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The Thickness of the Cartesian Product of Two Graphs

Yichao Chen and Xuluo Yin

Abstract. The thickness of a graph G is the minimum number of planar subgraphs whose union is G. A t-minimal graph is a graph of thickness t that contains no proper subgraph of thickness t. In this paper, upper and lower bounds are obtained for the thickness, $t(G \square H)$, of the Cartesian product of two graphs G and H, in terms of the thickness $t(G)$ and $t(H)$. Furthermore, the thickness of the Cartesian product of two planar graphs and of a t-minimal graph and a planar graph are determined. By using a new planar decomposition of the complete bipartite graph $K_{4k,4k}$, the thickness of the Cartesian product of two complete bipartite graphs $K_{n,n}$ and $K_{n,n}$ is also given for $n \neq 4k+1$.

1 Introduction

In this paper all graphs are simple. A graph G is often denoted by $G = (V, E)$, where $V(G)$ is the vertex set and $E(G)$ is the edge set. The order and the size of G are denoted by $\nu(G)$ and $\varepsilon(G)$, respectively. *A complete graph* is a graph in which any two vertices are adjacent. A complete graph on n vertices is denoted by K_n. A *complete bipartite graph* is a graph whose vertex set can be partitioned into 2 parts such that every edge has its ends in different parts and every two vertices in different parts are adjacent. We use K_{p_1,p_2} to denote a complete bipartite graph in which the i-th part contains p_i $(1 \leq i \leq 2)$ vertices.

A graph is said to be *planar* if it can be drawn on the plane in such a way that its edges intersect only at their endpoints. Such a drawing is called a *plane graph*. A planar graph is *maximal planar* if it is not possible to add an edge such that the graph is still planar. The *thickness* $t(G)$ of a graph G is the minimum number of planar spanning subgraphs into which G can be decomposed. The thickness of a graph was inaugurated by W. T. Tutte [15] in 1963. As a topological invariant of a graph, it plays an important role in graph drawing and VLSI circuit design [1]. In [11], Mansfield proved that determining the thickness of a graph is NP-complete. Thus, it is very difficult to get the exact value of thickness for arbitrary graphs. The only types of graphs whose thickness have been obtained are complete graphs [2, 4, 16], complete bipartite graphs [5], and hypercubes [10]. The reader is referred to [6, 12, 18] for more background and results about the thickness problem.

The *cartesian product* of graphs G and H is a graph $G \square H$ with vertex set $V(G \square H) = V(G) \times V(H)$, that is the set $\{(g,h) \mid g \in G, h \in H\}$. The edge set of $G \square H$

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consists of all pairs $(g,h)(g',h')$ of vertices with $gg' \in E(G)$ and $h = h'$ or $hh' \in E(H)$ and $g = g'$. For any $h \in V(H)$, we denote by G^h the subgraph of $G \square H$ induced by $V(G) \times \{h\}$; it is isomorphic to G and called a *G-fiber*. The *H-fiber* ${}^gH$ is defined analogously, where $g \in G$. The Cartesian product is a very important graph operation; we refer the reader to [9] for topics on Cartesian product of graphs.

In the past forty years, the topological invariants of the Cartesian product of two graphs, *e.g.*, genus (see [13,17] etc.) crossing number (see [9] etc.) were often discussed in topological graph theory. In this paper, the thickness of the Cartesian product of arbitrary two graphs is studied. This paper is organized as follows. In Section 2, the upper and lower bounds for $t(G \square H)$ are given. When m or s is even, the value of $t(K_{m,n} \square K_{s,t})$ is determined when n and t are large enough. The thickness of the Cartesian product of two planar graphs is given in Section 3. In Section 4, the thickness for the Cartesian product of a t-minimal graph and a planar graph is presented. In the final section, we show that $t(K_{n,n} \square K_{n,n}) = \lceil \frac{n+1}{2} \rceil$, for $n \neq 4k+1$.

2 Bounds for the Thickness of the Cartesian Product of Two Graphs

The *union* $G \cup H$ of two graph G and H is the graph $(V(G) \cup V(H), E(G) \cup E(H))$. The *intersection* $G_1 \cap G_2$ of G_1 and G_2 is defined analogously. The *join* $G + H$ of two vertex disjoint graphs G and H is obtained from $G \cup H$ by joining every vertex of G to every vertex of H. In [19], Yang and Chen presented an explicit formula for the thickness of the Cartesian product $K_n \square P_m$, for $m \geq 2$ and $n \neq 6p+3$. We have the following general bounds for the thickness of the Cartesian product of two arbitrary graphs.

Theorem 2.1 *The thickness of $G \square H$ satisfies the inequality*

$$\mathrm{Max}\{t(G), t(H)\} \leq t(G \square H) \leq t(G) + t(H).$$

Proof Since both G and H are subgraphs of $G \square H$, we have that

$$t(G \square H) \geq \mathrm{Max}\{t(G), t(H)\}.$$

Suppose that $V(G) = \{v_1, v_2, \dots, v_n\}$ and $V(H) = \{u_1, u_2, \dots, u_m\}$. From the definition of the Cartesian product of two graphs, the graph $G \square H$ contains $\nu(G)$ number of disjoint copies ${}^gH$ of H and $\nu(H)$ number of disjoint copies G^h of G, where $g \in V(G)$ and $h \in V(H)$. Let $\{G_1^{u_i}, G_2^{u_i}, \dots, G_{t(G)}^{u_i}\}$ be a planar decomposition of G^{u_i}, for $i = 1,2,\dots,m$, and let $\{{}^{v_j}H_1, {}^{v_j}H_2, \dots, {}^{v_j}H_{t(H)}\}$ be a planar decomposition of ${}^{v_j}H$, for $j = 1,2,\dots,n$. Define $G_j = G_j^{u_1} \cup G_j^{u_2} \cup \cdots \cup G_j^{u_m}$, for $j = 1,2,\dots,t(G)$ and $H_i = {}^{v_1}H_i \cup {}^{v_2}H_i \cup \cdots \cup {}^{v_n}H_i$, for $i = 1,2,\dots,t(H)$.

It is easy to see that G_j, for $j = 1,2,\dots,t(G)$, and H_i, for $i = 1,2,\dots,t(H)$, are planar subgraphs. Thus, $\{G_1, G_2, \dots, G_{t(G)}, H_1, H_2, \dots, H_{t(H)}\}$ is a planar decomposition of $G \square H$, which shows that $t(G \square H) \leq t(G) + t(H)$. Summarizing the above, the result follows. ∎

Let G_1 and G_2 be subgraphs of a graph G. If $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \{v\}$ (a vertex of G), then we say that G is the *vertex amalgamation* of G_1 and G_2 at vertex v, denoted $G = G_1 \vee_v G_2$.

