INDUCTION OF CHARACTERS AND FINITE p-GROUPS

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Abstract. Let $G$ be a finite $p$-group, where $p$ is an odd prime number, $H$ a subgroup of $G$ and $\theta \in \text{Irr}(H)$ an irreducible character of $H$. Assume also that $|G : H| = p^2$. Then the character $\theta^G$ of $G$ induced by $\theta$ is either a multiple of an irreducible character of $G$, or has at least $\frac{p + 1}{2}$ distinct irreducible constituents.

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1. Introduction. Let $G$ be a finite group. Denote by $\text{Irr}(G)$ the set of irreducible complex characters of $G$. Throughout this work, we use the notation of [2]. In addition, we are going to denote by $\text{Lin}(G) = \{ \lambda \in \text{Irr}(G) \mid \lambda(1) = 1 \}$ the set of linear characters.

Let $\Gamma$ be a character of $G$. Then $\Gamma$ can be expressed as a nontrivial integral linear combination of distinct irreducible characters of $G$. Denote by $\eta(\Gamma)$ the number of distinct irreducible constituents of $\Gamma$.

Let $G$ be a finite $p$-group, where $p$ is a prime number, $H$ be a subgroup of $G$ and $\theta \in \text{Irr}(H)$. Denote by $\theta^G$ the character of $G$ induced by $\theta$. If $H$ is a normal subgroup, then either $\eta(\theta^G) = 1$, i.e. $\theta^G$ is a multiple of an irreducible, or $\eta(\theta^G) \geq p$, i.e. $\theta^G$ is an integral linear combination of at least $p$ distinct irreducible characters of $G$ (see Lemma 2.2). In Theorem 4.15, it is shown that given any prime $p > 2$ and any integer $l \geq 2$, there exist a $p$-group $G$, a subgroup $H$ of $G$ with $|G : H| = p^l$ and $\theta \in \text{Irr}(H)$ such that $\eta(\theta^G) = \frac{p + 1}{2}$. Therefore Lemma 2.2 does not remain true without the hypothesis that $H$ is normal in $G$. But given any prime $p > 2$ and any integer $n > 0$, do there exist a $p$-group $G$, a subgroup $H$ of $G$ and $\theta \in \text{Irr}(H)$ with $\eta(\theta^G) = n$? If we also required, in addition, that $|G : H| = p^2$ and $1 < n < \frac{p + 1}{2}$, then the answer is no. More specifically:

**Theorem A.** Let $G$ be a finite $p$-group, where $p$ is an odd prime number, $H$ be a subgroup of $G$ and $\theta \in \text{Irr}(H)$. Assume also that $|G : H| = p^2$. Then either $\eta(\theta^G) = 1$ or $\eta(\theta^G) \geq \frac{p + 1}{2}$.

For a fixed prime $p > 3$, Theorem A implies that there exists a “gap” among the possible values that $\eta(\theta^G)$ can take for any finite $p$-group $G$, any subgroup $H$ of $G$ with $|G : H| = p^2$, and any character $\theta \in \text{Irr}(H)$. But, do there exist a $p$-group $G$, a subgroup $H$ of $G$ and $\theta \in \text{Irr}(H)$ with $1 < \eta(\theta^G) < \frac{p + 1}{2}$ and $|G : H| > p^2$? The answer is yes. In Theorem 4.23, given any prime $p$ such that 3 divides $p - 1$, we provide a $p$-group $G$, a subgroup $H$ of $G$ with $|G : H| = p^3$ and a character $\lambda \in \text{Lin}(H)$ such that $\eta(\lambda^G) = \frac{p + 2}{2}$. Does it mean then that, for a fixed prime $p > 5$, there are no “gaps” among the possible values that $\eta(\theta^G)$ can take for any finite $p$-group $G$, any subgroup $H$ of $G$ with $|G : H| = p^3$, and any character $\theta \in \text{Irr}(H)$? We do not know the answer of that question.
2. Preliminaries.

**Lemma 2.1.** Let $G$ be a finite group, $N$ be a normal subgroup of $G$ and $\theta \in \text{Irr}(N)$. Let $G_{\theta}$ be the stabilizer of $\theta$ in $G$. Then $\eta(\theta^G) = \eta(\theta^{G_{\theta}})$.

*Proof.* Observe that all the irreducible constituents of $\theta^G$ lie above $\theta$. Thus by Clifford theory it follows that $\eta(\theta^G) = \eta(\theta^{G_{\theta}})$. \hfill \Box

**Lemma 2.2.** Let $G$ be a finite $p$-group, $H$ be a normal subgroup of $G$ and $\theta \in \text{Irr}(H)$. Then either $\eta(\theta^G) = 1$ or $\eta(\theta^G) \geq p$.

*Proof.* In [1, Lemma 4.1], it is proved that, if in addition to the previous hypothesis, $\theta$ is $G$-invariant, then $\eta(\theta^G) = 1$ or $\eta(\theta^G) \geq p$. Thus by induction on $|G : H|$ and Lemma 2.1, the result follows. \hfill \Box

Let $G$ be a group, $H$ be a subgroup of $G$ and $\theta \in \text{Irr}(H)$. Denote by $\text{Irr}(G \mid \theta) = \{\chi \in \text{Irr}(G) \mid [\chi_H, \theta] \neq 0\}$ the set of irreducible characters of $G$ lying above $\theta$.

