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The growth rate of trajectories of a quadratic differential

HOWARD MASUR†

Department of Mathematics, University of Illinois at Chicago, Chicago, Illinois 60680, *USA*

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Abstract. Suppose q is a holomorphic quadratic differential on a compact Riemann surface of genus $g \geq 2$. Then q defines a metric, flat except at the zeroes. A saddle connection is a geodesic joining two zeroes with no zeroes in its interior. This paper shows the asymptotic growth rate of the number of saddles of length at most T is at most quadratic in T. An application is given to billiards.

0. *Introduction*

Let X be a closed Riemann surface of genus $g > 1$ and $q = q(z)\,dz^2$ a nonzero holomorphic quadratic differential on X such that $\int_X |q| = 1$. A *trajectory* is an arc along which $\arg q(z)\,dz^2$ is constant. The trajectory is *regular* if it does not contain zeroes of q and *singular* if the two endpoints of the arc are zeroes and there are no zeroes in the interior of the arc. Singular trajectories are also called saddle connections. If there is a closed regular trajectory there is a parallel family of freely homotopic trajectories of equal $|q^{1/2}\,dz|$ length sweeping out an annulus. On the boundary of the annulus are saddle connections. A geodesic for the metric $|q^{1/2}\,dz|$ is made up of pieces of trajectories that make an angle of at least π at a zero. Between any two points there is a unique geodesic in every homotopy class.

This paper is concerned with the asymptotics of the number of parallel families of closed regular trajectories and the number of saddle connections. Specifically, let $N_1(T)$ denote the number of families of length $\leq T$ and $N_2(T)$ the number of saddle connections of length $\leq T$. Since the boundary of the annulus contains a saddle connection of smaller length, clearly $N_1(T) \leq N_2(T)$.

THEOREM 1.
$$\varlimsup_{T\to\infty} \frac{N_1(T)}{T^2} \leq \varlimsup_{T\to\infty} \frac{N_2(T)}{T^2} < \infty.$$

Remark 1. This result is perhaps surprising in that the growth rate does not depend on the genus. For example, for any metric the growth rate of homotopy classes of simple curves of length $\leq T$ is T^{6g-6} ([**R**] Lemma 2.4). However the geodesic for $|q^{1/2}||dz|$ in a homotopy class is often made of many pieces of saddle connections

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so the number of saddle connections of length $\leq T$ is less than the number of geodesics, in fact grows at most quadratically. The question of lower bounds will be investigated in a later paper.

Remark 2. For $g=1$ there are of course no singular trajectories. Suppose X is $\mathbf{C}$ modulo $z \to z+1$ and $z \to z+i$. Then a trajectory of $q = dz^2$ is closed if the tangent of the argument is a rational, p/q, and the length is $(p^2+q^2)^{1/2}$. Then $N_1(T)$ is the set of integer lattice points (p, q) inside a disc of radius T; p, q relatively prime. It is easy to see that $N_1(T)$ grows quadratically with T.

We give an application to the study of billiards. Let Δ be a connected polygon in the plane. A broken line formed by line segments $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ will be called a generalized diagonal of Δ if it lies inside Δ except for the points x_i. The points x_0, x_n are vertices, $x_1, \ldots, x_{n-1}$ lie on the sides and for $i = 1, \ldots, n-1$ the segments $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ form the same angle with the side of Δ on which x_i lies. The number of generalized diagonals is always infinite.

Katok [**K**] raised the question of the asymptotic count of generalized diagonals. Let $D_T(\Delta)$ be the number of length $\leq T$. He proved

$$\overline{\lim_{T\to\infty}} \frac{\log D_T(\Delta)}{T} = 0.$$

Let G be the group generated by reflection in lines through the origin parallel to the sides. If G is finite Δ is said to be a rational billiard table.

THEOREM 2. $$\overline{\lim_{T\to\infty}} \frac{D_T(\Delta)}{T^2} < \infty \quad \text{if } \Delta \text{ is rational.}$$

It was shown by Boshernitzan [**B**] that Theorem 2 implies that the geodesic flow on a rational billiard table is uniquely ergodic in almost every direction, a result first proved in [**K-M-S**].