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Lemma 2.2 ([19]) *If G is the vertex amalgamation of G_1 and G_2, $t(G_1) = n_1$ and $t(G_2) = n_2$, then $t(G) = \max\{n_1, n_2\}$.*

Let G_1 be a graph with a vertex v of degree k and $N_G(v) = \{u_1, u_2, \ldots, u_k\}$. Let G_2 be a graph with a vertex v of degree k and $N_H(v) = \{w_1, w_2, \ldots, w_k\}$. Delete the vertex v from G_1 and G_2. Then construct a graph G by adding k edges $u_1w_1, u_2w_2, \ldots, u_kw_k$. The edges $u_1w_1, u_2w_2, \ldots, u_kw_k$ are called the *product edges* and the resulting graph G is called a *dot product* of G_1 and G_2, denoted by $G = G_1 \circ G_2$, as in Figure 1.

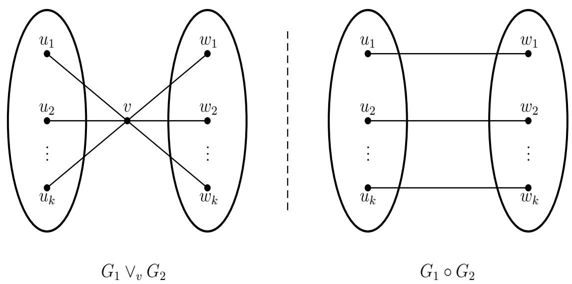

Figure 1: The dot product $G_1 \circ G_2$ of G_1 and G_2.

In [19], Yang and Chen obtained the thickness for $K_n \square K_2$. By generalizing the techniques in [19], we have the following result.

Lemma 2.3 *Let the graph $G + v$ denote the join of the graph G and a vertex v; then the thickness of $G \square K_2$ equals $t(G + v)$.*

Proof Suppose that $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(K_2) = \{x, y\}$. Given a planar decomposition of $G \square K_2$, by contracting the subgraph from G^x (or G^y) to a single vertex in every planar subgraph, we can obtain a planar decomposition of $G + v$, *i.e.*, $t(G + v) \le t(G \square K_2)$.

Let G' be a disjoint copy of G and $V(G') = \{v'_1, v'_2, \ldots, v'_n\}$. Let $H = G + v \vee_v G' + v$. From Lemma 2.2, we infer that $t(H) = t(G + v)$. We now construct a planar decomposition of $G \square K_2$ from H. Let $\{G_1, G_2, \ldots, G_n\}$ be a planar decomposition of $G + v$. For $1 \le i \le n$, let G'_i be a copy of G_i such that $G_i \cap G'_i = \{v\}$ and $G_i - v$ is isomorphic to $G'_i - v$. Thus, $\{G'_1, G'_2, \ldots, G'_n\}$ is a planar decomposition of $G' + v$. Defining $H_i = G_i \circ G'_i$, for $i = 1, 2, \ldots, n$, it is easy to see that $\{H_1, H_2, \ldots, H_n\}$ is a planar decomposition of $G \square K_2$. Thus, we have $t(G + v) \ge t(G \square K_2)$. Combining the above, we have the desired result. ∎

Theorem 2.4 *If $\varepsilon(G) \ge 1$ and $\varepsilon(H) \ge 1$, the thickness of $G \square H$ satisfies the inequality*

$$t(G \square H) \ge \mathrm{Max}\{t(G + v), t(H + v)\}.$$

Proof Since both $G \square K_2$ and $H \square K_2$ are subgraphs of $G \square H$, from Lemma 2.3, $t(G \square H) \ge \mathrm{Max}\{t(G + v), t(H + v)\}$. The result follows. ∎

The bounds in Theorem 2.1 are best possible if one of the graphs G and H is empty, since the empty graph has thickness 0. The following theorem gives another example to show that upper bound of Theorem 2.1 is sharp.

Theorem 2.5 *Suppose that at least one of the numbers m and s is even. Then*

$$t(K_{m,n} \square K_{s,t}) = t(K_{m,n}) + t(K_{s,t}) = \left\lceil \frac{m+s}{2} \right\rceil$$

if $n \geq 2m^2 - m$ and $t \geq 2s^2 - s$.

Proof From the definition of Cartesian product of two graphs, we have

$$\nu(K_{m,n} \square K_{s,t}) = (m+n)(s+t),$$
$$\varepsilon(K_{m,n} \square K_{s,t}) = mn(s+t) + (m+n)st.$$

From Euler's formula, the maximum planar subgraph of $K_{m,n} \square K_{s,t}$ contains at most $2(m+n)(s+t) - 4$ edges. Thus, we have

$$\text{(2.1)} \quad t(K_{m,n} \square K_{s,t}) \geq \left\lceil \frac{mn(s+t) + (m+n)st}{2(m+n)(s+t) - 4} \right\rceil$$
$$= \left\lceil \frac{m+s}{2} - \frac{m^2(s+t) - 2m}{2(m+n)(s+t) - 4} - \frac{s^2(m+n) - 2s}{2(m+n)(s+t) - 4} \right\rceil.$$

The following two cases are considered.

(a) Both m and s are even. If $n \geq m^2 - m$ and $t \geq s^2 - s$, then

$$\frac{m^2(s+t) - 2m}{2(m+n)(s+t) - 4} + \frac{s^2(m+n) - 2s}{2(m+n)(s+t) - 4} < 1.$$

Combining the inequality (2.1), we have

$$t(K_{m,n} \square K_{s,t}) \geq \left\lceil \frac{m+s}{2} \right\rceil.$$

From [5, Theorem 1], $t(K_{p_1,p_2}) = \lceil \frac{p_1}{2} \rceil$ when p_1 is even and $p_2 > \frac{1}{2}(p_1 - 2)^2$, or p_1 is odd and $p_2 > (p_1 - 1)(p_1 - 2)$. By Theorem 2.1,

$$t(K_{m,n} \square K_{s,t}) \leq t(K_{m,n}) + t(K_{s,t})$$

Thus,

$$\left\lceil \frac{m+s}{2} \right\rceil \leq t(K_{m,n} \square K_{s,t}) \leq \left\lceil \frac{m}{2} \right\rceil + \left\lceil \frac{s}{2} \right\rceil,$$

where $n \geq m^2 - m$ and $t \geq s^2 - s$. Since both m and s are even, we have $\lceil \frac{m+s}{2} \rceil = \lceil \frac{m}{2} \rceil + \lceil \frac{s}{2} \rceil$, and the result follows.