**Lemma 2.3.** Let $G$ be a finite $p$-group, $H$ be a subgroup of $G$ and $\theta \in \text{Irr}(H)$. Let $Z_1$ be a subgroup of the center $Z(G)$ of $G$ such that $|HZ_1 : H| = p$. Then $\theta$ extends to $HZ_1$ and

$$\eta(\theta^G) = \sum_{\nu \in \text{Irr}(HZ_1 \mid \theta)} \eta(\nu^G).$$

In particular, if $\nu \in \text{Irr}(HZ_1 \mid \theta)$ we have that

$$\eta(\theta^G) \geq \eta(\nu^G) + (p - 1). \quad (2.4)$$

*Proof.* Observe that $\theta$ extends to $HZ_1$ since $Z_1 \leq Z(G)$ and $|HZ_1 : H| = p$. Thus there are exactly $p$ characters in $\text{Irr}(HZ_1 \mid \theta)$. Let $\alpha \in \text{Lin}(H \cap Z_1)$ be the unique character such that $\theta_H^Z_1 = \theta(1)\alpha$. Since $(\theta^H)_Z_1 = (\theta_H^Z_1)^{Z_1}$, we have that $(\theta^H)_Z_1 = \theta(1)\sum_{\nu \in \text{Lin}(Z_1 \mid \theta)} \nu$. Therefore

$$\text{for any } \nu, \mu \in \text{Irr}(HZ_1 \mid \theta), \text{ if } \nu \neq \mu \text{ then } \nu Z_1 \neq \mu Z_1. \quad (2.5)$$

Observe that for any $\chi \in \text{Irr}(G)$ and any $\beta \in \text{Lin}(Z_1)$, if $[\chi_Z_1, \beta] \neq 0$ then $\chi_Z_1 = \chi(1)\beta$. By (2.5), it follows that if $\chi, \psi \in \text{Irr}(G), \nu, \mu \in \text{Irr}(HZ_1 \mid \theta), \nu \neq \mu, [\chi Z_1, \nu] \neq 0$ and $[\psi Z_1, \mu] \neq 0$, then $\chi \neq \psi$. Thus the irreducible constituents of $\theta^G$ lying over distinct extensions of $\theta$ in $HZ_1$ are distinct characters. It follows that

$$\eta(\theta^G) = \sum_{\nu \in \text{Irr}(HZ_1 \mid \theta)} \eta(\nu^G).$$

Since $\eta(\nu^G) \geq 1$ for any $\nu \in \text{Irr}(HZ_1)$, (2.4) follows. \hfill \Box

3. Proof of Theorem A. Let $G$ and $\theta \in \text{Irr}(H)$ be a minimal counterexample of the statement of Theorem A with respect to the order $|G|$ of $G$. That is we are assuming that

$$|G : H| = p^2, \quad 1 < \eta(\theta^G) < \frac{p + 1}{2} \quad (3.1)$$
and for any finite $p$-group $G_1$, any subgroup $H_1$ of $G_1$, and any $\theta_1 \in \text{Irr}(H_1)$, if

$$|G_1 : H_1| = p^2 \text{ and } |G_1| < |G| \text{ then either } \eta(\theta_1^{G_1}) = 1 \text{ or } \eta(\theta_1^{G_1}) \geq \frac{p+1}{2}.$$  \hspace{1cm} (3.2)

Set $\overline{L} = L/\text{core}_G(\text{Ker}(\theta))$ for any subgroup $L$ of $G$ such that $L \geq \text{core}_G(\text{Ker}(\theta))$. Observe that $H \geq \text{core}_G(\text{Ker}(\theta))$ and $|\overline{G} : \overline{H}| = |G : H|$. Observe also that we can regard $\theta$ as a character of $H/\text{core}_G(\text{Ker}(\theta))$ and $\eta(\theta^{\overline{G}}) = \eta(\theta^{\overline{G}})$.

By working with the group $G/\text{core}_G(\text{Ker}(\theta))$ and (3.2), we may assume that

$$\text{core}_G(\text{Ker}(\theta)) = 1.$$  

Thus $\overline{L} = L$ for all subgroups $L$ of $G$.

Denote by $Z$ the center $Z(G)$ of $G$.

**CLAIM 3.3.** $Z < H$. Let $\nu \in \text{Lin}(Z)$ be the unique character of $Z$ lying below $\theta$. Then $\nu \in \text{Lin}(Z)$ is a faithful character of $Z$ and $Z$ is a cyclic group.

**Proof.** Suppose $Z$ is not contained in $H$. Let $Z_1 \leq Z$ be such that $|HZ_1 : H| = p$. Lemma 2.3 implies that $\eta(\theta^Z) \geq p$, a contradiction with (3.1). Thus $Z \leq H$. Since $Z = H$ implies that $H$ is normal, by Lemma 2.2 we must have that $Z < H$.

Since $\text{Ker}(\theta) \cap Z$ is normal in $G$ and $\text{core}_G(\text{Ker}(\theta)) = 1$, it follows that $\theta_Z$ is a faithful character of $Z$. Therefore $\nu \in \text{Lin}(Z)$ is faithful and $Z$ is cyclic. \hfill $\square$

**CLAIM 3.4.** $\text{core}_G(H) = Z$.

**Proof.** Assume that there exists a normal subgroup $N$ of $G$ such that $N \leq H$ and $N/Z$ is a chief factor of $G$. Fix $\beta \in \text{Irr}(N)$ such that $[\theta_N, \beta] \neq 0$. Since $\nu \in \text{Lin}(Z)$ is a faithful character, we can check that $C_G(N)$ is a normal subgroup of $G$ of index $p$. Also the stabilizer $G_\beta$ of $\beta$ in $G$ is $C_G(N)$.

If $H \cap C_G(N) < H$, by Clifford theory we have that there exists some $\alpha \in \text{Irr}(H \cap C_G(N))$ such that $\alpha^H = \theta$. Thus $\eta(\theta^G) = \eta(\alpha^G)$. Since $|C_G(N)| < |G|$ and $|C_G(N) : H \cap C_G(N)| = p^2$, by (3.2) we have that $\eta(\alpha^{C_G(N)}) = 1$ or $\eta(\alpha^{C_G(N)}) \geq \frac{p+1}{2}$. By Lemma 2.1 we then have that $\eta(\alpha^G) = 1$ or $\eta(\alpha^G) \geq \frac{p+1}{2}$ and therefore $\eta(\theta^G) = 1$ or $\eta(\theta^G) \geq \frac{p+1}{2}$, a contradiction with (3.1). We may assume then that $H < C_G(N)$.