It is easy to see that classical integrable billiards such as a rectangle, equilateral triangle satisfy the quadratic estimate $D_T(\Delta) \sim cT^2$. This was generalized by Gutkin [**G**] to a broader class of 'almost integrable' billiards.

Finally let $P_T(\Delta)$ be the number of parallel families of periodic orbits of length $\leq T$ for a rational billiard. Since $P_T(\Delta) \leq D_T(\Delta)$ we have

COROLLARY. $$\overline{\lim_{T\to\infty}} \frac{P_T(\Delta)}{T^2} < \infty.$$

Theorem 2 is a direct consequence of Theorem 1 once it is understood how a billiard flow on a rational polygon gives rise to trajectories of a quadratic differential on a closed Riemann surface. A generalized diagonal gives rise to a saddle connection. This is described in detail in [**K-M-S**].

The idea behind Theorem 1 is as follows. We will fix certain constants $\varepsilon > 0$, $0 < \sigma < 1$ and $C > 1$ and consider saddle connections β whose length is in the interval $[\sigma n, n]$. It is enough to show there are $0(n^2)$ such β. For each β we will choose the argument so that $\arg q(z)\, dz^2$ is π along β; that is, β is a vertical trajectory. We then contract the length of β to ε via a Teichmüller map which contracts along vertical trajectories and expands along horizontal trajectories. On this image

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Riemann surface under the Teichmüller map, if every saddle connection crossing β has length $\geq C\varepsilon$ we will say β is ε-isolated and ε-wide on the domain surface. A simple calculation, Proposition 2.1, shows there is a lower bound of at least constant divided by n^2 between the arguments of crossing ε-wide saddle connections and thus $0(n^2)$ such saddle connections, where the bound depends only on constants.

For example, suppose we have a parallel family of closed trajectories of length T on a torus. When the length is contracted to ε, any closed curve crossing has length $\geq 1/\varepsilon$ since the family fills out the surface and the area is one. For suitably chosen C, ε this is $\geq C\varepsilon$.

On a surface of higher genus the analogous statement may be false since the area of a parallel family may be much less than one; it may fill up only a small part of the surface. For any such β_0 we will associate a sequence $Y_1, \ldots, Y_p$ of surfaces where p is bounded in terms of the genus. Each Y_i will have a boundary of saddle connections and a distinguished saddle connection β_i on it. Each β_i will be in the interior of Y_{i+1} for $i = 0, \ldots, p-1$.

After a suitable Teichmüller map the saddle connections on the boundary of Y_i can simultaneously be made short while β_p itself is ε-wide. This allows us to use Proposition 2.1 to count the number of β_p that occur and since β_p is an element of a system of curves that simultaneously become short, we will be able (Proposition 2.5) to count the number of boundaries of Y_p, hence the number of Y_p that occur.

Next we will compute (Theorem 4.1) for a given surface with boundary the number of saddle connections in the interior of a given length. This will allow us for each Y_p to compute the number of β_{p-1} that occur and again by Propoisition 2.5 the number of Y_{p-1} for given Y_p. The product of this with the number of Y_p gives the total number of Y_{p-1}. We then proceed inductively to find the number of β_0 that define any sequence. The technical difficulties of this paper stem from defining such sequences and more specifically considering not just a single saddle connection but systems of saddle connections. Accordingly § 1 contains preliminary definitions and constructions necessary for the construction of the Y_i. § 2 contains the computations for ε-wide saddle connections, § 3 the above mentioned construction of the Y_i and § 4 the computations that are the proofs of the main theorems.

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1. *Preliminaries*

This section is devoted to definitions and various constructions needed later in the proof of Theorem 4.1 and the main theorem. For the reader unfamiliar with quadratic differentials we refer to [S] as a basic reference and to [**K–M–S**] as well.

In the remainder of the paper, Y with subscripts will refer to a (variable) compact Riemann surface of genus p and $n \geq 0$ boundary components satisfying $2p-2+n \leq 2g-2$ and $\varphi = \varphi\, dz^2$ will refer to a quadratic differential on Y with $\|\varphi\| = \int_Y |\varphi| = 1$. We require the boundary ∂Y to be a union of saddle connections. Y will often contain a distinguished saddle connection α, possibly in its interior.

