Canad. Math. Bull. Vol. **43** (1), 2000 pp. 69–73

Type II Spectral Flow and the Eta Invariant

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Abstract. The relative eta invariant of Atiyah-Patodi-Singer will be shown to be expressible in terms of the notion of Type I and Type II spectral flow.

1 Introduction

In this note we will show that the continuous part of the relative eta invariant of Atiyah-Patodi-Singer can be expressed as the Type II spectral flow of a loop of self-adjoint Fredholm operators in a II_∞ factor.

This is accomplished in two steps. The first is to use the data required to construct the relative eta invariant to obtain a loop of self-adjoint Fredholm operators in a II_∞ factor. The second is to use index theory to show that the homotopy class of this loop in $\pi_1(\mathcal{F}^{\mathrm{sa}}_{\mathrm{II}}) \cong \mathbb{R}$ is equal to the relative eta invariant. We make use of results of A. Carey and J. Phillips to accomplish this.

2 Type II Spectral Flow

We review the notion of spectral flow relative to a Type II_∞ factor introduced by Vic Perera in [11]. Let $\mathcal{M}$ be a II_∞ factor with trace τ. The Breuer compacts in $\mathcal{M}$ is the norm closed two sided ideal, $\mathcal{K}_{\mathcal{M}}$, generated by the operators of finite trace. An operator in $\mathcal{M}$ is Breuer Fredholm if it has a two sided inverse modulo $\mathcal{K}_{\mathcal{M}}$. For such an operator the projections onto its kernel and kernel of its adjoint have finite trace, but, as opposed to ordinary (or Type I) Fredholm operators, its range may not be closed. Let $\mathcal{F}^{\mathrm{sa}}_{\mathrm{II}}$ denote the self-adjoint Breuer Fredholm operators in $\mathcal{M}$. This space has three components. Two of them are contractible, while the third is a classifying space for $K^{-1}(X;\mathbb{R})$, in the sense that $[X, \mathcal{F}^{\mathrm{sa}}_{\mathrm{II}}] \cong K^{-1}(X;\mathbb{R})$ [3]. It follows from this that $\pi_1(\mathcal{F}^{\mathrm{sa}}_{\mathrm{II}}) \cong \mathbb{R}$. Thus, if α_t is a loop based at the identity operator in $\mathcal{F}^{\mathrm{sa}}_{\mathrm{II}}$, then its homotopy class corresponds to a real number via the above isomorphism. One defines the Type II spectral flow of such a loop α_t to be this real number,

$$sf_{\mathrm{II}}(\alpha_t) = [\alpha_t] \in \pi_1(\mathcal{F}^{\mathrm{sa}}_{\mathrm{II}}) \cong \mathbb{R}.$$

We note that this is a natural generalization of the usual notion of spectral flow. Indeed, if one has a loop of self-adjoint elliptic differential operators on a closed manifold then the spectral flow of the loop, as defined by Atiyah and Lustig in [2], can be obtained as follows. Let $f(x) = x/\sqrt{1+x^2}$. Then $f(D_t)$ is a loop of self-adjoint Fredholm operators. Its homotopy class defines an element of $\pi_1(\mathcal{F}^{\mathrm{sa}}) \cong \mathbb{Z}$. The integer it defines agrees with the

Received by the editors March 2, 1998; revised January 22, 1999.
The first author was supported in part by NSF Grant DMS-9401457.
AMS subject classification: 19K56, 46L80.

more geometric notion presented in [2]. The definition of spectral flow for the unbounded family D_t is equal to the homotopy class of the loop $[f(D_t)] \in \pi_1(\mathcal{F}_{\mathrm{I}}^{\mathrm{sa}}) \cong \mathbb{R}$ (*cf.* [4], [1], [12], [13]).

3 The Type II_∞ Factor

In this section we will describe how one constructs the II_∞ factor, $\mathcal{M}$, and certain loops in $\mathcal{F}_{\mathrm{II}}^{\infty}$ which will lead to the connection between Type II spectral flow and the eta invariant. We start with the following data. Let M be a closed smooth odd-dimensional Riemannian manifold. Let $\alpha\colon \pi_1(M) \to U_n$ be a unitary representation of its fundamental group and let P_α denote the associated flat, principle U_n bundle. There exists an integer k so that the bundle $P_{k\alpha}$ is trivial and we fix a trivialization $\theta\colon P_{k\alpha} \to M \times U_{nk}$. For now, we will suppress the notation involving k and simply assume that the bundle P_α is trivial, although k will play a role later.

