroup satisfying the identity \(q = r\) is the transitive closure \(\psi^*\) of the relation

\[ \psi = \bigcup \left\{ p(q(a_1, \ldots, a_n)) \times p(r(a_1, \ldots, a_n)) \mid p \in P_H^1(\mathcal{P}^*(H, \cdot)), \text{ and } a_1, \ldots, a_n \in H \right\}. \]

Since this relation is strongly regular, the factor semihypergroup is a group. It means that \(\psi^*\) contains the smallest relation \(\alpha_{qr}^*\) of the hypergroup \((H, \cdot, /, \setminus)\) with the property that the factor hypergroup is a group satisfying the identity \(q = r\). Since

\[ \alpha_{qr} = \bigcup \left\{ p(q(a_1, \ldots, a_n)) \times p(r(a_1, \ldots, a_n)) \mid p \in P_H^1(\mathcal{P}^*(H, \cdot, /, \setminus)), \text{ and } a_1, \ldots, a_n \in H \right\} \]

and \(P_H^1(\mathcal{P}^*(H, \cdot)) \subseteq P_H^1(\mathcal{P}^*(H, \cdot, /, \setminus))\), it follows that \(\psi \subseteq \alpha_{qr}\), thus \(\psi^* \subseteq \alpha_{qr}^*\) and we obtain \(\psi^* = \alpha_{qr}^*\).

So, the smallest strongly regular equivalence on \(H\) for which the factor hypergroup satisfies the identity \(q = r\) can be obtained by considering in Theorem 4.4 only those polynomial functions \(p\) that are obtained with the multioperation \(\cdot\) (in other words, it is not necessary to use the multioperations \(\setminus\) and \(\setminus\) in the construction of \(p\)).
It is easy to observe that Theorem 4.4 and Remark 14 lead to the following characterization of the fundamental relation of a multialgebra.

**Corollary 4.5.** The fundamental relation $a^*$ of a multialgebra $\mathcal{A}$ is the transitive closure of the relation $a' \subseteq A \times A$ defined by $xa'y$ if and only if there exist $p \in P_A^{(1)}(\mathcal{P}^*(\mathcal{A}))$ and $a \in A$ such that $x, y \in p(a)$.

5. Identities and factor multialgebras

Let $n \in \mathbb{N}$ and $q, r \in P^{(n)}(\tau)$. Let $\mathcal{B}$ be a universal algebra and $\rho$ an equivalence relation on $B$. We denote by $\rho_{qr}$ the smallest equivalence relation on $B$ containing $\rho$ and all the pairs $(q(b_0, \ldots, b_{n-1}), r(b_0, \ldots, b_{n-1}))$ with $b_0, \ldots, b_{n-1} \in B$. We denote by $\theta(\rho_{qr})$ the smallest congruence relation on $\mathcal{B}$ containing $\rho_{qr}$. Clearly $\theta(\rho_{qr})$ is the smallest congruence relation on $\mathcal{B}$ containing

$$\rho \cup \{(q(b_0, \ldots, b_{n-1}), r(b_0, \ldots, b_{n-1})) \mid b_0, \ldots, b_{n-1} \in B\}.$$

Theorem 10.4 from [9] presents a characterization for the smallest congruence relation of a universal algebra, which contains a given relation. According to this, $x\theta(\rho_{qr})y$ if and only if there exist $m \in \mathbb{N}^*$, a sequence $x = t_0, t_1, \ldots, t_m = y$, and pairs of elements

$$(x_i, y_i) \in \rho \cup \{(q(b_0, \ldots, b_{n-1}), r(b_0, \ldots, b_{n-1})) \mid b_0, \ldots, b_{n-1} \in B\}$$

and unary algebraic functions $p_i, i \in \{1, \ldots, m\}$, such that

$$\{p_i(x_i), p_i(y_i)\} = \{t_{i-1}, t_i\}, i \in \{1, \ldots, m\}.$$

Clearly, if we take $q = r = x_0$, then $\theta(\rho_{qr})$ is the smallest congruence relation on $\mathcal{B}$ which contains $\rho$. We denote it by $\theta(\rho)$.

**Lemma 5.1.** Let $\rho$ be an equivalence relation on a universal algebra $\mathcal{B}$. If $n \in \mathbb{N}$, $p \in P_{B/\rho}^{(n)}(\mathcal{P}^*(\mathcal{B}/\rho))$, and $x, y, z_0, \ldots, z_{n-1} \in B$ are such that

$$\rho(x), \rho(y) \in p(\rho(z_0), \ldots, \rho(z_{n-1})),$$

then $x\theta(\rho)y$.