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For any saddle connection β of φ define $h(\beta)=\int_\beta |\mathrm{Re}\,\varphi^{1/2}\,dz|$ and $v(\beta)=\int |\mathrm{Im}\,\varphi^{1/2}\,dz|$. They are referred to as the horizontal and vertical lengths, respectively. The length $\int_\beta |\varphi|^{1/2}|dz|=|\beta|=(h^2(\beta)+v^2(\beta))^{1/2}$.

Two saddle connections are *disjoint* if they intersect at most at one or two common zeroes. Otherwise they *cross. A basic fact used throughout this paper is that there are at most* $C_1=C_1(p,n)$ disjoint singular trajectories independent of the metric. A system $\Gamma=(\gamma_1,\ldots,\gamma_p)$ will always refer to a collection of *disjoint* trajectories and $|\Gamma|=\max|\gamma_i|$, $v(\Gamma)=\max v(\gamma_i)$.

The basic tool in this paper is the Teichmüller map. For $t\in\mathbf{R}$ let $f_t: Y\to Y_t$ denote the Teichmüller map determined by φ with maximal dilatation $K=e^t$. Denote by φ_t the unit norm terminal quadratic differential on Y_t. We denote by $|\beta|_t$, $v_t(\beta)$, $h_t(\beta)$ the corresponding length functions on Y_t. They are given by:

$$h_t(\beta)=\mathrm{e}^{t/2}h(\beta),\ v_t(\beta)=\mathrm{e}^{-t/2}v(\beta),\ |\beta|_t=(h_t^2(\beta)+v_t^2(\beta))^{1/2}.$$

Let $\varphi_\theta=\mathrm{e}^{i\theta}\varphi$. Then $h_{\theta,t}(\beta)$, $v_{\theta,t}(\beta)$ and $|\beta|_{\theta,t}$ will denote the effect on β of first multiplying by $\mathrm{e}^{i\theta}$ and then applying a Teichmüller map. If θ is understood we will drop that subscript. The absence of a t variable refers to $t=0$. Denote by θ_β the angle for which $h_{\theta_\beta}(\beta)=0$. We call θ_β the *vertical* angle for β. Lengths of course are not invariant under $\varphi\to\varphi_t$ but areas are.

We next define the cross product of two saddle connections. Define

$$|\beta_1\times\beta_2|=\left|\int_{\beta_1}\mathrm{Re}\,\varphi^{1/2}\,dz\int_{\beta_2}\mathrm{Im}\,\varphi^{1/2}\,dz-\int_{\beta_2}\mathrm{Re}\,\varphi^{1/2}\,dz\int_{\beta_1}\mathrm{Im}\,\varphi^{1/2}\,dz\right|.$$

Note that the absolute value does not depend on choice of branch of $\varphi^{1/2}$ along β_i nor does it depend on the orientation of β_i. Notice also $|\beta_1\times\beta_2|$ is invariant under both $\varphi\to\varphi_\theta$ and $\varphi\to\varphi_t$.

A notion basic for counting saddle connections on a surface with boundary is the following.

Definition. Let $D>0$. We say β_1 and β_2 are D-close if $|\beta_1\times\beta_2|\le D$. If Γ is a system of curves, β is D-close to Γ if β is D-close to each $\gamma\in\Gamma$.

LEMMA 1.1. *Suppose α is D_1-close to β and β is D_2-close to γ. Further suppose $|\alpha|\le M_1|\beta|$ and $|\gamma|\le M_2|\beta|$. Then α is $M_1D_2+M_2D_1$-close to γ.*

Proof. Since cross products and lengths, $|\ |$, are invariant under rotations $\varphi\to\varphi_\theta$, assume $h(\beta)=0$, $v(\beta)=|\beta|$. Then $h(\alpha)\le D_1/|\beta|$, $h(\gamma)\le D_2/|\beta|$. This implies $|\alpha\times\gamma|\le v(\alpha)h(\gamma)+h(\alpha)v(\gamma)\le M_1D_2+M_2D_1$.