The total space of the flat principle bundle, $P_\alpha = \tilde{M} \times_\alpha U_n$, can be identified with $M \times U_n$ via the trivialization θ. This induces a horizontal foliation, which will be denoted $\mathcal{F}_\alpha$, of $M \times U_n$ and whose leaves are quotients of the images of $\tilde{M} \times U_n$. One can associate a C^*-algebra and a von Neumann algebra to $\mathcal{F}_\alpha$. For the C^*-algebra one takes the reduced algebra of Connes, $C^*(\mathcal{F}_\alpha)$. For the von Neumann algebra, $\mathcal{W}^*(\mathcal{F}_\alpha)$ we will take the weak closure of $C^*(\mathcal{F}_\alpha)$ in its regular representation. In fact, we will actually have to consider variants of these constructions which take into account a vector bundle E on $M \times U_n$. For this one completes the smooth convolution algebra $C_c^\infty\big(G, s^*(E) \otimes r^*(E)\big)$, where G is the holonomy groupoid. Its regular representation is on the Hilbert space of sections $\mathcal{H} = L^2\big(G, r^*(E)\big)$. The von Neumann algebra obtained as the weak closure of $C^*(\mathcal{F}_\alpha, E)$ is isomorphic to $\mathcal{W}^*(\mathcal{F}_\alpha)$. Indeed, it follows from [10] that $C^*(\mathcal{F}_\alpha) \cong C^*(\mathcal{F}_\alpha, E)$.

The type of the von Neumann algebra, $\mathcal{W}^*(\mathcal{F}_\alpha)$, depends on the representation α. If the image of $\alpha\colon \pi_1(M) \to U_n$ is infinite, then $\mathcal{W}^*(\mathcal{F}_\alpha)$ is a Type II_∞ algebra. If, in addition, the image of α is dense, it is a II_∞ factor [15]. We will concentrate on this case.

Let $\mathcal{D}\colon C^\infty(E) \to C^\infty(E)$ denote a first order, essentially self-adjoint, elliptic differential operator on M. For example, if we assume M is odd dimensional, then the Dirac operator is a possible choice for $\mathcal{D}$. One can apply the suspension process of [7], [6] which lifts $\mathcal{D}$ to a leafwise elliptic operator, $\mathcal{D}_\alpha\colon L^2(M \times U_n, \tilde{E}) \to L^2(M \times U_n, \tilde{E})$ on $M \times U_n$, where $\tilde{E}$ is the bundle on $M \times U_n$ obtained from E in the natural way [8].

Proposition 3.1 *The operator $\mathcal{D}_\alpha$ defines an unbounded self-adjoint operator on the Hilbert space $\mathcal{H}$. It is affiliated to the von Neumann algebra $\mathcal{W}^*(\mathcal{F}_\alpha)$ and it has a parametrix modulo the Breuer ideal $\mathcal{B} \subset \mathcal{W}^*(\mathcal{F}_\alpha)$. Moreover, $\mathcal{D}_\alpha(I + {\mathcal{D}_\alpha}^2)^{-\frac{1}{2}}$ is an element of $\mathcal{F}_{\mathrm{II}}^{\mathrm{sa}}$ and has a parametrix modulo p-summable operators, for a suitable p.*

Proof This follows from the results in [7], [6], [8]. ∎

Let $\phi\colon M \times U_n \to U_k$ be a continuous unitary valued function. Then $1 \otimes \phi$ defines an automorphism of the bundle $\tilde{E} \otimes \epsilon^k$. All the previous constructions extend to the case where $\tilde{E}$ is replaced by $\tilde{E} \otimes \epsilon^k$. Define $\mathcal{D}_\alpha^\phi$ by the formula

$$\mathcal{D}_\alpha^\phi = (1 \otimes \phi)^*(\mathcal{D}_\alpha \otimes 1)(1 \otimes \phi), \tag{3.1}$$

where the operators are acting on the bundle $\tilde{E} \otimes \epsilon^k$. Since $\mathcal{D}_\alpha \otimes 1$ and $\mathcal{D}_\alpha^\phi$ have the same principle symbol there is a unique homotopy class of paths between them. We take the straight line path,

$$\mathcal{D}_t = (1-t)\mathcal{D}_\alpha + t\mathcal{D}_\alpha^\phi = \mathcal{D}_\alpha + (1-t)(1 \otimes \phi)^*[\mathcal{D}_\alpha, 1 \otimes \phi], \tag{3.2}$$

and convert it to a path of bounded operators using the following result of Phillips and Carey [13], [5].