Since $|C_G(N) : H| = p$, $H$ is normal in $C_G(N)$ and thus by Lemma 2.2 we have that either $\eta(\theta^{C_G(N)}) = 1$ or $\eta(\theta^{C_G(N)}) = p$. By Lemma 2.1 and the previous statement, we have that $\eta(\theta^G) = 1$ or $\eta(\theta^G) \geq p$, a contradiction with (3.1). Thus such $N$ cannot exist and so $\text{core}_G(H) = Z$. \hfill $\square$

Let $Y/Z$ be a chief factor of $G$. By the previous claim, it follows that $HY > H$.

Since $Y/Z$ has order $p$, we have that $|HY : H| = p$. Since $|G : H| = p^2$, it follows that $|G : HY| = p$ and thus $HY$ is a normal subgroup of $G$.

Set $C = C_G(Y)$.

**CLAIM 3.5.** $|G : C| = p$. Also, given any $\mu \in \text{Lin}(Y)$ which is an extension of the faithful character $\nu \in \text{Lin}(Z)$, we have that the stabilizer $G_\mu$ of $\mu$ in $G$ is $C$.

**Proof.** Since $\nu \in \text{Lin}(Z)$ is a faithful character of the center $Z$ of $G$ and $Y/Z$ is a chief factor of the $p$-group $G$, it follows that the index of the centralizer $C$ of $Y$ in $G$ is $p$.

**Claim 3.6.** $HY/Z$ is an elementary abelian $p$-group. Also, we may assume that $Z(HY) \geq Y$ and thus $C = HY$. \hfill $\square$
Proof. Since $|HY:H|=p$, we have that $(HY) = ([h,k] | h,k \in HY) \leq H$. Observe that $(HY)$ is normal in $G$ since $HY$ is normal in $G$ and $(HY)$ is a characteristic subgroup of $HY$. Since $\text{core}_G(H) = Z$, it follows then that $(HY)' \leq Z$. Also, since $Y/Z$ is of order $p$ and $Z < H$, $(HY)^p = \langle k^p | k \in HY \rangle$ is a characteristic subgroup of the normal subgroup $HY$ of $G$ and it is contained in $H$. It follows then that $(HY)^p \leq Z$ and thus $HY/Z$ is an elementary abelian $p$-group.

Observe that the center $Z(HY)$ of $HY$ contains $Z$. If $Z(HY) = Z$, then there is a unique character in $\text{Irr}(H)$ lying above $v$ since $HY/Z$ is an elementary abelian $p$-group and $v \in \text{Lin}(Z)$ is a faithful character, and so $\eta(\theta^G) = 1$ or $\eta(\theta^G) = p$, that is a contradiction with (3.1) and therefore it must follow that $Z(HY) > Z$. By replacing $Y$ for a normal subgroup of $G$ contained in $Z(HY)$ if necessary, we may assume then that $Y \leq Z(HY)$ and thus $C_G(Y) = HY$.

CLAIM 3.7. The character $\theta \in \text{Irr}(H)$ extends to $HY = C$. Thus $\theta^C$ is the sum of the $p$ distinct extensions of $\theta$.

Proof. Since $|HY:H|=p$, we have that either $\theta^H \in \text{Irr}(HY)$ or $\theta^HY$ is the sum of the $p$ distinct extensions of $\theta$.

Suppose that $\theta^C \in \text{Irr}(C)$. Let $\mu \in \text{Lin}(Y)$ be the unique character of $Y$ such that $[(\theta^HY)_Y, \mu] \neq 0$. Since $G_{\mu} = C$, then $\theta^G \in \text{Irr}(G)$. Thus $\theta^HY$ is the sum of the $p$ distinct extensions of $\theta$.

Let $\rho_1, \ldots, \rho_p \in \text{Irr}(HY)$ be the $p$ distinct extensions of $\theta$. Since $|G:HY| = p$, by Lemma 2.2 we must have that

$$\rho_i^G \in \text{Irr}(G). \quad (3.8)$$

Since $Z(C) \geq Y$, there is a unique character $\mu_i \in \text{Lin}(Y)$ lying below $\rho_i$.

CLAIM 3.9. $Z(C) = Y$.

Proof. Clearly $Y \leq Z(C)$. Assume that $Y < Z(C)$. Let $X \leq Z(C)$ such that $X/Y$ is a chief factor of $G$ and $Y < X \leq HY = C$. Observe that such $X$ exists since $HY$ is normal in $G$, and $X$ is abelian since $X \leq Z(C)$. We are going to conclude that $v \in \text{Lin}(Z)$ is not a faithful character, which is a contradiction with Claim 3.3.


Proof. Since $Y$ and $X$ are normal subgroups of $G$ with $Y \triangleleft X$ and $|X/Y| = p$, the chief factor $X/Y$ of the $p$-group $G$ is centralized by $G$. So $[X, G] \leq Y$. Suppose that $[X, G]Z < Y$. Since $|Y/Z| = p$, we must have $[X, G] \leq Z = Z(G)$. So commutation in $G$ induces a bilinear map

$$d : (xZ, gC_G(X)) \mapsto [x, g]$$

of $X/Z \times G/C_G(X)$ into the cyclic group $Z$. This map $d$ is non-singular on the right since $[X, g] = 1$ if and only if $g \in C_G(X)$. It is non-singular on the left since $[x, G] = 1$ if and only if $x \in Z$. Because $|X : Z| = p^m$ and $d$ is a non-singular bilinear form of $X/Z \times G/C_G(X)$ into the cyclic group $Z$, we have $|G : C_G(X)| = p^2$. Since $\lambda \in \text{Lin}(X | v)$ extends the faithful character $v \in \text{Irr}(Z)$, this implies that $C_G(X) = G_\lambda$. Thus $|G : G_\lambda| = p^2$. Since $X \leq Z(C)$, $C$ fixes $\lambda$. But then $|G : C| = p$, $C \leq G_\lambda$ and $|G : G_\lambda| = p^2$. This contradiction proves the claim.
Given any character $\rho \in \text{Irr}(C)$, since $X \leq Z(C)$, we have that $\frac{1}{\rho|_{X}}\rho_{X} \in \text{Lin}(X)$ is the unique character lying below $\rho$.