The next lemma says that if β is close to α, and γ and β can simultaneously be made short then γ is close to α.

LEMMA 1.2. *Suppose β is D-close to α and $|\beta|\ge|\alpha|/M_1$. Further suppose for some $t>0$, M_2 and γ, $\varepsilon/2\le v_t(\beta)\le|\beta|_t\le\varepsilon$ and $|\gamma|_t\le M_2\varepsilon$. Then there is a $D'=D'(M_1,\varepsilon,M_2,D)$ such that γ and α are D'-close.*

Proof. Since $t>0$ and $|\beta|_t\le\varepsilon$ we have $h(\beta)\le\varepsilon$. Then

$$v^2(\beta)\ge|\beta|^2-\varepsilon^2\ge|\alpha|^2/M_1^2-\varepsilon^2\ge\frac{v^2(\alpha)}{M_1^2}-\varepsilon^2.$$

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Thus $v(\alpha)\le M_1(v(\beta)+\varepsilon)$ so $v_t(\alpha)\le M_1(v_t(\beta)+\varepsilon)\le 2M_1\varepsilon$. Then $h_t(\beta)v_t(\alpha)\le 2M_1\varepsilon^2$. Since closeness is preserved under $\varphi\to\varphi_t$ and β and α are D-close, $v_t(\beta)h_t(\alpha)\le D+v_t(\alpha)h_t(\beta)\le D+2M_1\varepsilon^2$. Thus

$$h_t(\alpha)\le\frac{2D+4M_1\varepsilon^2}{\varepsilon}.$$

We conclude that

$$|\gamma\times\alpha|\le v_t(\gamma)h_t(\alpha)+v_t(\alpha)h_t(\gamma)\le(M_2\varepsilon)\left(\frac{2D+4M_1\varepsilon^2}{\varepsilon}\right)+(M_2\varepsilon)(2M_1\varepsilon).$$

Suppose Γ_1 and Γ_2 are two systems with mutually disjoint saddle connections.

Definition. A system Γ separates Γ_1 from Γ_2 if every arc intersecting both Γ_1 and Γ_2 intersects Γ. Γ is allowed to contain saddle connections of Γ_i but no saddle connection of Γ crosses a saddle connection of Γ_i. A system Γ properly separates if $\Gamma\ne\Gamma_i$ and if Γ_i is a single saddle connection β, $\beta\notin\Gamma$.

Definition. Γ M - separates Γ_1 from Γ_2 if it separates and $|\Gamma|<|\Gamma_2|/M$.

We will use this notion often when Γ_1 and Γ_2 are saddle connections β and α. β is *not* M-separated from α if there is no Γ separating β from α with $|\Gamma|<|\alpha|/M$.

We will often need to combine two systems to form a new one.

PROPOSITION 1.3. *Suppose* Γ_1 *and* Γ_2 *are each systems dividing* Y *into two or more components. There is a system* Γ *called the combination of* Γ_1 *and* Γ_2 *with the properties that*

(i) *if* α *is* D_i*-close to* Γ_i *then* α *is* $C_1(D_1+D_2)$*-close to* Γ.

(ii) $|\Gamma|\le C_1(|\Gamma_1|+|\Gamma_2|)$,

(iii) *if* α *is disjoint from* Γ_1 *and* Γ_2 *it is disjoint from* Γ *and* Γ *separates* α *from each* Γ_i.

Proof. Denote by $*$ juxtaposition of two arcs. Consider a component U of the complement of $\Gamma_1\cup\Gamma_2$ and Ω a component of the boundary of U. Ω consists of a union of subarcs of saddle connections of Γ_i. First we include in Γ any arc of Ω that is an entire saddle connection. Now suppose $\gamma_1\in\Omega$ is a proper subarc of a saddle connection of Γ_1, with one endpoint a nonzero endpoint of a proper subarc $\gamma_2\in\Omega$ of a saddle connection in Γ_2. Replace $\gamma_1*\gamma_2$ by the geodesic γ in U in the same homotopy class joining the endpoints of $\gamma_1*\gamma_2$. Now consider γ to be in Ω instead of $\gamma_1*\gamma_2$ thereby reducing U. γ is a union of saddle connections and subarcs of trajectories.