**Proof.** We prove this lemma by induction over the steps of construction of the polynomial functions from $P_{B/\rho}^{(n)}(\mathcal{P}^*(\mathcal{B}/\rho))$.

Step 1. If $p = c^a_{\rho(b)}$ for some $b \in B$, then $\rho(x) = \rho(y) = \rho(b)$ and hence $x\theta(\rho)y$. 

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Step 2. If \( p = e^a_i \) for some \( i \in \{0, \ldots, n - 1\} \), then \( \rho(x) = \rho(y) = \rho(z_i) \) and hence \( x \theta(p) y \).

Step 3. We consider the statement proved for \( p_0, \ldots, p_{n-1} \) (\( \gamma < o(\tau) \)) and we take \( p = f_\gamma(p_0, \ldots, p_{n_\gamma-1}) \). Since

\[
p(p(z_0), \ldots, p(z_{n-1})) = f_\gamma(p_0(p(z_0), \ldots, p(z_{n-1})), \ldots, p_{n_\gamma-1}(p(z_0), \ldots, p(z_{n-1}))),
\]

from \( \rho(x), \rho(y) \in p(p(z_0), \ldots, p(z_{n-1})) \), we deduce that there exist some elements \( x_i, y_i \in B \) with \( \rho(x_i), \rho(y_i) \in p_\gamma(p(z_0), \ldots, p(z_{n-1})) \) such that

\[
\rho(x) \in f_\gamma(p(x_0), \ldots, p(x_{n_\gamma-1})), \quad \rho(y) \in f_\gamma(p(y_0), \ldots, p(y_{n_\gamma-1})).
\]

From the definition of the multioperation \( f_\gamma \) in \( \mathcal{B}/\rho \) it follows that there exist \( x'_i, y'_i \in B \) with \( x_i \rho x'_i \) and \( y_i \rho y'_i \) (\( i \in \{0, \ldots, n_\gamma - 1\} \)) such that \( x = f_\gamma(x'_0, \ldots, x'_{n_\gamma - 1}) \), and \( y = f_\gamma(y'_0, \ldots, y'_{n_\gamma - 1}) \). Since the statement holds for \( p_0, \ldots, p_{n_\gamma-1} \), it follows that \( x_i \theta(p) y_i \), for all \( i \in \{0, \ldots, n_\gamma - 1\} \). Hence \( x_i \rho x_i, x_i \theta(p) y_i, y_i \rho y_i \) which implies \( x_i \theta(p) y_i \) for all \( i \in \{0, \ldots, n_\gamma - 1\} \). However, \( \theta(p) \) is a congruence on \( \mathcal{B} \), thus

\[
x = f_\gamma(x'_0, \ldots, x'_{n_\gamma - 1}) \theta(p) f_\gamma(y'_0, \ldots, y'_{n_\gamma - 1}) = y,
\]

which ends the proof of the lemma. \( \square \)

**Lemma 5.2.** Let \( n \in \mathbb{N}, q, r \in P^{(n)}(\tau) \) and let \( \rho \) be an equivalence relation on the universal algebra \( \mathcal{B} \). If \( p \in P^{(1)}(\mathcal{B}/\rho) \) and \( x, y, z_0, \ldots, z_{n-1} \in B \) are such that

\[
\rho(x) \in p(q(\rho(z_0), \ldots, \rho(z_{n-1}))) \quad \text{and} \quad \rho(y) \in p(r(\rho(z_0), \ldots, \rho(z_{n-1}))),
\]

then \( x \theta(p_{qr}) y \).

**Proof.** We prove this lemma by using the steps of construction of the polynomial functions from \( P^{(1)}(\mathcal{B}/\rho) \).

Step 1. If \( p = e_\rho^b \) for some \( b \in B \), then \( \rho(x) = \rho(y) = \rho(b) \) and hence \( x \theta(p_{qr}) y \).

Step 2. If \( p = e_\rho^b \), then \( \rho(x) \in q(\rho(z_0), \ldots, \rho(z_{n-1})) \) and \( \rho(y) \in r(\rho(z_0), \ldots, \rho(z_{n-1})) \). According to Remark 7 we also have \( \rho(q(z_0, \ldots, z_{n-1})) \in q(\rho(z_0), \ldots, \rho(z_{n-1})) \) and \( \rho(r(z_0, \ldots, z_{n-1})) \in r(\rho(z_0), \ldots, \rho(z_{n-1})) \), so, using the previous lemma, it follows that \( x \theta(p) q(z_0, \ldots, z_{n-1}) \) and \( y \theta(p) r(z_0, \ldots, z_{n-1}) \). However, \( \theta(p) \subseteq \theta(p_{qr}) \) and \( q(z_0, \ldots, z_{n-1}) \theta(p_{qr}) r(z_0, \ldots, z_{n-1}), \) thus \( x \theta(p_{qr}) y \).