**Step 3.11.** There exist some $\lambda \in \text{Lin}(X)$, some $g \in G\backslash C$ and $i \in \{2, \ldots, p - 1\}$ such that $[(\theta^{C})_{X}, \lambda] \neq 0$, $[(\theta^{C})_{X}, \lambda^{g}] \neq 0$ and $[(\theta^{C})_{X}, \lambda^{g^{i}}] \neq 0$.

**Proof.** Since $1 < \eta(\theta^{G}) < \frac{p - 1}{2}$ and $\rho_{1}^{G}, \ldots, \rho_{p}^{G}$ are the irreducible constituents of $\theta^{G}$, there exist at least 3 distinct $j, k, l \in \{1, 2, \ldots, p\}$ such that $\rho_{j}^{G} = \rho_{k}^{G} = \rho_{l}^{G}$. Since $X$ is normal in $G$, by Clifford Theory it follows that $\frac{1}{\rho|_{\langle \rho \rangle}}(\rho)_{X}$ and $\frac{1}{\rho|_{\langle \rho \rangle}}(\rho)_{Y}$ are $G$-conjugates. Set $\lambda = \frac{1}{\rho|_{\langle \rho \rangle}}(\rho)_{X}$. Then there exists some $g \in G\backslash C$ such that $\lambda^{g} = \frac{1}{\rho|_{\langle \rho \rangle}}(\rho)_{Y}$. Since $X \leq Z(C)$ and $|G : C| = p$, there exists some $i \in \{2, \ldots, p - 1\}$ such that $(\lambda^{g^{i}}) = \frac{1}{\rho|_{\langle \rho \rangle}}(\rho)_{Y}$.

Fix $g \in G\backslash C$ as in 3.11. Since $X / Y$ is cyclic of order $p$, $H \cap X > Z$, and $H \cap Y = Z$ we may choose $x \in H$ such that $X = \langle x, Y \rangle$. (3.12)

Since $X \leq Z(C)$, we have $[X, C] = 1$. Suppose that $[x, g^{-1}] \in Z$. Then $x$ centralizes both $g^{-1}$ and $C$ modulo $Z$. Hence $xZ \in Z(G / Z)$, which is false by Step 3.10. Hence $[x, g^{-1}] \in Y \backslash Z$ and so $Y = Z \langle y \rangle$ is generated over $Z$ by $y = [x, g^{-1}]$. (3.13)

Since $[Y, G] \leq Z$ we have that $z = [y, g^{-1}] \in Z$. If $z = 1$, then $G = C \langle g \rangle$ centralizes $Y = Z \langle y \rangle$, since $C$ centralizes $Y < X$ because $X \leq Z(C)$, and $G$ centralizes $Z$. This is impossible because $Z = Z(G) < Y$. Thus $z = [y, g^{-1}]$ is a non-trivial element of $Z$. (3.14)

By (3.13) we have $y = [x, g^{-1}] = x^{-1} \theta^{g^{-1}}$. Finally $z^{g^{-1}} = z$ since $z \in Z$. Since $X = Z \langle x, y \rangle$ is abelian since $X \leq Z(C)$, it follows that $z^{g^{-1}} = z$, $y^{g^{-1}} = yz^{i}$ and $x^{g^{-1}} = xy^{i}z^{(j)}$, (3.15)

for any integer $j = 0, 1, \ldots, p - 1$. Because $g^{-p} \in C$ centralizes $X$ since $X \leq Z(C)$, we have $z^{p} = 1$ and $y^{p}z^{(j)} = 1$. (3.16)

Since $p > 2$ is odd by hypothesis, $p$ divides $(\frac{p}{2}) = \frac{p(p - 1)}{2}$ and $z^{(j)} = 1$. Therefore $y^{p} = z^{p} = 1$. It follows that $y^{i}$, $z^{i}$ and $z^{(j)}$ depend only on the residue of $i$ modulo $p$, for any integer $i \geq 0$. such that $X = Y \langle x \rangle$ and $x \in C$. Thus by (3.14) we have that $z^{(j)} \neq 1$ for any integer $0 < j < p$.

Let $\lambda \in \text{Lin}(X)$ and $i \in \{2, \ldots, p - 1\}$ be as in Step 3.11. Set $\sigma = \frac{1}{\theta|_{\langle \theta \rangle}}(\theta_{X \cap H})$. We can check that $\sigma \in \text{Lin}(X \cap H)$. Since $(\theta^{C})_{X} = (\theta_{H \cap X})^{X}$, we have that $\lambda$, $\lambda^{g}$ and $\lambda^{g^{i}}$ are...
extensions of \( \sigma \). Since \( x \in (H \cap X) \), by the previous statement we have that
\[
\lambda(x) = \lambda^g(x) = \lambda^{\sigma}(x).
\]  
(3.17)

By (3.15) we have that
\[
\lambda^g(x) = \lambda(x^{g^{-1}}) = \lambda(xy) = \lambda(x)\lambda(y).
\]
Thus by (3.17), we get
\[
\lambda(y) = 1.
\]  
(3.18)

Therefore
\[
\lambda^g(x) = \lambda(x^{g^{-1}})
= \lambda(xy) = \lambda(x)\lambda(y) = \lambda(x)\lambda(z^{(\lambda)})
= \lambda(x)\lambda(z^{(\lambda)}),
\]
where the last line follows from (3.18). By (3.17), we have that \( \lambda(z^{(\lambda)}) = 1 \). But \( \lambda_Z = v \in \text{Lin}(Z) \) is a faithful character and \( z^{(\lambda)} \neq 1 \) by (3.16). This is a contradiction and the claim is proved.