Notice if α, a saddle connection, is disjoint from γ_1 and γ_2 it is disjoint from γ. Moreover any arc from α to γ_1 in U must cross γ. If γ is a simple closed regular trajectory, replace γ in U by the geodesics of saddle connections in the homotopy class on the boundary of the annulus. Again if α is disjoint from γ_1 and γ_2 it is disjoint from γ and any arc in U to γ_1 crosses γ. If γ is not a closed trajectory, include all saddle connections of γ in Γ. If there is an arc σ of γ which is not a saddle connection, an endpoint of σ is a nonzero endpoint of an arc $\gamma_1'\in\Omega$ of Γ_1 or $\gamma_2'\in\Omega$ of Γ_2. Replace $\sigma*\gamma_1'$ (respectively $\sigma*\gamma_2'$) by the geodesic γ' in the

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homotopy class joining the endpoints. We continue in this manner of replacing a juxtaposition of arcs with geodesics γ' until one of three possibilities occurs.

(i) γ' is a saddle connection which we then include in Γ.

(ii) γ' is a closed regular trajectory. Include in Γ the geodesic of saddle connections homotopic to γ', in U.

(iii) Ω reduces to a triangle and none of the arcs are saddle connections. In that case eliminate all such arcs.

This process is repeated for each subarc for each γ_i of Ω not previously eliminated and then for each boundary component Ω of each complementary U to form Γ. We have seen that if α is disjoint from Γ_i it is disjoint from Γ. We next show Γ separates α from Γ_1. Suppose σ is an arc from α to Γ_1 missing Γ and hitting α and Γ_1 only at its endpoints. If α is in U and σ stays in U except for its endpoints then we have seen σ must intersect Γ. Thus σ must leave U a first time crossing a subarc γ_2 of Γ_2. It must therefore intersect a curve γ homotopic to $\gamma_1 * \gamma_2$, $\gamma_1 \in \Gamma_1$. If γ itself is not in Γ then σ must cross a curve homotopic to $\gamma * \gamma_1'$ or $\gamma * \gamma_2'$ where $\gamma_i' \in \Gamma_i$ since by assumption σ does not cross γ_i' as it leaves U for the first time at γ_2. Continuing, this argument shows σ intersects a $\gamma \in \Gamma$ proving (iii).

Now if γ is constructed out of pieces $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ then $|\alpha \times \gamma| \leq |\alpha \times \gamma_1| + |\alpha \times \gamma_2|$. Now for each saddle $\beta \in \Gamma_i$ written as

$$\beta = \gamma_i^1 * \gamma_i^2 * \quad * \gamma_i^p,$$

a juxtaposition of pieces of β, $|\alpha \times \beta| = \sum_{j=1}^{p} |\alpha \times \gamma_i^j|$. Since there are at most C_1 saddle connections in Γ_i, for any $\gamma \in \Gamma$, $|\alpha \times \gamma| \leq C_1(D_1 + D_2)$ proving (i).

Further each γ constructed out of γ_1 and γ_2 satisfied $|\gamma| \leq |\gamma_1| + |\gamma_2|$. Since Γ_1 and Γ_2 consist of at most C_1 saddle connections each, we have each $\gamma \in \Gamma$ satisfies $|\gamma| \leq C_1(|\Gamma_1| + |\Gamma_2|)$, proving (ii).

Remark. If all components of the complement of $F_1 \cup F_2$ are simply connected and contain no saddle connections, then $\Gamma = \emptyset$. This possibility will not occur in this paper.

COROLLARY 1.4. *Suppose Γ_i, $i = 1, \ldots, p$, $p \leq C_1$ are dividing systems $|\Gamma_i| \leq M$ and α is disjoint from each. Then there is a system Γ, which combines the Γ_i; Γ separates α from each Γ_i; Γ and α are disjoint and $|\Gamma| \leq (2C_1)^{C_1} M$.*

Proof. Take the combination of Γ_1 and Γ_2, the combination of that with Γ_3 and so forth.