Proposition 3.2 *The map $\alpha(t) = \mathcal{D}_t(1 + \mathcal{D}_t{}^2)^{-\frac{1}{2}}$ defines a continuous path in $\mathcal{F}_{\mathrm{II}}^{\mathrm{sa}}$.*

We want to make this into a loop, so consider the operator $(1 \otimes \phi)$. It is an element of the unitary group of $\mathcal{W}^*(\mathcal{F}_\alpha)$ which is path connected. Let $\beta(t)$ be a path of unitaries from $(1 \otimes \phi)$ to I. The required loop is obtained by following the path $\mathcal{D}_t$ by the path $\beta(t)^*(\mathcal{D}_\alpha \otimes 1)\beta(t)$. We will denote this loop by $\lambda(\mathcal{D}^\phi, \alpha, \theta)$. It is easy to see that the homotopy class $[\lambda(\mathcal{D}^\phi, \alpha, \theta)] \in \pi_1(\mathcal{F}_{\mathrm{II}}^{\mathrm{sa}})$ depends only on the data $(\mathcal{D}, \alpha, \theta, \phi)$ and does not depend on the other choices made in its construction.

Now, suppose we have the data $\mathcal{D}$ and (α, θ). We may associate a loop of self-adjoint Type II Fredholms to this by taking for ϕ the projection $\phi = pr_1 \colon M \times U_n \to U_n$. This depends on the trivialization θ so we denote the corresponding element of $\pi_1(\mathcal{F}_{\mathrm{II}}^{\mathrm{sa}})$ by $\lambda(\mathcal{D}, \alpha, \theta)$. Using the fact that $\pi_1(\mathcal{F}_{\mathrm{II}}^{\mathrm{sa}}) \cong \mathbb{R}$ we will define the resulting real number to be the Type II spectral flow associated to our data,

$$sf_{\mathrm{II}}(\mathcal{D}, \alpha, \theta) = [\lambda(\mathcal{D}_\alpha, \theta)] \in \pi_1(\mathcal{F}_{\mathrm{II}}^{\mathrm{sa}}) \cong \mathbb{R}. \tag{3.3}$$

Remark 3.3 There is a classifying space for data such as (α, θ). Let U_n^δ denote the unitary group with the discrete topology and let $\mathrm{BU}_n^\delta \to \mathrm{BU}_n$ be the induced map on classifying spaces. Let $\overline{\mathrm{BU}_n}$ be the homotopy fiber. Then there is a one-one correspondence between homotopy classes of maps $M \to \overline{\mathrm{BU}_n}$ and homotopy classes of pairs (α, θ) [6]. Indeed, there is a pairing

$$\chi \colon K_1(M) \times [M, \overline{\mathrm{BU}_n}] \to \pi_1(\mathcal{F}_{\mathrm{II}}^{\mathrm{sa}}), \tag{3.4}$$

which, upon fixing the operator $\mathcal{D}$, yields a map

$$\chi_{\mathcal{D}} \colon [M, \overline{\mathrm{BU}_n}] \to \pi_1(\mathcal{F}_{\mathrm{II}}^{\mathrm{sa}}) \tag{3.5}$$

which is given by $\chi_{\mathcal{D}}\big([(\alpha, \theta)]\big) = [\lambda(\mathcal{D}, \alpha, \theta)]$.