Since \( Z(HY) = Y \), we have that \( Z(H) = Z \). Thus \( HY \) is a class 2 group with \( HY/Z \) elementary abelian. Therefore \( \theta \in \text{Irr}(H) \) is the only character in \( H \) lying above \( v \in \text{Lin}(Z) \). Hence an irreducible character of \( G \) lies over \( \theta \) if and only if it lies over \( v \). Since \( \text{Irr}(G | v) \) has either 1 element or at least \( p \) by Lemma 2.2, it follows that \( \eta(v^G) = 1 \) or \( \eta(v^G) \geq p \), and therefore either \( \eta(\theta^G) = 1 \) or \( \eta(\theta^G) \geq p \). But \( 1 < \eta(\theta^G) < \frac{p+1}{2} \), and that is our final contradiction and thus the statement of Theorem A holds.

4. Examples. In this section, we will prove that the group \( G \), the subgroup \( H \) and the character \( \lambda \in \text{Lin}(H) \) that satisfy Hypothesis 4.1 have the properties that \( |G : H| = p^2 \) and \( \eta(\lambda^G) = \frac{p+1}{2} \). And then, given any integer \( n \geq 2 \), we construct a group \( G \) with a subgroup \( H \) and a character \( \lambda \in \text{Lin}(H) \) such that \( |G : H| = p^n \) and \( \eta(\lambda^G) = \frac{p^n+1}{2} \).

**Hypothesis 4.1.** Fix an odd prime \( p \). Let \( G \) be the semidirect product of a cyclic group \( C \) of order \( p \) and an elementary abelian group \( A \) of order \( p^2 \). Assume \( C = \langle a \rangle \) and
\[
A = \langle a \rangle \times \langle [a, c] \rangle \times \langle [a, c, c] \rangle,
\]  
(4.2)

for some \( a \) in \( A \). Observe that the subgroup \( \{e\} \times \{e\} \times \langle [a, c, c] \rangle \) is the center of the group \( G \). Set \( Z = \{e\} \times \{e\} \times \langle [a, c, c] \rangle \).

Fix \( \omega \) a primitive complex \( p \)-th root of unity. Let \( \alpha \in \text{Lin}([a]) \), \( \beta \in \text{Lin}([a, c]) \) and \( \gamma \in \text{Lin}([a, c, c]) \) be the unique linear characters such that \( \alpha(a) = \beta([a, c]) = \gamma([a, c, c]) = \omega \).

Set
\[
H = \langle a \rangle \times \{e\} \times \langle [a, c, c] \rangle \text{ and } \lambda = 1_{[a]} \times 1_{[c]} \times \gamma \in \text{Lin}(H).
\]  
(4.3)
Observe that $H$ is a subgroup of $A$ of index $p$. Thus $|G : H| = p^2$. Observe also that $\lambda$ extends to $A$ and there are exactly $p$ distinct extensions of $\lambda$ to $A$; namely

$$\operatorname{Irr}(A \mid \lambda) = \{1_{(a)} \times \beta^r \times \gamma \mid r = 0, 1, \ldots, p - 1\}. \quad (4.4)$$

Set $\Lambda_r = 1_{(a)} \times \beta^r \times \gamma$.

**Lemma 4.5.** Assume Hypothesis 4.1. Given any integer $i$ with $0 < i$, we have that

$$(\Lambda_r)^i = a^{r + \frac{ik-1}{2}} \times \beta^{r+i} \times \gamma.$$

**Proof.** Observe that $(\Lambda_r)^i = a^{r} \times \beta^{i} \times \gamma = a^{r} \times \beta^{r+1} \times \gamma$ since $a^{r} = a[a, c]$ and $[a, c] = [a, c][a, c]$. Assume by induction that

$$(\Lambda_r)^{r+1} = a^{r+1} \times \beta^{r+1} \times \gamma. \quad (4.6)$$

Then

$$(\Lambda_r)^{r+1} = (\Lambda_r)^{i} = \frac{a^{r+1} \times \beta^{r+i} \times \gamma}{a^{r+1} \times \beta^{r+1} \times \gamma} = a^{r+1} \times \beta^{r+1} \times \gamma,$$

where the last line follows since $a^{r} = a[a, c]$ and $[a, c] = [a, c][a, c]$. We can check that $rn + \frac{n(n-1)}{2} + r + n = r(n + 1) + \frac{(n+1)n}{2}$. Thus

$$(\Lambda_r)^{r+1} = a^{(r+1)+\frac{n+1}{2}} \times \beta^{r+(n+1)} \times \gamma,$$

and the result follows by induction. $\square$

**Lemma 4.7.** Assume Hypothesis 4.1. Let $r$ be an integer such that $0 < r < p$. Then $(\Lambda_r)^i$ is an extension of $\lambda$ if and only if either $j \equiv 0 \mod p$ or $j \equiv (1-2r) \mod p$. If $i \equiv (1-2r) \mod p$ then $(\Lambda_r)^i = \Lambda_{1-r}$.