Our next objective is Proposition 1.6, a construction which will be used in Theorem 4.1 to reduce a general counting problem to counting only saddle connections that are D-close to each other, D a universal number depending only on the genus not on the quadratic differential. We begin with

LEMMA 1.5. *Suppose β_1 and β_2 bound two sides of a triangle with no vertices in the interior. Then $|\beta_1 \times \beta_2| \leq 2$.*

Proof. $|\beta_1 \times \beta_2|$ is the area of a parallelogram which is twice that of the triangle. The triangle has at most area one.

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Now an arbitrary pair of saddle connections may not form two sides of a triangle and therefore may not be 2-close. In fact there is no universal D such that any two saddle connections are D-close. The aim here is to prove.

PROPOSITION 1.6. *Assume β and α are disjoint saddle connections on Y and β is not M-separated from α. Then there are D, M_1 depending on C_1, M and a sequence*

$$\beta=\beta_0, \beta_1, \ldots, \beta_n=\alpha \quad n \le C_1$$

of disjoint saddle connections such that β_i is D-close to β_{i+1} and is not M_1-separated from it $i=0, \ldots, n-1$.

We first prove

PROPOSITION 1.7. *Suppose β is a saddle connection, Ω_0 is a system such that either $\beta \in \Omega_0$ or they are disjoint. Further assume for some M, $|\beta| \ge |\Omega_0|/M$. Then for any $M'>1$ there is a $D=D(M, M', C_1)$ such that if β is not D-close to Ω_0 there is a Ω_0' properly separating β from $\Omega_0-\beta$ such that $|\beta| \ge M'|\Omega_0'|$ and β D-close to Ω_0'.*

Proof. If β is $D=2$-close to Ω_0 we have nothing to show. Suppose not. The proof will consist of adding and eliminating disjoint saddle connections at most C_1 times. The constant D produced may increase by a fixed multiplicative constant each time. The final constant will not depend on $|\beta|$ or $|\Omega_0|$, only on M, M' and C_1. We will refer to all such constants by D even as then change.

We begin by considering triangles with β as one edge whose other edges are either disjoint from $\Omega_0-\beta$ or coincide with saddle connections in $\Omega_0-\beta$. Any such edge is 2-close to β by Lemma 1.5.

If we can separate β from $\Omega_0-\beta$ by Γ_1 by adding 1 or 2 such triangles whose longest side γ_1 distinct from β satisfies $|\beta| \ge |\gamma_1| M'$ we are done. Thus assume for any such separating Γ_1, $|\gamma_1| \ge |\beta|/M'$. Since $|\beta| \ge |\Omega_0|/M$ we have $|\gamma_1| \ge |\Omega_0|/MM'$. Suppose inductively we have constructed $\beta=\Gamma_0, \Gamma_1, \ldots, \Gamma_k$, $M_i=M_i(M, M')$, and D_i, $i=0, \ldots, k$ such that

(i) Γ_{i+1} properly separates Γ_i from $\Omega_0-\beta$.
(ii) $|\Gamma_i-\Gamma_{i-1}| \le M'|\Gamma_{i+1}-\Gamma_i|$.
(iii) the longest curve $\gamma_i \in \Gamma_i-\Gamma_{i-1}$ is D_i close to $\Gamma_{i+1}-\Gamma_i$.
(iv) $|\gamma_{i+1}| \ge |\Omega_0|/M_i$.

We have given the construction for $k=1$. Suppose γ_k is 2-close to $\Omega_0-\Gamma_k$. Then by Lemma 1.1, applied with $\alpha=\gamma_{k-1}$, $\beta=\gamma_k$, $\gamma \in \Omega_0-\Gamma_k$, and (ii), (iii), and (iv) above we have that γ_{k-1} is $D=D_{k-1}M_{k-1}+2M'$-close to $\Omega_0-\Gamma_k$.