(b) One of m and s is even and the other is odd. In this case, if $n \geq 2m^2 - m$ and $t \geq 2s^2 - s$, then

$$\frac{m^2(s+t) - 2m}{2(m+n)(s+t) - 4} + \frac{s^2(m+n) - 2s}{2(m+n)(s+t) - 4} < \frac{1}{2}.$$

Combining inequality (2.1), we have

$$t(K_{m,n} \square K_{s,t}) \geq \left\lceil \frac{m+s}{2} \right\rceil.$$

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In a similar way to case (a) above, we have $t(K_{m,n}\square K_{s,t}) = \lceil \frac{m+s}{2} \rceil$, for $n \geq 2m^2 - m$ and $t \geq 2s^2 - s$.

Summarizing the above, the proof is completed. ∎

3 The Thickness of the Cartesian Product of Two Planar Graphs

In this section, we determine the thickness of the Cartesian product of two planar graphs. We will provide more examples to show that the bounds in Theorem 2.1 are sharp. In [3], Behzad and Mahmoodian proved the follows two theorems.

Theorem 3.1 ([3]) *Let G and H be connected graphs on at least three vertices. Then $G \square H$ is planar if only if both G and H are paths or one is a path and the other is a cycle.*

Theorem 3.2 ([3]) *Let G be an outerplanar graph. Then $G \square K_2$ is planar if only if G is outerplanar.*

We have the following theorem.

Theorem 3.3 *Let G and H be two planar graphs. The thickness of $G \square H$ is*

$$t(G \square H) = \begin{cases} 1 & \text{if both two graphs are paths,} \\ 1 & \text{if one is a path and the other is a cycle,} \\ 1 & \text{if one is outerplanar and the other is } K_2, \\ 2 & \text{otherwise.} \end{cases}$$

Proof From Theorem 2.1, we have $1 \leq t(G \square H) \leq 2$. However, from Theorems 3.1 and 3.2, we infer that the only planar Cartesian products are $P_m \square P_n$, $P_m \square C_n$ and $G \square K_2$, where G is outerplanar. The result follows. ∎

4 The Thickness of the Cartesian Product of a t–minimal Graph and a Planar Graph

A graph G is said to be *t-minimal* if all of its proper subgraphs have thickness less than t. This concept was introduced by Tutte [15] in 1963. In [15], Tutte also proved that every graph G with thickness $t > k$, contains a k-minimal subgraph of G. In [8], Hobbs and Grossman proved that there exists a t-minimal graph with connectivity 2, for every $t \geq 2$. In [14], Širáň and Horák gave an explicit construction of an infinite number of t-minimal graphs with connectivity 2, edge-connectivity t, and minimum degree t. In [19], Yang and Chen determined the thickness for the Cartesian product of a t-minimal graph and an outerplanar graph. We have the following result.

Theorem 4.1 *Let G be t-minimal graph and H be a planar graph; then $t(G \square H) = t(G)$.*

Proof Suppose that $V(G) = \{v_1, v_2, \ldots, v_m\}$ and $V(H) = \{u_1, u_2, \ldots, u_n\}$. For $1 \leq i \leq n$, let the vertex set of the G-fiber graph G^{u_i} be $G \times \{u_i\}$. For $1 \leq j \leq m$, let the vertex set of the H-fiber ${}^{v_j}H$ be $\{v_j\} \times H$. Suppose that $\{G_{1,i}, G_{2,i}, \dots, G_{t,i}\}$ be a planar decomposition of G^{u_i}, for $i = 1, 2, \dots, n$. Since G^{u_i} is a t−minimal graph, without the loss of generality, we suppose that the graph $G_{t,i}$ contains only one edge $(v_1, u_i)(v_2, u_i)$, for $i = 1, 2, \dots, n$, and where $v_1 v_2$ is an edge of G. In the following discussion, we will construct a planar subgraph decomposition of $G \,\square\, H$ with t planar subgraphs $G_1, G_2, \dots, G_t$.

Defining

$$\begin{aligned} G_1 &= G_{1,1} \cup G_{1,2} \cup \cdots \cup G_{1,n} \cup {}^{v_1}H, \\ G_j &= G_{j,1} \cup G_{j,2} \cup \cdots \cup G_{j,n}, \\ G_t &= G_{t,1} \cup G_{t,2} \cup \cdots \cup G_{t,n} \cup {}^{v_2}H \cup \cdots \cup {}^{v_m}H \end{aligned}$$

for $2 \le j \le t-1$. Now let us show that $\{G_1, G_2, \dots, G_t\}$ is a planar decomposition of $G \,\square\, H$.

(a) Let $V({}^{v_1}H) = \{(v_1, u_i) | i = 1, 2, \dots, n\}$. Since the graphs $G_{1,1}, G_{1,2}, \dots, G_{1,n}$ are disjoint and $V({}^{v_1}H) \cap V(G_{1,i}) = \{(v_1, u_i)\}$, we amalgamate the two planar graphs $G_{1,i}$ and ${}^{v_1}H$ at the vertex (v_1, u_i) for $i = 1, 2, \dots, n$, and denote the resulting graph by G_1. Since the amalgamation of two planar graphs is still planar, G_1 is planar.

(b) Since the graphs $G_{j,1}, G_{j,2}, \dots, G_{j,n}$ are mutually disjoint planar graphs, this implies that the graph G_j is planar, for $j = 2, 3, \dots, t-1$.

(c) Recall that the planar subgraphs ${}^{v_2}H, {}^{v_3}H, \dots, {}^{v_m}H$ are mutually disjoint and each graph $G_{t,i}$ contains only one edge $\{(v_1, u_i)(v_2, u_i)\}$, for $i = 1, 2, \dots, n$. Since $V(G_{t,i}) \cap V({}^{v_2}H) = \{(v_2, u_i)\}$, for each $i (1 \le i \le n)$ we amalgamate the graph $G_{t,i}$ and ${}^{v_2}H$ at the vertex (v_2, u_i), the union $G_{t,1} \cup G_{t,2} \cup \cdots \cup G_{t,n} \cup {}^{v_2}H$ is still a planar graph. From the fact that $V(G_{t,i}) \cap V({}^{v_j}H) = \varnothing$, for $i = 1, 2, \dots, n$, and $j = 3, 4, \dots, m$, we infer that the graphs

$$G_{t,1} \cup G_{t,2} \cup \cdots \cup G_{t,n} \cup {}^{v_2}H, {}^{v_3}H, {}^{v_4}H, \dots, {}^{v_m}H$$

are mutually disjoint, thus the graph G_t is planar.