**Proof.** By Lemma 4.5, we have that $(\Lambda_r)^i$ is an extension of $\lambda$ if and only if $a^{r+\frac{it-1}{2}} = 1_{(a)}$. Since $a$ is a faithful linear character of a cyclic group of order $p$, $a^{r+\frac{it-1}{2}} = 1_{(a)}$ if and only if $(ir + \frac{i(t-1)}{2}) \equiv 0 \mod p$. Observe that $(ir + \frac{i(t-1)}{2}) \equiv 0 \mod p$ if and only if either $i \equiv 0 \mod p$ or $r + \frac{t-1}{2} \equiv 0 \mod p$. Therefore $(\Lambda_r)^i$ is an extension of $\lambda$ if and only if either $i \equiv 0 \mod p$ or $i \equiv (1-2r) \mod p$.

If $i \equiv (1-2r) \mod p$, then $(\Lambda_r)^i = \Lambda_{1-r}$ by Lemma 4.5. $\square$

**Lemma 4.8.** Assume Hypothesis 4.1. Then $1 < \eta(\lambda^G) \leq \frac{p+1}{2}$.

**Proof.** By the previous lemma, it follows that the stabilizer of $\Lambda_r$ is a proper subgroup of $G$. Since $|G : A| = p$ and $\Lambda_r \in \operatorname{Lin}(A)$, we have that

$$(\Lambda_r)^G \in \operatorname{Irr}(G) \text{ for any integer } r. \quad (4.9)$$

Since $p > 2$, it follows that there exist two distinct integers $k, l$ such that $0 < k, l < p$ and $k \neq (1-2l) \mod p$. Thus by Lemma 4.7 we have that $\Lambda_k$ and $\Lambda_l$ are not
Assume that core \(G\) and \(\Lambda_k\) lie above \(\lambda\), we have that \(\eta(G) \geq 2\).

Observe that \(r \equiv (1 - r) \mod p\) if and only if \(2r \equiv 1 \mod p\). Thus given any \(r\) such that \(0 < r < p\) and \(2r \equiv 1 \mod p\), by Lemma 4.7 we have that \(\Lambda, \Lambda_{1-r} \in \text{Irr}(A)\) are two distinct \(G\)-conjugate extensions of \(\lambda\). Thus \(\eta(G) \leq \frac{p+1}{2}\).

**Proposition 4.10.** Assume Hypothesis 4.1. Then \(|G : H| = p^2\) and \(\eta(G) = \frac{p+1}{2}\).

**Proof.** By Lemma 4.8, we have that \(1 < \eta(G) \leq \frac{p+1}{2}\). Thus by Theorem A, it follows that \(\eta(G) = \frac{p+1}{2}\).

Denote by \(1_H\) the principal character of \(H\).

**Lemma 4.11.** Let \(p\) be a prime number, \(G\) be a \(p\)-group and \(H\) be a subgroup of \(G\) with \(|G : H| = p^n\). Then \(\eta((1_H^G)^G) \geq n(p - 1) + 1\).

**Proof.** We are going to use a double induction, first on \(|G|\) and then on \(n\), where \(|G : H| = p^n\). Using induction on the order of \(G\), without lost of generality we may assume that \(\text{core}_G(H) = 1\).

Let \(Z_1\) be a subgroup of the center \(Z(G)\) of \(G\) with \(|Z_1| = p\). Observe that \(H \cap Z_1 = 1\) since \(\text{core}_G(H) = 1\). Thus \(|HZ_1 : H| = p\). By Lemma 2.3, we have that

\[
\eta((1_H^G)^G) \geq \eta((1_{HZ_1}^G)^G) + (p - 1).
\]

Since \(|G : HZ_1| = p^{n-1}\), by induction on \(n\) we have that

\[
\eta((1_{HZ_1}^G)^G) \geq (n - 1)(p - 1) + 1.
\]

The result follows by (4.12) and the previous statement. \(\square\)

**Lemma 4.13.** Let \(G_0\) be a \(p\)-group and \(\Gamma\) be a character of \(G_0\). Assume that \([\Gamma, 1_{G_0}] = 0\). Let \(N = G_0 \times G_0 \times \cdots \times G_0\) be the direct product of \(p\)-copies of \(G_0\). Set

\[
\Delta = \Gamma \times 1_{G_0} \times \cdots \times 1_{G_0}.
\]

Let \(C = \langle c \rangle\) be a cyclic group of order \(p\). Observe that \(C\) acts on \(N\) by

\[
c : (n_0, n_1, \ldots, n_{p-1}) \mapsto (n_{p-1}, n_0, \ldots, n_{p-2})
\]

for any \((n_0, n_1, \ldots, n_{p-1}) \in N\).

Let \(G\) be the direct product of \(N\) and \(C\), i.e. \(G\) is the wreath product of \(G_0\) and \(C\). Then \(\eta(\Delta^G) = \eta(\Gamma)\).

**Proof.** Let \(\delta \in \text{Irr}(N)\) be a constituent of \(\Delta\). Observe that \(\delta\) is of the form \(\gamma \times 1_{G_0} \times \cdots \times 1_{G_0}\), for some \(\gamma \in \text{Irr}(G_0)\) such that \([\gamma, \Gamma] \neq 0\). Observe that \(\gamma \neq 1_{G_0}\) since \([\Gamma, 1_{G_0}] = 0\). By (4.14), we have that \(\delta\) is \(G\)-invariant if and only if \(\gamma = 1_{G_0}\). Thus \(\delta^G \in \text{Irr}(G)\) for any constituent \(\delta \in \text{Irr}(N)\) of \(\Delta\). Observe that the \(G\)-orbit of \(\delta \in \text{Irr}(N)\) is

\[
\{ \gamma \times 1_{G_0} \times \cdots \times 1_{G_0}, 1_{G_0} \times \gamma \times \cdots \times 1_{G_0}, \cdots, 1_{G_0} \times \cdots \times 1_{G_0} \times \gamma \}.
\]
Thus if $\delta, \epsilon \in \text{Irr}(N)$ are two distinct constituents of $\Delta$, then $\delta^G \neq \epsilon^G$. It follows that $\eta(\Delta^G) = \eta(\Gamma)$. \hfill $\square$

**Theorem 4.15.** Let $p$ be an odd prime number and $n \geq 2$ be an integer. There exist a $p$-group $G$, a subgroup $H$ of $G$ and $\lambda \in \text{Lin}(H)$, such that $|G : H| = p^n$ and $\eta(\lambda^G) = \frac{p+1}{2}$.