Then since $\Omega_0-\Gamma_{k-1} \subseteq (\Omega_0-\Gamma_k) \cup (\Gamma_k-\Gamma_{k-1})$, and γ_{k-1} is D-close to both $\Omega_0-\Gamma_k$ and $\Gamma_k-\Gamma_{k-1}$ it is D-close to $\Omega_0-\Gamma_{k-1}$. Repeating this argument with the hypothesis γ_{k-1} D-close to $\Omega_0-\Gamma_{k-1}$ replacing γ_k $D=2$-close to $\Omega_0-\Gamma_k$, we find eventually that β D-close to $\Omega_0-\beta$ to begin with.

Thus we may assume γ_k is not 2-close to $\Omega_0-\Gamma_k$. From Γ_k we wish to either construct a new sequence $\beta=\Gamma_0, \ldots, \Gamma_l$ satisfying (i)-(iv) and such that Γ_l properly separates Γ_k from $\Omega_0-\beta$ or to simply find the Ω_0' that we desired. To do that we add a triangle with γ_k as one side arriving at Γ_{k+1} separating Γ_k from $\Omega_0-\beta$ with γ_k 2-close to $\Gamma_{k+1}-\Gamma_k$.

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If now $|\gamma_k| \le M'|\Gamma_{k+1} - \Gamma_k|$ then (ii) and (iv) are satisfied as well as (i) and (iii) for the sequence $\Gamma_0, \ldots, \Gamma_k, \Gamma_{k+1}$.

If, on the other hand, $|\gamma_k| \ge M'|\Gamma_{k+1} - \Gamma_k|$, then again by applying Lemma 1.1, with $\alpha = \gamma_{k-1}$, $\beta = \gamma_k$, $\gamma \in \Gamma_{k+1} - \Gamma_{k-1}$, the assumption $|\gamma_k| \ge M'|\Gamma_{k+1} - \Gamma_k|$ and (ii), (iii), give

$$\gamma_{k-1} \quad D = DM' + 2M'\text{-close to } \Gamma_{k+1} - \Gamma_{k-1}.$$

If now $|\gamma_{k-1}| \le M'|\Gamma_{k+1} - \Gamma_{k-1}|$, then applying Lemma 1.1 again using $\alpha = \gamma_{k-2}$, $\beta = \gamma_{k-1}$ and $\gamma \in \Gamma_{k+1} - \Gamma_{k-1}$ we find γ_{k-2} D-close to $\Gamma_{k+1} - \Gamma_{k-1}$. Since $\Gamma_{k+2} - \Gamma_{k-1} \subset (\Gamma_{k+1} - \Gamma_{k-1}) \cup (\Gamma_{k-1} - \Gamma_{k-2})$, γ_{k-2} is D-close to $\Gamma_{k+2} - \Gamma_{k-1}$. If $|\Gamma_{k+2} - \Gamma_{k-2}| \le |\gamma_{k-2}|/M'$ then γ_{k-3} D-close to $\Gamma_{k+1} - \Gamma_{k-2}$. Either we can continue in this manner for all k; we have β is D-close to Γ_{k+1} and we have found the $\Omega_0' = \Gamma_{k+1}$ that we desired, or there is a Γ_j such that

$$|\Gamma_{k+1} - \Gamma_j| \ge |\gamma_j|/M'.$$

In that case we will have constructed a sequence $\Gamma_0, \Gamma_1, \ldots, \Gamma_j, \Gamma_{k+1}$ relabelled $\Gamma_0, \ldots, \Gamma_l$ satisfying (i)-(iv). Thus either there is an Ω_0' or we have found a new sequence. If the latter, there must be a maximal Γ_p constructed after at most C_1 steps; namely a sequence $\Gamma_0, \ldots, \Gamma_p$ satisfying (i)-(iv) such that there is no sequence $\Gamma_0, \ldots, \Gamma_l$ satisfying (i)-(iv) such that Γ_l separates Γ_p from $\Omega_0 - \beta$. We will use this maximal sequence to find Ω_0'.