Summarizing the above, a planar decomposition of $G \,\square\, H$ with t subgraphs $G_1, G_2, \dots\ G_t$ is constructed, which shows $t(G \,\square\, H) \le t$. On the other hand, $G \subset G \,\square\, H$, so we have $t(G \,\square\, H) \ge t$. The theorem follows. ∎

5 The Thickness of $K_{n,n} \,\square\, K_{n,n}$

In [5], Beineke, Harary, and Moon constructed a planar decomposition of $K_{m,n}$ when m is even. By using the planar decomposition, they determined the thickness for $K_{m,n}$ for most values of m and n. Up to now, determining the thickness of bipartite graph $K_{m,n}$ is still open, when m and n are odd and there exists an integer k satisfying $n = \lfloor \frac{2k(m-2)}{(m-2k)} \rfloor$. The theorem of Beineke, Harary, and Moon implies the following result.

Theorem 5.1 ([5]) *The thickness of the complete bipartite graph $K_{n,n}$ is*

$$t(K_{n,n}) = \left\lceil \frac{n+2}{4} \right\rceil.$$

Theorem 5.2 *The thickness of the Cartesian product of two complete bipartite graphs $K_{n,n}$ and $K_{n,n}$ satisfies the inequality*

$$t(K_{n,n} \square K_{n,n}) \geq \left\lceil \frac{n+1}{2} \right\rceil .$$

Proof It is easy to see that $\nu(K_{n,n} \square K_{n,n}) = 4n^2$ and $\varepsilon(K_{n,n} \square K_{n,n}) = 4n^3$. Suppose H be a maximum planar subgraph of $K_{n,n} \square K_{n,n}$. Since the graph $K_{n,n} \square K_{n,n}$ does not contain triangles, from Euler's Formula, we have $|E(H)| \leq 8n^2 - 4$.

For $n \geq 1$, we have $0 \leq \frac{1}{2} - \frac{n}{4n^2-2} < \frac{1}{2}$, thus

$$t(K_{n,n} \square K_{n,n}) \geq \left\lceil \frac{4n^3}{8n^2-4} \right\rceil = \left\lceil \frac{n+1}{2} + \frac{1}{2} - \frac{n}{4n^2-2} \right\rceil = \left\lceil \frac{n+1}{2} \right\rceil . \qquad \blacksquare$$

5.1 A Planar Decomposition for $K_{4k,4k}$

Let the 2-partite sets of $K_{4k,4k}$ be $U = \{u_1, u_2, \dots, u_{4k}\}$ and $V = \{v_1, v_2, \dots, v_{4k}\}$. We will construct a new planar decomposition for the complete bipartite graph $K_{4k,4k}$. Let $\{G_1, G_2, \dots, G_{k+1}\}$ be the planar decomposition of $K_{4k,4k}$. The construction has three steps.

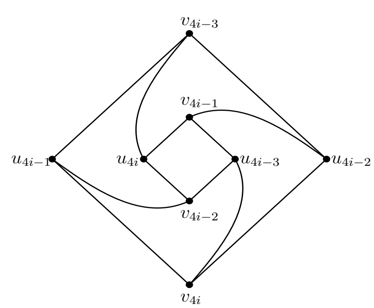

Figure 2: The graph H_i.

(a) We first construct a subgraph H_i of G_i, for $i = 1, 2, \dots, k$. The vertex set $V(H_i)$ of H_i, for $i = 1, 2, \dots, k$, consists of the vertices u_{4i-3}, u_{4i-2}, u_{4i-1}, u_{4i}, v_{4i-3}, v_{4i-2}, v_{4i-1}, and v_{4i}. The edge set $E(H_i)$ consists of two 4-cycles and four independent edges between them. The two 4-cycles are $u_{4i-3}v_{4i-2}u_{4i}v_{4i-1}u_{4i-3}$ and $v_{4i-3}u_{4i-2}v_{4i}u_{4i-1}v_{4i-3}$. The four independent edges are $v_{4i-3}u_{4i}$, $v_{4i-2}u_{4i-1}$, $v_{4i}u_{4i-3}$, and $v_{4i-1}u_{4i-2}$, as shown in Figure 2.

(b) Add $2k-2$ parallel edges between v_{4i-3} and v_{4i-1} in H_i and insert $2k-2$ new vertices

$$\bigcup_{\substack{r=1 \\ r \neq i}}^{k} \{u_{4r-3}, u_{4r-2}\}$$

on these $2k-2$ parallel edges respectively. In a similar way, we do this for the vertex pairs $\{v_{4i-2}, v_{4i}\}$, $\{u_{4i-3}, u_{4i-2}\}$ and $\{u_{4i-1}u_{4i}\}$ in H_i, and insert $6k-6$ vertices

$$\bigcup_{\substack{r=1\\ r\neq i}}^{k}\{u_{4r-1}, u_{4r}\}, \quad \bigcup_{\substack{r=1\\ r\neq i}}^{k}\{v_{4r-2}, v_{4r}\}, \quad \bigcup_{\substack{r=1\\ r\neq i}}^{k}\{v_{4r-3}, v_{4r-1}\}$$

on these $6k-6$ parallel edges, respectively. The resulting graph is denoted by G_i, for $i = 1, 2, \ldots, k$.

(c) The graph G_{k+1} consists of $4k$ independent edges $u_1v_1, u_2v_2, \ldots, u_{4k}v_{4k}$; *i.e.*, $G_{k+1} = \bigcup_{i=1}^{4k}\{u_iv_i\}$.

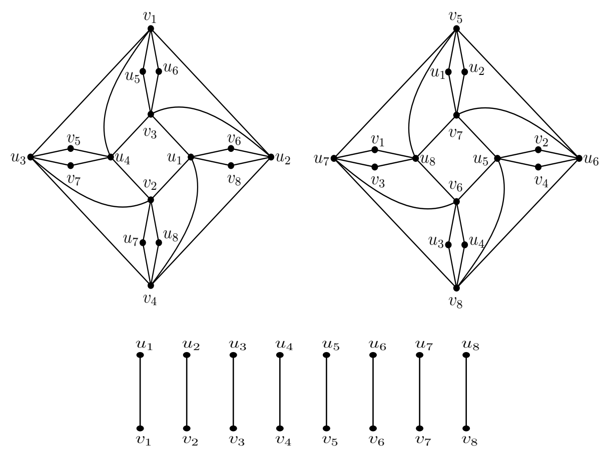

Figure 3: A planar decomposition of $K_{8,8}$.

Now a planar decomposition $\{G_1, G_2, \ldots, G_{k+1}\}$ of $K_{4k,4k}$ is completed. By using the construction above, a planar decomposition of the graph $K_{8,8}$ is shown in Figure 3.