**Proof.** If $n = 2$, then the result follows by Lemma 4.10. By induction on $n$, we may assume that the result holds for any integer $n$ such that $n-1 \geq 2$.

Fix a $p$-group $G_0$, a subgroup $H_0 \leq G_0$ and $\lambda_0 \in \text{Lin}(H_0)$ such that:

$$|G_0 : H_0| = p^{n-1} \quad \text{and} \quad \eta(\lambda_0^{G_0}) = \frac{p+1}{2}. \quad (4.16)$$

Let $N$ and $G$ be as in Lemma 4.13. Let

$$H = H_0 \times G_0 \times \ldots \times G_0.$$ 

Then $H$ is a subgroup of $N$ and $|G : H| = |G : N||N : H_0| = p|G_0 : H_0| = p^n$.

Set $\lambda = \lambda_0 \times 1_{G_0} \times \ldots \times 1_{G_0}$. Observe that $\lambda \in \text{Lin}(H)$ since $\lambda_0 \in \text{Lin}(H_0)$. We can check that $\eta(\lambda^N) = \eta(\lambda_0^{G_0})$. Thus by (4.16) we have that $\eta(\lambda^N) = \frac{p+1}{2}$.

By Lemma 4.11, we have that $\lambda_0 \neq 1_{H_0}$. Thus $[\lambda_0^{G_0}, 1_{G_0}] = 0$. By Lemma 4.13 we have then that $\eta(\lambda^N) = \eta(\lambda^G)$ and the result is proved. \hfill $\square$

**Lemma 4.17.** Let $p$ be a prime number such that $p - 1$ is divisible by $3$. Fix $r \in \{1, \ldots, p - 1\}$. Then the set \(\{r(1 - i^3) \mod p \mid i = 0, \ldots, p - 1\}\) has $\frac{p+2}{3}$ elements. Also, given any $e \in \{r(1 - i^3) \mod p \mid i = 1, \ldots, p - 1\}$, there are exactly $3$ distinct solutions in $\{1, \ldots, p - 1\}$ of the equation $e \equiv r(1 - x^3) \mod p$.

**Proof.** Let $u$ be a generator of the units of the field $F$ of $p$ elements. Then $U = \langle u^{\frac{r-1}{3}} \rangle$ is a subgroup of order $3$ and any element in $U$ is a solution of $x^3 \equiv 1 \mod p$. Thus given any integer $n \neq r$, if the equation $x^3 \equiv r - n \mod p$ has a solution, then it has exactly $3$ distinct solutions in $F$. Therefore the set \(\{r(1 - i^3) \mod p \mid i = 1, \ldots, p - 1\}\) has $\frac{p-1}{3}$ distinct elements. Since $0^3 = 0$, the set \(\{r(1 - i^3) \mod p \mid i = 0, \ldots, p - 1\}\) has $\frac{p-1}{3} + 1 = \frac{p+2}{3}$ elements. \hfill $\square$

**Hypothesis 4.18.** Let $p > 3$ be a prime number such that $p - 1$ is divisible by $3$. Let $F$ be a field of $p$ elements and $F[x]$ be the truncated polynomial algebra generated over $F$ by some $x$ satisfying only $x^4 = 0$. So $F[x]$ is a vector space of dimension $4$ over $F$ with $1, x, x^2$ and $x^3$ as a basis. Let $m$ be an isomorphism of the additive group $F[x]^+$ of $F[x]$ onto a multiplicative group $M$. Then $M$ is an elementary abelian multiplicative group of order $p^4$ with $m(1), m(x), m(x^2), m(x^3)$ as generators. Let $U$ be the subgroup of the unit group $F[x]^+$ generated by $1 + x$ and $1 + x^2$. The general element of $U$ is

$$(1 + x)^i(1 + x^2)^j = 1 + ix + \left(\binom{i}{2} + j\right)x^2 + \left(\binom{i}{3} + ij\right)x^3 \quad (4.19)$$

for arbitrary integers $i, j$, since $x^4 = 0$. Because $p > 3$, it follows that $U$ is elementary abelian of order $p^2$, and that (4.19) holds for any $i, j \in F$. The group $U$ acts naturally on the group $M$, so that

$$m(y)^u = m(yu) \quad (4.20)$$
for all \( y \in F[x] \) and \( u \in U \). Let \( G \) be the semidirect product of \( M \) and \( U \). Then \( G \) is a multiplicative group with order \( p^6 \).

Let \( H \) be the subgroup

\[
H = \langle m(1), m(x), m(x^3) \rangle = \{ m(a_0 + a_1 x + a_3 x^3) \mid a_0, a_1, a_3 \in F \}. \tag{4.21}
\]

Fix a primitive \( p \)-th root of unity \( \omega \). Fix an integer \( r > 0 \) such that \( 3r \equiv -1 \mod p \). Thus \( r \equiv \frac{-1}{3} \mod p \) and \( r \not\equiv 0 \mod p \). Let \( \lambda \in \text{Lin}(H) \) be the character given by

\[
\lambda( m(a_0 + a_1 x + a_3 x^3) ) = \omega^{ra_0 + ra_1 + a_3}. \tag{4.22}
\]

**Theorem 4.23.** Assume Hypothesis 4.18. Then

\[
\lambda^G = \chi_0 + 3 \sum_{i=1}^{\frac{p-1}{3}} \chi_i \tag{4.24}
\]

where \( \chi_i \in \text{Irr}(G) \) and \( \chi_i \neq \chi_j \) if \( i \neq j \) for \( i, j = 0, 1, \ldots, \frac{p-1}{3} \). Thus \( \eta(\lambda) = \frac{p+2}{3} \).