Step 3. We consider the statement proved for \( p_0, \ldots, p_{n_\gamma-1} \) (\( \gamma < o(\tau) \)) and we take \( p = f_\gamma(p_0, \ldots, p_{n_\gamma-1}) \). Since

\[
p(q(\rho(z_0), \ldots, \rho(z_{n-1}))) \\
= f_\gamma(p_0(q(\rho(z_0), \ldots, \rho(z_{n-1}))), \ldots, p_{n_\gamma-1}(q(\rho(z_0), \ldots, \rho(z_{n-1}))))
\]
and
\[ p(r(\rho(z_0), \ldots, \rho(z_{n-1}))) = f_y(p_0, \ldots, p_{n-1})(r(\rho(z_0), \ldots, \rho(z_{n-1}))) = f_y(p(r(\rho(z_0), \ldots, \rho(z_{n-1}))), \ldots, p_{n-1}(r(\rho(z_0), \ldots, \rho(z_{n-1})))) , \]
from \( \rho(x) \in p(q(\rho(z_0), \ldots, \rho(z_{n-1}))) \) and \( \rho(y) \in p(r(\rho(z_0), \ldots, \rho(z_{n-1}))) \), we deduce that there exist \( x_i, y_i \in B \) with \( \rho(x_i) \in p_q(\rho(z_0), \ldots, \rho(z_{n-1})) \) and \( \rho(y_i) \in p_r(\rho(z_0), \ldots, \rho(z_{n-1})) \) such that
\[ \rho(x) \in f_y(\rho(x_0), \ldots, \rho(x_{n-1})), \quad \rho(y) \in f_y(\rho(y_0), \ldots, \rho(y_{n-1})). \]
From the definition of the multioperation \( f_y \) in \( B/\rho \) it results that there exist \( x'_i, y'_i \in B \) with \( x_i \rho x'_i \) and \( y_i \rho y'_i \) \((i \in \{0, \ldots, n_y - 1\})\) such that \( x = f_y(x'_0, \ldots, x'_{n-1}), \) and \( y = f_y(y'_0, \ldots, y'_{n-1}). \) Since the statement holds for \( p_0, \ldots, p_{n_y-1} \) it follows that \( x_i \theta(\rho qr) y_i \) for all \( i \in \{0, \ldots, n_y - 1\}. \) Hence \( x'_i \rho x_i, x_i \theta(\rho qr) y_i, y_i \rho y'_i \) which implies \( x'_i \theta(\rho qr) y'_i \) for all \( i \in \{0, \ldots, n_y - 1\}. \) However, \( \theta(\rho qr) \) is a congruence on \( B, \) thus
\[ x = f_y(x'_0, \ldots, x'_{n-1}) \theta(\rho qr) f_y(y'_0, \ldots, y'_{n-1}) = y, \]
which ends the proof of the lemma. \( \square \)

Now we can prove the main result of this paper.

**Theorem 5.3.** Let \( n \in \mathbb{N} \) and \( q, r \in P^{(n)}(\tau). \) If \( \rho \) is an equivalence relation on a universal algebra \( B, \) then \( (B/\rho)/\alpha^{r*}_{qr} \cong B/\theta(\rho qr). \)

**Proof.** First we will prove that the correspondence \( \alpha^{r*}_{qr} \langle \rho(a) \rangle \mapsto \theta(\rho qr) \langle a \rangle \) defines a bijective map \( h : (B/\rho)/\alpha^{r*}_{qr} \to B/\theta(\rho qr). \)