4 Type II Spectral Flow and the Eta Invariant

In this section we will relate Type II spectral flow and the relative eta invariant. Recall that what we are calling the "relative eta invariant" is the difference of ξ invariants for an operator $\mathcal{D}$ and the same operator twisted by a flat bundle. It was studied in [2] and expressed topologically via the Flat Bundle index theorem. To be more specific, to a self-adjoint operator $\mathcal{D}$ and a unitary representation $\alpha \colon \pi_1(M) \to U_n$, Atiyah-Patodi-Singer associate an element in $\mathbb{R}/\mathbb{Z}$ via the formula

$$\eta(\mathcal{D}, \alpha) = \overline{\xi(\mathcal{D} \otimes E_\alpha) - \xi(\mathcal{D} \otimes \epsilon^n)} \in \mathbb{R}/\mathbb{Z} \tag{4.1}$$

where

$$\xi(\mathcal{D}) = \frac{1}{2}\big(\eta(\mathcal{D}) + \dim\ker(\mathcal{D})\big), \tag{4.2}$$

E_α is the flat vector bundle defined by α, and $\eta(\mathcal{D})$ is the eta invariant. On the other hand, if one chooses an integer k so that the bundle classified by $k\alpha$ is trivial, and then fixes a trivialization of that bundle θ, then in the previous section we have defined a real number $sf_{\mathrm{II}}(\mathcal{D}, \alpha, \theta)$. Our goal is to establish a relation between these invariants.

The approach used in [2] to study $\eta(\mathcal{D}, \alpha)$ starts by breaking it into two parts. One component, which we will denote $sf_{\mathrm{I}}(\mathcal{D}, k\alpha, \theta)$, is defined in terms of the Type I spectral flow of a loop $\lambda_{\mathrm{I}}(\mathcal{D}, \alpha, \theta)$ of self-adjoint Fredholm operators defined as follows. Consider the operator $\mathcal{D}\colon C^\infty(E) \to C^\infty(E)$. As above, we twist it by the flat bundle defined by α and use the trivialization θ to represent this operator and the operator $\mathcal{D} \otimes 1$ on the same Hilbert space. One then forms a loop, $\lambda_{\mathrm{I}}(\mathcal{D}, \alpha, \theta)$, as before, but in this case it is in the usual self-adjoint Fredholm operators on a separable Hilbert space. Thus, we have $[\lambda_{\mathrm{I}}(\mathcal{D}, \alpha, \theta)] \in \pi_1(\mathcal{F}_{\mathrm{I}}^{sa}) \cong \mathbb{Z}$. We then define

$$sf_{\mathrm{I}}(\mathcal{D}, k\alpha, \theta) = \overline{\frac{1}{k}[\lambda_{\mathrm{I}}(\mathcal{D}, k\alpha, \theta)]} \in \mathbb{Q}/\mathbb{Z}, \tag{4.3}$$

where the bar means projecting to a coset in $\mathbb{Q}/\mathbb{Z}$.

The second component was defined by first constructing an element $\nu \in K^1(TM;\mathbb{R})$ in the following way. Let $\mathrm{Tch}(k\alpha, \theta) \in H^*(M, \mathbb{R})$ be the transgressed Chern character associated to the trivialized flat bundle, and let $\nu = ch^{-1}\big(\mathrm{Tch}(k\alpha, \theta)\big) \in K^1(M, \mathbb{R})$. Let $[\sigma(\mathcal{D})] \in K^1(TM, \mathbb{R})$ be the symbol class of the operator $\mathcal{D}$. Consider the class $\nu \cdot [\sigma(\mathcal{D})] \in K^0(TM, \mathbb{R})$. The desired number is the index of this class as an element of $\mathbb{R}$. It is this second component which will be interpreted as Type II spectral flow.

Recall that the Flat Bundle Index Theorem states that the sum of these components when projected into $\mathbb{R}/\mathbb{Z}$ is equal to $\eta(\mathcal{D}, \alpha) = \overline{\xi(\mathcal{D} \otimes E_\alpha) - \xi(\mathcal{D} \otimes \epsilon^n)}$. The main result of this section is the following.

Theorem 4.1 *One has*

$$\eta(\mathcal{D}, \alpha) = \overline{\frac{1}{k}\big(sf_{\mathrm{I}}(\mathcal{D}, k\alpha, \theta) + sf_{\mathrm{II}}(\mathcal{D}, k\alpha, \theta)\big)} \in \mathbb{R}/\mathbb{Z}. \tag{4.4}$$