Remark 5.3 For $1 \le i \le 4k$, we first connect u_{4k+1} and v_{4k+1} to v_i and u_i by new edges $u_{4k+1}v_i$ and $v_{4k+1}u_i$ in G_{k+1}, respectively, then connect u_{4k+1} to v_{4k+1} by a new edge $u_{4k+1}v_{4k+1}$. Thus, the planar decomposition of $K_{4k,4k}$ above implies a planar decomposition of $K_{4k+1,4k+1}$.

5.2 The Thickness of $K_{n,n} \square K_{n,n}$

In this subsection, we will determine the thickness of $K_{n,n} \square K_{n,n}$, for $n \neq 4k+1$. Let the 2-partite sets of $K_{n,n}$ be $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_n\}$. Let the G-fibers of $K_{n,n} \square K_{n,n}$ be $K_{n,n}^{u_i}$ and $K_{n,n}^{v_i}$, for $i = 1, 2, \ldots, n$, and let the H-fibers of $K_{n,n} \square K_{n,n}$ be ${}^{u_j}K_{n,n}$ and ${}^{v_j}K_{n,n}$, for $j = 1, 2, \ldots, n$.

We first construct a planar decomposition for $K_{4k,4k}$. From the construction of Subsection 5.1 for $K_{4k,4k}$, for $1 \le i \le n$, we suppose that $\{G_{1,i}, G_{2,i}, \ldots, G_{k,i}, G_{k+1,i}\}$

and $\{G'_{1,i}, G'_{2,i}, \dots, G'_{k,i}, G'_{k+1,i}\}$ are the planar decompositions of $K^{u_i}_{n,n}$ and $K^{v_i}_{n,n}$, respectively. Similarly for $1 \le j \le n$, let

$$\{\overline{G}_{1,j}, \overline{G}_{2,j}, \dots, \overline{G}_{k,j}, \overline{G}_{k+1,j}\} \quad \text{and} \quad \{\overline{G'}_{1,j}, \overline{G'}_{2,j}, \dots, \overline{G}'_{k,j}, \overline{G'}_{k+1,j}\}$$

be the planar decompositions for ${}^{u_j}K_{n,n}$ and ${}^{v_j}K_{n,n}$ respectively.

Defining

$$G_j = \bigcup_{i=1}^{n} G_{j,i} \cup \bigcup_{i=1}^{n} G'_{j,i}, \quad G_{k+j} = \bigcup_{i=1}^{n} \overline{G}_{j,i} \cup \bigcup_{i=1}^{n} \overline{G'}_{j,i},$$
$$G_{2k+1} = \bigcup_{i=1}^{n} G_{k+1,i} \cup \bigcup_{i=1}^{n} G'_{k+1,i} \cup \bigcup_{i=1}^{n} \overline{G}_{k+1,i} \cup \bigcup_{i=1}^{n} \overline{G'}_{k+1,i}$$

for $1 \le j \le k$. Let us show that $\{G_1, G_2, \dots, G_{2k+1}\}$ is a planar decomposition of $K_{n,n} \square K_{n,n}$. There are three cases.

- The graph G_j is planar, for $1 \le j \le k$, because the planar graphs $G_{j,1}, G_{j,2}, \dots, G_{j,n}, G'_{j,1}, G'_{j,2}, \dots, G'_{j,n}$ are mutually disjoint.
- From the planar graphs $\overline{G}_{j,1}, \overline{G}_{j,2}, \dots, \overline{G}_{j,n}, \overline{G'}_{j,1}, \overline{G'}_{j,2}, \dots, \overline{G'}_{j,n}$ are mutually disjoint, the graph G_{j+k} is also planar, for $1 \le j \le k$.
- Recall that

$$G_{k+1,i} = \bigcup_{j=1}^{4k} (u_j, u_i)(v_j, u_i), \qquad G'_{k+1,i} = \bigcup_{j=1}^{4k} (u_j, v_i)(v_j, v_i),$$
$$\overline{G}_{k+1,i} = \bigcup_{j=1}^{4k} (u_i, u_j)(u_i, v_j), \qquad \overline{G'}_{k+1,i} = \bigcup_{j=1}^{4k} (v_i, u_j)(v_i, v_j),$$

  for $i = 1, 2, \dots, 4k$. In this case the graph G_{2k+1} is the union of $16k^2$ disjoint 4-cycles $(u_i, u_j)(v_i, u_j)(v_i, v_j)(u_i, v_j)(u_i, u_j)$, for $i, j = 1, 2, \dots, 4k$, thus G_{2k+1} is planar.

Summarizing the above, $K_{4k,4k} \square K_{4k,4k}$ can be decomposed into $2k+1$ planar subgraphs $G_1, G_2, \dots, G_{2k+1}$, which shows that

$$t(K_{4k,4k} \square K_{4k,4k}) \le 2k+1. \tag{5.1}$$

By using the procedure above, a planar decomposition

$$G_1 = \bigcup_{i=1}^{4} G_{1,i} \cup \bigcup_{i=1}^{4} G'_{1,i}, \quad G_2 = \bigcup_{i=1}^{4} \overline{G}_{1,i} \cup \bigcup_{i=1}^{4} \overline{G'}_{1,i},$$
$$G_3 = \bigcup_{i=1}^{4} G_{k+1,i} \cup \bigcup_{i=1}^{4} G'_{k+1,i} \cup \bigcup_{i=1}^{4} \overline{G}_{k+1,i} \cup \bigcup_{i=1}^{4} \overline{G'}_{k+1,i}$$

of the graph $K_{4,4} \square K_{4,4}$ is shown in Figures 4, 5, and 6.

We now turn to construction of a planar decomposition for $K_{4k-1,4k-1} \square K_{4k-1,4k-1}$. For $1 \le i \le n$, let

$$\{H_{1,i}, H_{2,i}, \dots, H_{k,i}, H_{k+1,i}\} \quad \text{and} \quad \{H'_{1,i}, H'_{2,i}, \dots, H'_{k,i}, H'_{k+1,i}\}$$

be the planar decompositions of $K^{u_i}_{n,n}$ and $K^{v_i}_{n,n}$, respectively. For $1 \le i \le n$, let

$$\{\overline{H}_{1,i}, \overline{H}_{2,i}, \dots, \overline{H}_{k,i}, \overline{H}_{k+1,i}\} \quad \text{and} \quad \{\overline{H}'_{1,i}, \overline{H}'_{2,i}, \dots, \overline{H'}_{k,i}, \overline{H'}_{k+1,i}\}$$

be the planar decompositions of ${}^{u_i}K_{n,n}$ and ${}^{v_i}K_{n,n}$ respectively. From [7], we know $K_{4k-1,4k-1}$ is a $k+1$-minimal graph, hence we suppose that each of the graphs

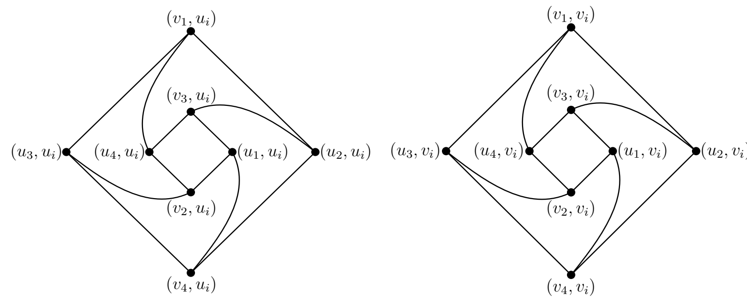

Figure 4: The graphs $G_{1,i}$ and $G'_{1,i}$ for $i = 1, 2, 3, 4$.