For this, we will show that if \( a, b \in B, \) we have \( \rho(a) \alpha^{r*}_{qr} \rho(b) \) if and only if \( a \theta(\rho qr) b. \) If \( \rho(a) \alpha^{r*}_{qr} \rho(b), \) then there exist \( m \in \mathbb{N} \) and \( z_0, \ldots, z_m \in B \) such that
\[ \rho(a) = z_0 \alpha_{qr} a \quad \rho(z_1) \alpha_{qr} \ldots \alpha_{qr} \rho(z_m) = \rho(b). \]
Thus for each \( i \in \{1, \ldots, m\} \) there exist \( p_i \in P^{(i)}_{B/\rho}(\mathcal{B}^{*}(\mathcal{B}/\rho)) \) and \( z'_0, \ldots, z'_{n-1} \in B \) such that
\[ \rho(z_{i-1}) \in p_i(q(\rho(z'_0), \ldots, \rho(z'_{n-1}))), \quad \rho(z_i) \in p_i(r(\rho(z'_0), \ldots, \rho(z'_{n-1}))) \quad \text{or} \]
\[ \rho(z_{i-1}) \in p_i(r(\rho(z'_0), \ldots, \rho(z'_{n-1}))), \quad \rho(z_i) \in p_i(q(\rho(z'_0), \ldots, \rho(z'_{n-1}))). \]
According to Lemma 5.2 it follows that for any \( i \in \{1, \ldots, m\} \) we have \( z_{i-1} \theta(\rho qr) z_i. \) We deduce that \( z_0 \theta(\rho qr) z_m. \) However, \( a \rho z_0, z_m \rho b \) and \( \rho \subseteq \theta(\rho qr), \) thus \( a \theta(\rho qr) b. \)
Conversely, if \( a \theta (\rho_{qr}) b \), there exist \( m \in \mathbb{N} \), a sequence \( a = t_0, t_1, \ldots, t_m = b \), pairs of elements \((x_i, y_i) \in \rho \cup \{(q(b_0, \ldots, b_{n-1}), r(b_0, \ldots, b_{n-1})) \mid b_0, \ldots, b_{n-1} \in B\} \), and unary algebraic functions \( p_i, \ i \in \{1, \ldots, m\}, \) such that

\[
\{t_{i-1}, t_i\} = \{p_i(x_i), p_i(y_i)\}, \quad i \in \{1, \ldots, m\}.
\]

We have \( \{\rho(t_{i-1}), \rho(t_i)\} = \{\rho(p_i(x_i)), \rho(p_i(y_i))\} \).

If \((x_i, y_i) \in \rho\), then \( \rho(x_i) = \rho(y_i) \), and since

\[
\rho(p_i(x_i)) \in p_i(\rho(x_i)) = p_i(\rho(y_i)) \ni \rho(p_i(y_i)),
\]

we deduce that \( \rho(t_{i-1}) \alpha^* \rho(t_i) \), thus \( \rho(t_{i-1}) \alpha_{qr}^* \rho(t_i) \).

If \((x_i, y_i) = (q(b_0, \ldots, b_{n-1}), r(b_0, \ldots, b_{n-1})) \) for some \( b_0, \ldots, b_{n-1} \in B \), from

\[
\rho(p_i(x_i)) = \rho(p_i(q(b_0, \ldots, b_{n-1}))) \in p_i(\rho(q(b_0, \ldots, b_{n-1})))
\]

\[
\subseteq p_i(q(\rho(b_0), \ldots, \rho(b_{n-1}))),
\]

\[
\rho(p_i(y_i)) = \rho(p_i(r(b_0, \ldots, b_{n-1}))) \in p_i(\rho(r(b_0, \ldots, b_{n-1})))
\]

\[
\subseteq p_i(r(\rho(b_0), \ldots, \rho(b_{n-1}))),
\]

it follows that \( \rho(t_{i-1}) \alpha_{qr}^* \rho(t_i) \).

So, we have proved that \( \rho(a) = \rho(t_0) \alpha_{qr}^* \rho(t_m) = \rho(b) \).

We deduce that \( h \) is well defined and also that \( h \) is injective. Its surjectivity is obvious.

Now, we can prove that the map \( h \) is an isomorphism between the universal algebras \((B/\rho)/\alpha_{qr}^* \) and \( B/\theta(\rho_{qr}) \).

Indeed, let us consider \( \gamma < o(\tau) \) and \( b_0, \ldots, b_{n-1} \in B \). Since

\[
f_\gamma(\alpha_{qr}^* (\rho(b_0)), \ldots, \alpha_{qr}^* (\rho(b_{n-1}))) = \alpha_{qr}^* (\rho(b))
\]

for any \( \rho(b) \in f_\gamma(\rho(b_0), \ldots, \rho(b_{n-1})) \) and since

\[
\rho(f_\gamma(b_0, \ldots, b_{n-1})) \in f_\gamma(\rho(b_0), \ldots, \rho(b_{n-1})),
\]

we have \( f_\gamma(\alpha_{qr}^* (\rho(b_0)), \ldots, \alpha_{qr}^* (\rho(b_{n-1}))) = \alpha_{qr}^* (\rho(f_\gamma(b_0, \ldots, b_{n-1}))) \), thus

\[
h(f_\gamma(\alpha_{qr}^* (\rho(b_0)), \ldots, \alpha_{qr}^* (\rho(b_{n-1})))) = \theta(\rho_{qr})(f_\gamma(b_0, \ldots, b_{n-1})).
\]