Proof It suffices to show that the second component described above is equal to Type II spectral flow. It follows from Proposition 3.1 that the operator $\mathcal{D}_\alpha$ defines a pre-Breuer-Fredholm module in the sense of Carey and Phillips [5]. We may form the loop $\mathcal{D}_t = \lambda(\mathcal{D}, \alpha, \theta)(t)$ and consider the associated path of bounded operators, $f(\mathcal{D}_t)$, where $f(x) = x/\sqrt{1+x^2}$. The Type II spectral flow associated to that loop can be expressed as the index of a certain Toeplitz operator. To do this, one needs to perturb the original pre-Fredholm module to a genuine Fredholm module. This can always be done, and the resulting projection in this case is the positive spectral projection, P, for the leafwise self-adjoint elliptic operator $\mathcal{D}_\alpha$. The Toeplitz operator we will consider, T_ϕ, is that obtained by compressing

https://doi.org/10.4153/CMB-2000-010-6 Published online by Cambridge University Press

multiplication by $1 \otimes \phi$ to the range of P. The index theory for such operators was studied in [7], [6] where it was shown the its index is given by the following topological formula,

$$\operatorname{Index}_{\mathrm{II}}(T_\phi) = \langle \mathrm{Tch}(\alpha, \theta) \cup \Phi^{-1} ch\big(\sigma(\mathcal{D})\big) \cup \mu, [M \times U_n] \rangle, \tag{4.5}$$

where μ is the Poincaré dual of the Ruelle-Sullivan current. From this it follows easily that 4.5 is equal to

$$\langle \mathrm{Tch}(\alpha, \theta) \cup \Phi^{-1} ch\big(\sigma(\mathcal{D})\big), [M] \rangle.$$

But by [2, p. 324] this is equal to $\operatorname{Index}\big(\nu \cdot [\sigma(\mathcal{D})]\big)$. This completes the proof. ■

Remark 4.2 There is a related paper by S. Hurder, [9], in which a notion of Type II spectral flow arises which appears to be quite different from the one used in this paper. We have chosen to use the general and abstract definition of Type II spectral flow as developed in [11]. It follows from the results of the present paper, along with the work in [9] and [14], that the two notions agree. Indeed, Hurder shows that his definition of Type II spectral flow agrees with the real valued index of a leafwise Toeplitz operator, while we observe that ours agrees with that same index.

References

[1] M. F. Atiyah and I. M. Singer, *Index theory for skew-adjoint Fredholm operators*. Inst. Hautes Études Sci. Publ. Math. **37**(1969), 305–326.

[2] M. F. Atiyah, I. M. Singer and V. K. Patodi, *Spectral asymmetry and Riemannian geometry, III*. Math. Proc. Camb. Phil. Soc. **79**(1976), 71–99.

[3] D. Basu, *A classifying space for $K^1(X;\mathbb{R})$*. Integral Equations Operator Theory **31**(1998), 287–298.

[4] B. Boos-Bavnbek and K. P. Wojciechowski, *Elliptic boundary problems for Dirac operators.* Birkhäuser, 1993.

[5] A. Carey and J. Phillips, *Unbounded Fredholm modules and spectral flow.* Preprint, 1997.

[6] R. G. Douglas, S. Hurder and J. Kaminker, *Cyclic cocycles, renormalization and eta-invariants.* Invent. Math. **103**(1991), 101–181.

[7] ———, *The longitudinal cocycle and the index of Toeplitz operators.* J. Funct. Anal. **101**(1991), 120–144.

[8] R. G. Douglas and J. Kaminker, *Induced cyclic cocycles and higher eta invariants.* J. Funct. Anal. **147**(1997), 301–326.

[9] S. Hurder, *Eta invariants and the odd index theorem for coverings.* Contemp. Math. **105**(1990), 47–82

[10] H. Moriyoshi and T. Natsume, *The Godbillon-Vey cyclic cocycle and longitudinal Dirac operators.* Pacific J. Math. **172**(1996), 483–539.

[11] V. Perera, *Real valued spectral flow.* Contemp. Math. **185**(1993), 307–318.

[12] J. Phillips, *Self-adjoint Fredholm operators and spectral flow.* Canad. Math. Bull. **39**(1996), 460–467.

[13] ———, *Spectral flow in type I and II factors—a new approach.* Cyclic Cohomology and Noncommutative Geometry (Waterloo, 1995), Fields Inst. Commum. **17**(1997), 137–153.

[14] Mohan Ramachandran, *Von Neumann index theorems for manifolds with boundary.* J. Differential Geom. **38**(1993), 315–349.

[15] M. Rieffel, *The type of group measure space von Neumann algebras.* Monatsh. Math (2) **85**(1978), 149–162.

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