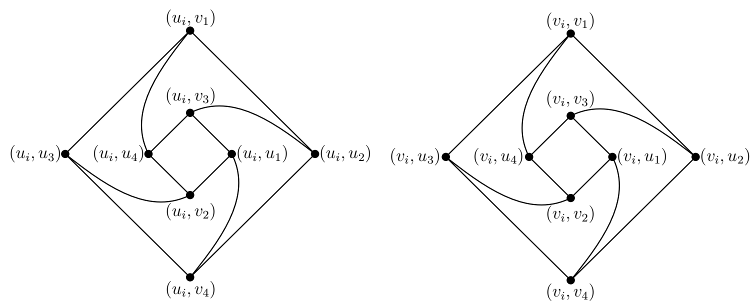

Figure 5: The graphs $\overline{G}_{1,i}$ and $\overline{G}'_{1,i}$, for $i = 1, 2, 3, 4$.

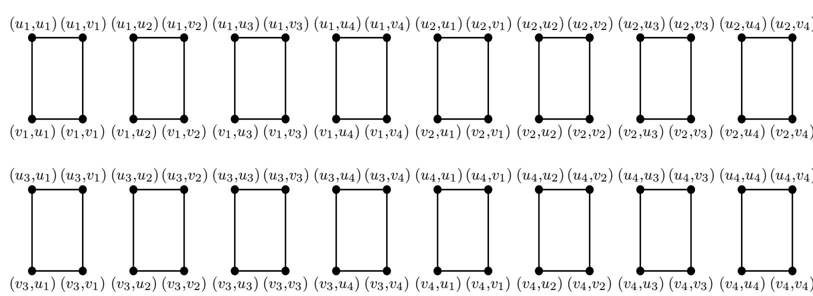

Figure 6: The graph G_3.

$H_{k+1,i}$, $H'_{k+1,i}$, $\overline{H}_{k+1,i}$, and $\overline{H'}_{k+1,i}$ contains only one edge. In the following discussion, we suppose the subscripts u_i, v_i in $K^{u_i}_{n,n}$, $K^{v_i}_{n,n}$, ${}^{u_i}K_{n,n}$ and ${}^{v_i}K_{n,n}$ are taken modulo n for $i > n$.

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First, we define

$$H_{k+1,i} = \{(u_{i+1}, u_i)(v_{i+2}, u_i)\}, \qquad H'_{k+1,i} = \{(u_{i+1}, v_i)(v_{i+2}, v_i)\},$$
$$\overline{H}_{k+1,i} = \{(u_i, u_i)(u_i, v_i)\}, \qquad \overline{H'}_{k+1,i} = \{(v_i, u_{n+i-1})(v_i, v_{n+i-1})\}.$$

Suppose that

$$(u_i, u_i)(v_{i+1}, u_i) \in H_{1,i}, \qquad (u_i, v_i)(v_{i+1}, v_i) \in H'_{1,i},$$
$$(u_i, u_{n+i-1})(u_i, v_{n+i-1}) \in \overline{H}_{1,i} \qquad (v_i, u_{n+i-2})(v_i, v_{n+i-2}) \in \overline{H'}_{1,i}.$$

We further define

$$\widetilde{G}_1 = \bigcup_{i=1}^{n} H_{1,i} \cup \bigcup_{i=1}^{n} H'_{1,i} \cup \bigcup_{i=1}^{n} \overline{H}_{k+1,i} \cup \bigcup_{i=1}^{n} \overline{H'}_{k+1,i},$$
$$\widetilde{G}_j = \bigcup_{i=1}^{n} H_{j,i} \cup \bigcup_{i=1}^{n} H'_{j,i},$$
$$\widetilde{G}_{k+1} = \bigcup_{i=1}^{n} \overline{H}_{1,i} \cup \bigcup_{i=1}^{n} \overline{H'}_{1,i} \cup \bigcup_{i=1}^{n} H_{k+1,i} \cup \bigcup_{i=1}^{n} H_{k+1,i},$$
$$\widetilde{G}_{k+j} = \bigcup_{i=1}^{n} \overline{H}_{j,i} \cup \bigcup_{i=1}^{n} \overline{H'}_{j,i},$$

for $j = 2, 3, \dots, k$. We now show that $\widetilde{G}_j$ is planar, for $j = 1, 2, \dots, 2k$. There are four cases.

- For $1 \le i \le n$, let $\widehat{H}_i = H_{1,i} \cup H'_{1,i} \cup \overline{H}_{k+1,i} \cup \overline{H'}_{k+1,i+1}$. Suppose that the edge $(u_i, u_i)(v_{i+1}, u_i)$ lies in the outer face of the planar embedding of $H_{1,i}$. Recall that $H'_{1,i}$ is a copy of $H_{1,i}$; hence we assume that the edge $(u_i, v_i)(v_{i+1}, v_i)$ lies in the outer face of the planar embedding of $H'_{1,i}$. We join $H_{1,i}$ and $H'_{1,i}$ by two edges $(u_i, u_i)(u_i, v_i)$ and $(v_{i+1}, u_i)(v_{i+1}, v_i)$; the resulting graph is a planar embedding of $\widehat{H}_i$; *i.e.*, $\widehat{H}_i$ is a planar graph. Since the planar graphs $\widehat{H}_1, \widehat{H}_2, \dots, \widehat{H}_n$ are mutually disjoint and $\widetilde{G}_1 = \bigcup_{i=1}^{n} \widehat{H}_i$, the graph $\widetilde{G}_1$ is planar.
- Let $\widehat{H'}_i = \overline{H}_{1,i} \cup \overline{H'}_{1,i+1} \cup H_{k+1,n+i-1} \cup H'_{k+1,n+i-1}$. With a similar discussion to the case above, we have that $\widehat{H'}_i$ is a planar graph. Since $\widetilde{G}_{k+1} = \bigcup_{i=1}^{n} \widehat{H'}_i$ and the graphs $\widehat{H'}_1, \widehat{H'}_2, \dots, \widehat{H'}_n$ are mutually disjoint, we have that the graph $\widetilde{G}_{k+1}$ is planar.
- From the planar graphs $H_{j,1}, H_{j,2}, \dots, H_{j,n}, H'_{j,1}, H'_{j,2}, \dots, H'_{j,n}$ are mutually disjoint, we have that the graph $\widetilde{G}_j$ is planar, for $j = 2, 3, \dots, k$.
- For $2 \le j \le k$, the graph $\widetilde{G}_{j+k}$ is also planar, because the planar graphs

  $$\overline{H}_{j,1}, \overline{H}_{j,2}, \dots, \overline{H}_{j,n}, \overline{H'}_{j,1}, \overline{H'}_{j,2}, \dots, \overline{H'}_{j,n}$$

  are mutually disjoint.