We also have

\[
f_\gamma(h(\alpha_{qr}^* (\rho(b_0))), \ldots, h(\alpha_{qr}^* (\rho(b_{n-1}))))) = f_\gamma(\theta(\rho_{qr})(b_0), \ldots, \theta(\rho_{qr})(b_{n-1}))
\]

\[
= \theta(\rho_{qr})(f_\gamma(b_0, \ldots, b_{n-1})),
\]

hence \( h \) is a homomorphism. \( \Box \)
Let $\alpha^*$ be the fundamental relation of the multialgebra $\mathfrak{A}$. The universal algebra $\mathfrak{A}/\alpha^*$ will be denoted by $\overline{\mathfrak{A}}$ and it will be called the fundamental algebra of the multialgebra $\mathfrak{A}$. Since $\alpha^* = \alpha^*_{x_0x_0}$ and $\rho_{x_0x_0} = \rho$ we have the following result.

**Corollary 5.4.** Let $\rho$ be an equivalence relation on the universal algebra $\mathfrak{B}$ and let $\theta(\rho)$ be the smallest congruence relation on $\mathfrak{B}$ which contains $\rho$. Then $\overline{\mathfrak{B}}/\rho \cong \mathfrak{B}/\theta(\rho)$.

From Corollary 5.4, using the notations from the beginning of this section, we obtain the following.

**Corollary 5.5.** Let $n \in \mathbb{N}$ and $q, r \in P(n)(\tau)$. If $\rho$ is an equivalence relation on a universal algebra $\mathfrak{B}$, then $\overline{\mathfrak{B}}/\rho_{qr} \cong \mathfrak{B}/\rho_{qr}$.

From Theorem 5.3 and Corollary 5.5 we obtain the following consequence.

**Corollary 5.6.** If $\rho$ is an equivalence relation on the universal algebra $\mathfrak{B}$, then $(\mathfrak{B}/\rho)/\alpha^*_{qr} \cong \mathfrak{B}/\rho_{qr}$.

### 6. An application to hypergroups

A hypergroup is an $H_v$-group satisfying the associativity in a strong manner. A classical example of hypergroup is obtained in [8], by factorizing a group $(G, \cdot)$ through an equivalence relation determined by a subgroup $H$. The definition of the hyperproduct on $G/H = \{xH \mid x \in G\}$ is

$$(xH)(yH) = \{zH \mid z = x'y', x' \in xH, \ y' \in yH\}.$$ 

Clearly, $(G/H, \cdot)$ is a group if and only if $H$ is a normal subgroup of $G$.

Let $\gamma$ be the smallest strongly regular equivalence on $G/H$ such that the factor hypergroup is a commutative group. If $G'$ is the derived subgroup of $G$, then $G'H$ is the smallest normal subgroup $N$ of $G$ for which $H \subseteq N$ and $G/N$ is abelian. From Theorem 5.3 we obtain the group isomorphism

$$h : (G/H)/\gamma \to G/(G'H), \quad h(\gamma(xH)) = x(G'H).$$

The derived subhypergroup $D(K)$ of a hypergroup $(K, \cdot)$ is characterized in [7, Theorem 3.1] as being $\varphi_K^{-1}(1_{K/\gamma})$ where $\varphi_K : K \to K/\gamma$ is the canonical projection and $1_{K/\gamma}$ is the identity of the group $(K/\gamma, \cdot)$.

Let $\pi_H : G \to G/H$ and $\varphi_{G/H} : G/H \to (G/H)/\gamma$ be the canonical projections. Using [7, Theorem 3.1], a connection between the derived subhypergroup of $G/H$ and the derived subgroup of $G$ can be established as follows:

$$D(G/H) = (h \circ \varphi_{G/H})^{-1}(G') = \{xH \mid x \in G'H\} = (G'H)/H = \pi_H(G').$$
Of course, if $G' \subseteq H$, then $H$ is an normal subgroup of $G$ and $G/H$ is an abelian group, so $D(G/H) = (G/H)' = H$. If $H \subseteq G'$, then $D(G/H) = G'/H$.

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**References**