**Proof.** The center \( Z(G) \) of \( G \) is the subgroup \( \langle m(x^3) \rangle \) of order \( p \). Let \( \gamma \) be the faithful linear character of \( Z(G) \) sending \( m(x^3) \) to \( \omega \). Then \( \text{Lin}(M \mid \gamma) \) consists of the \( p^3 \) linear characters \( \mu_{f_0, f_1, f_2} \), for \( f_0, f_1, f_2 \in F \) given by

\[
\mu_{f_0, f_1, f_2}( m(a_0 + a_1 x + a_2 x^2 + a_3 x^3) ) = \omega^{f_0 a_0 + f_1 a_1 + f_2 a_2 + a_3} \tag{4.25}
\]

for all \( a_0, a_1, a_2, a_3 \in F \). If \( e, i, j \in F \), then (4.19) and (4.20) imply that the conjugate character \( \mu_{e, 0, 0}^{(1+x)^{-1}(1+x^2)^{-j}} \) to \( \mu_{e, 0, 0} \) sends

\[
\begin{align*}
m(1) &\mapsto \mu_{e, 0, 0}( m \left( 1 + i x + \left( \binom{i}{2} + j \right) x^2 + \left( \binom{i}{3} + ij \right) x^3 \right) ) = \omega^{e + ij}, \\
m(x) &\mapsto \mu_{e, 0, 0}( m \left( x + i x^2 + \left( \binom{i}{2} + j \right) x^3 \right) ) = \omega^{(i+j)}, \\
m(x^2) &\mapsto \mu_{e, 0, 0}( m(x^2 + i x^3) ) = \omega^i, \\
m(x^3) &\mapsto \mu_{e, 0, 0}( m(x^3) ) = \omega.
\end{align*}
\]

It follows that

\[
\mu_{e, 0, 0}^{(1+x)^{-1}(1+x^2)^{-j}} = \mu_{e + \binom{i}{2} + ij, (i+j), e, i} \tag{4.26}
\]

for any \( e, i, j \in F \). If we fix \( e \), then the above equation implies that distinct pairs \( (i, j) \in F \times F \) yield distinct conjugates \( \mu_{e, 0, 0}^{(1+x)^{-1}(1+x^2)^{-j}} \in \text{Lin}(M \mid \gamma) \). Hence the \( G \)-orbit \( L_e \) of \( \mu_{e, 0, 0} \) has exactly \( p^2 \) members. Furthermore the above equation implies that the only member of that orbit with the form \( \mu_{f, 0, 0} \) is \( \mu_{e, 0, 0} \). We conclude that the orbits
$L_e$, for $e \in F$, are $p$ distinct $G$-orbits in $\text{Lin}(M \mid \gamma)$, each with size $p^2$. Since the normal subgroup $M$ of index $p^2$ is exactly the stabilizer of $\mu_{e,0,0} \in \text{Lin}(M)$ in $G$, the induced characters

$$\chi_e = \mu_{e,0,0}^G$$

are precisely the distinct members of $\text{Irr}(G \mid \gamma)$. (4.27)

Then

$$\lambda^M = \sum_{f \in F} \mu_{r,rf}^G \quad \text{and} \quad \lambda^G = \sum_{f \in F} \mu_{r,rf}^G.$$  (4.28)

**Claim 4.29.** Let $i \in \{1, \ldots, p - 1\}$, $e = r(1 - i^3)$ and $j = r - (\binom{i}{2})$. Then

$$\mu_{e,0,0}^{(1+x)^{-1}(1+x^2)^{-1}} = \mu_{r,r,i}^G.$$  (4.30)

**Proof.** For a fixed $i$, we have

$$e + \binom{i}{3} + ij = e + \binom{i}{3} + i \left( r - \binom{i}{2} \right)$$

$$= e + i \left( \frac{i(i-1)(i-2)}{6} \right) + i \left( r - \frac{i(i-1)}{2} \right)$$

$$= i^3 \left( \frac{1}{6} - \frac{1}{2} \right) + i^2 \left( \frac{1}{2} - \frac{1}{2} \right) + i \left( r + \frac{1}{3} \right) + e$$

$$\equiv \frac{-i^3}{3} + e \mod p, \quad \text{since } r \equiv \frac{-1}{3} \mod p$$

$$\equiv \frac{-i^3}{3} + r(1 - i^3) \mod p, \quad \text{since } e = r(1 - i^3)$$

$$\equiv r - i^3 \left( r + \frac{1}{3} \right) \equiv r \mod p,$$

where the last line follows since $r \equiv \frac{-1}{3} \mod p$. Thus $(e + \binom{i}{3} + ij, \binom{i}{2} + j, i) = (r, r, i)$ in $F \times F \times F$ and so by (4.26) we get (4.30). \qed

By the previous claim and (4.28), we have that

$$\lambda^G = \sum_{i=0}^{p-1} \mu_{r(1-i^3),0,0}^G.$$  

By Lemma 4.17, we have then

$$\lambda^G = \mu_{r,0,0}^G + 3 \sum_{e \in \{r(1-i^3)\}_i=1,...,p-1} \mu_{e,0,0}^G.$$  (4.31)

By (4.27) we have that $\mu_{e,0,0}^G \in \text{Irr}(G)$ and $\mu_{e,0,0}^G \neq \mu_{f,0,0}^G$ if $e \neq f \mod p$. Thus by Lemma 4.17 and (4.31), we conclude that $\eta(\lambda^G) = \frac{p+2}{3}$ and the proof is complete. \qed
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