Summarizing the above, we obtain a planar decomposition $\{\widetilde{G}_1, \widetilde{G}_2, \dots, \widetilde{G}_{2k}\}$ of $K_{4k-1,4k-1} \,\square\, K_{4k-1,4k-1}$ with $2k$ planar subgraphs. Thus,

$$t(K_{4k-1,4k-1} \,\square\, K_{4k-1,4k-1}) \le 2k. \tag{5.2}$$

By using the procedure above, a planar decomposition $\{\widetilde{G}_1, \widetilde{G}_2\}$ of $K_{3,3} \,\square\, K_{3,3}$ is shown in Figures 7 and 8.

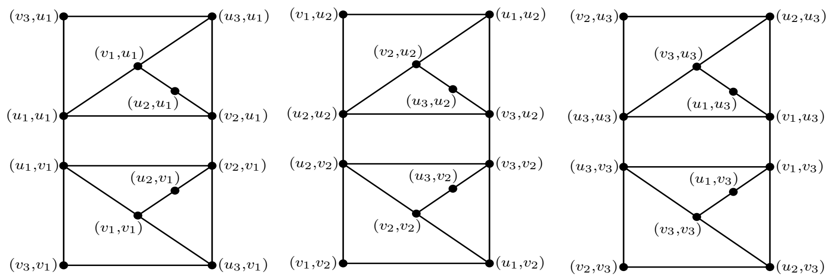

Figure 7: The graph $\widetilde{G}_1$.

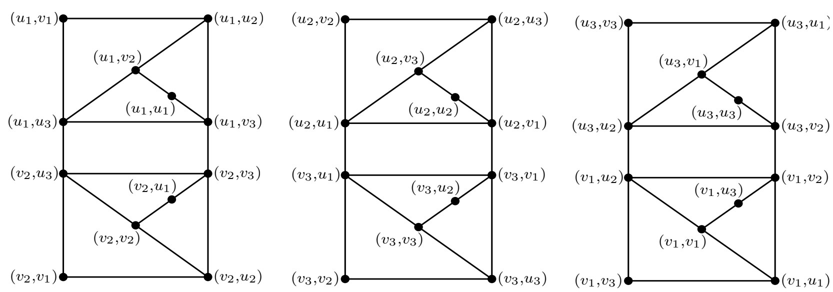

Figure 8: The graph $\widetilde{G}_2$.

Theorem 5.4 *The thickness of the Cartesian product of two complete bipartite graphs $K_{n,n}$ and $K_{n,n}$ is*

$$t(K_{n,n} \,\square\, K_{n,n}) = \left\lceil \frac{n+1}{2} \right\rceil \qquad (n \neq 4k+1).$$

Proof From Theorem 5.2 and inequality (5.1), we obtain $t(K_{4k,4k} \,\square\, K_{4k,4k}) = 2k+1$. From Theorem 5.2 and inequality (5.2), we obtain $t(K_{4k-1,4k-1} \,\square\, K_{4k-1,4k-1}) = 2k$. Because $K_{4k-2,4k-2} \,\square\, K_{4k-2,4k-2}$ is a subgraph of $K_{4k-1,4k-1} \,\square\, K_{4k-1,4k-1}$, combining Theorem 5.2, we know that $t(K_{4k-2,4k-2} \,\square\, K_{4k-2,4k-2}) = 2k$. Summarizing the discussion above, the theorem follows. ■

Remark 5.5 Though we fail to determine the value of $t(K_{4k+1,4k+1} \,\square\, K_{4k+1,4k+1})$, from Theorems 2.1, 5.1, and 5.2, we infer that

$$2k+1 \leq t(K_{4k+1,4k+1} \,\square\, K_{4k+1,4k+1}) \leq 2k+2.$$

For $k = 1$, we will construct a planar decomposition of $K_{5,5} \,\square\, K_{5,5}$ and show that $t(K_{5,5} \,\square\, K_{5,5}) = 3$. Suppose the 2-partite sets of $K_{5,5}$ are $U = \{u_1, u_2, \dots, u_5\}$ and $V = \{v_1, v_2, \dots, v_5\}$. For $1 \leq i, j \leq 5$, let $a_{ij} = (u_i, u_j)$, $a'_{ij} = (u_i, v_j)$, $b_{ij} = (v_i, u_j)$, and $b'_{ij} = (v_i, v_j)$; then a planar decomposition $\{B_1, B_2, B_3\}$ of $K_{5,5} \,\square\, K_{5,5}$ is shown in Appendices A, B, and C.

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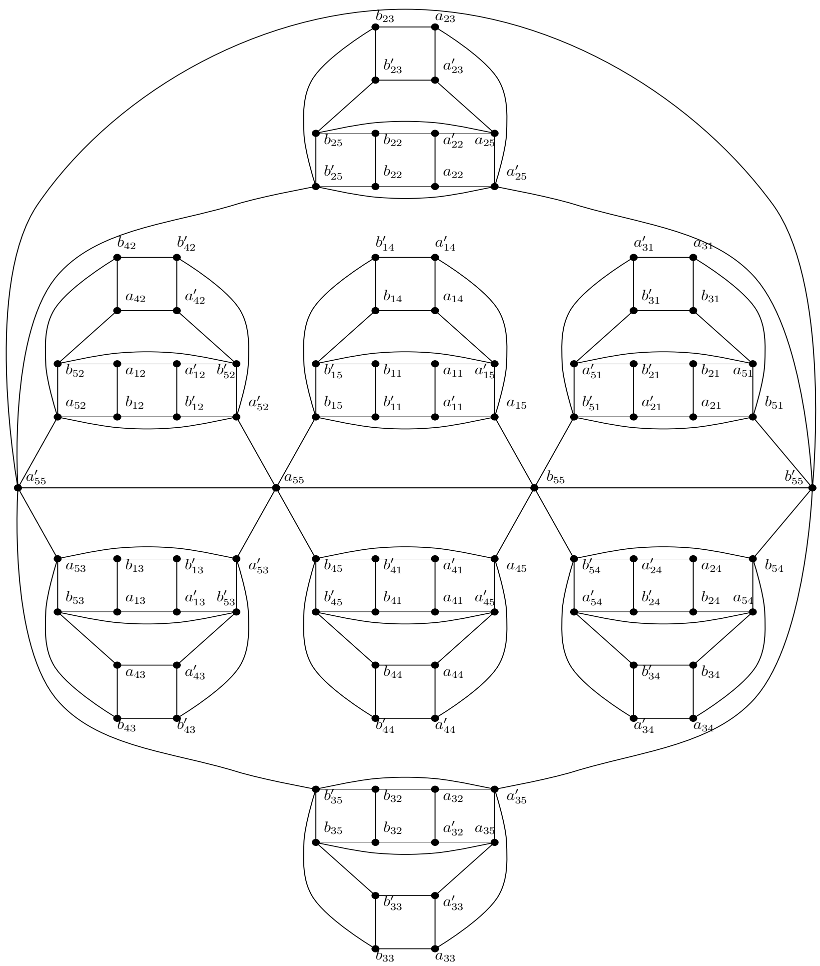
We pose the following problem for possible further study.

Problem 5.6 Find an explicit formula for $t(K_{m,n} \,\square\, K_{s,t})$, for any positive integers m, n, s, and t.

A The Graph B_1



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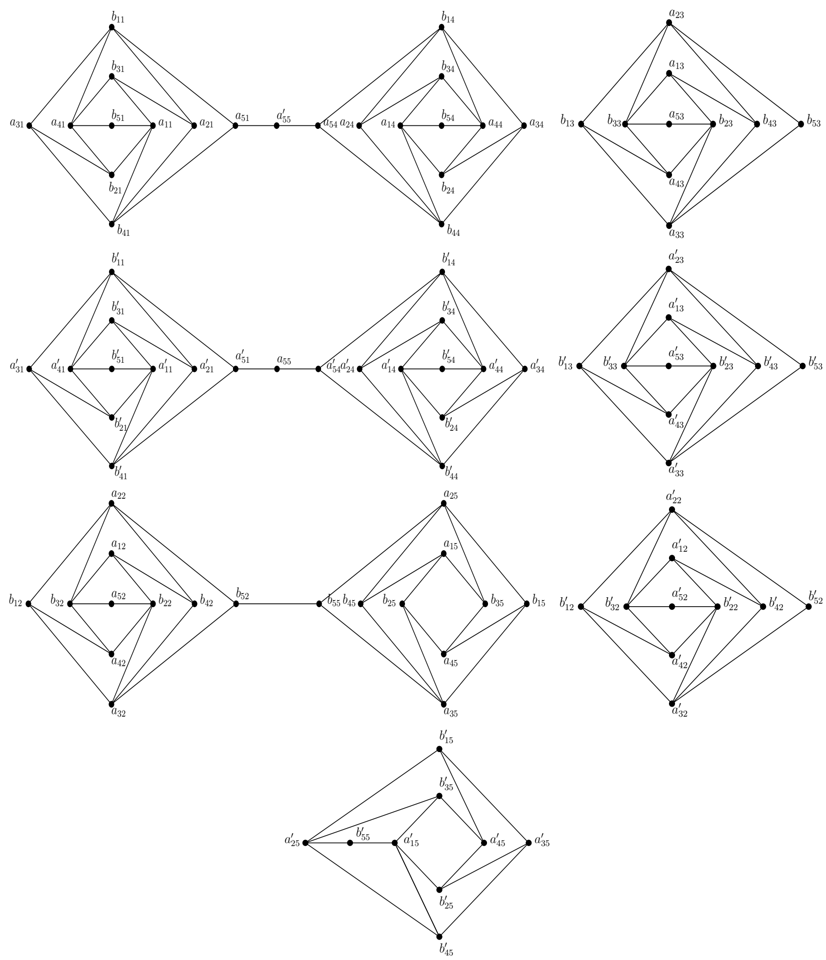
B The Graph B_2



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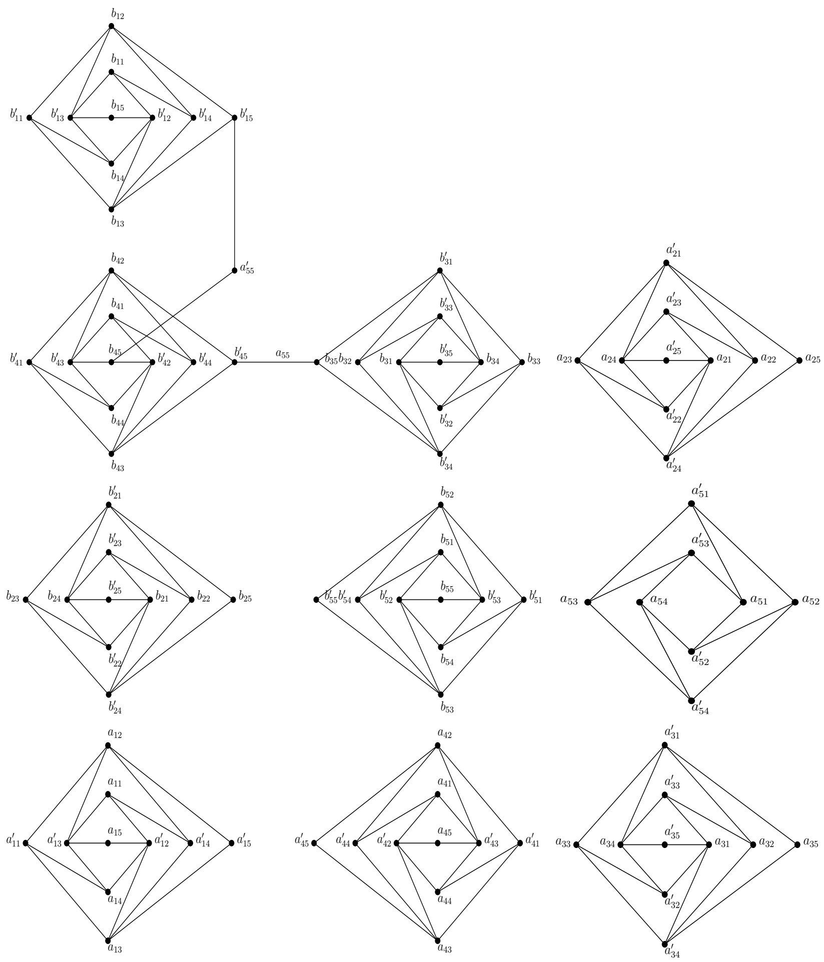
C The Graph B_3



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