Now if γ_p is 2-close to $\Omega_0 - \Gamma_p$, then just as in the first paragraph of the proof, Lemma 1.1 (ii), (iii) and (iv) show β D-close to Ω_0 to begin. Thus we may assume γ_p is not 2-close to $\Omega_0 - \Gamma_p$. Then construct Γ_{p+1} separating Γ_p from Ω_0 just as we constructed Γ_{k+1} from Γ_k. We must have $|\Gamma_{p+1} - \Gamma_p| \le |\gamma_p|/M' = |\Gamma_p - \Gamma_{p-1}|/M'$ for otherwise just as in the third paragraph $\Gamma_0, \ldots, \Gamma_{p+1}$ would be a new sequence satisfying (i)-(iv) contradicting the maximality of Γ_p. Then by (ii), (iii), (iv) and Lemma 1.1, γ_{p-1} D-close to $\Gamma_{p+1} - \Gamma_{p-1}$. Again $|\Gamma_{p+1} - \Gamma_{p-1}| \le |\gamma_{p-1}|/M'$ for otherwise $\Gamma_0, \ldots, \Gamma_{p-1}, \Gamma_{p+1}$ would satisfy (i)-(iv), contradicting Γ_p maximal. This implies γ_{p-2} D-close to $\Gamma_{p+1} - \Gamma_{p-2}$. Continuing, we find that β is D-close to Γ_{p+1} and we let $\Omega_0' = \Gamma_{p+1}$. □

Proof of Proposition 1.6. We let $M' = (2C_1)^{2C_1}$. Since β is not M-separated from α we have $|\beta| \ge |\alpha|/M$. We assume β is not $D = D(M, M', C_1)$-close to α, D given by Proposition 1.7. We will construct a sequence of disjoint systems $\beta = \Omega_0, \Omega_1, \ldots, \Omega_n = \alpha$ such that for $\iota \ge 1$, Ω_i separates Ω_{i-1} from α and for each $\omega_i \in \Omega_i - \Omega_{i-1}$ there is a $\omega_{i-1} \in \Omega_{i-1} - \Omega_i$ such that ω_{i-1} is D-close to ω_i and not M_1-separated from ω_i.

We begin by constructing Ω_1. By Proposition 1.7 there is a Γ_1 properly separating β from α such that

$$|\beta| \ge M'|\Gamma_1| \text{ and } \beta \quad D\text{-close to } \Gamma_1.$$

We may assume there is no Γ properly separating β from γ_1 with the same properties.

If Γ is any system property separating β from Γ_1 and $|\beta| \ge |\Gamma|/M$ then β is D-close to Γ. For if not, then again by Proposition 1.7, β could be properly separated from

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Γ by Γ' such that $|\beta| \geq M'|\Gamma'|$ and β D-close to Γ'. But then Γ' also properly separates β from Γ_1 and this contradicts the choice of Γ_1. Thus if there is some Γ properly separating β from Γ_1 with $|\Gamma| \leq 2C_1|\Gamma_1| \leq |\beta|$, replace Γ_1 with Γ and β is still D-close to Γ. If there is a further Γ' properly separating β from Γ with $|\Gamma'| \leq 2C_2|\Gamma| \leq |\beta|$ then we replace Γ with Γ'. We can do this at most C_1 times. By the choice of M' we eventually find Ω_1 properly separating β from Γ_1 such that

(i) $|\beta| \geq (M'/(2C_1)^{C_1})|\Omega_1|$

(ii) β D-close to Ω_1

(iii) there is no Γ properly separating β from Ω_1 with $|\Gamma| \leq 2C_1|\Omega_1|$.

We claim this implies

(iv) β is not C_1-separated by Γ' from any $\omega_1 \in \Omega_1$.

For if there were such an ω_1, we could combine Γ' and Ω_1 using Proposition 1.3 to find Γ separating β from Γ_1 such that

$$|\Gamma| \leq C_1(|\Gamma'| + |\Omega_1|) \leq C_1\left(\frac{|\Omega_1|}{C_1} + |\Omega_1|\right) \leq 2C_1|\Omega_1| < |\beta|.$$

Then $\beta \notin \Gamma$ so Γ properly separates β from Ω_1, a contradiction. Since each $\omega \in \Omega_1$ satisfies $|\omega| \leq |\beta|$ and each is D-close to β, Lemma 1.1 applied with $\alpha = \omega$, $\gamma = \omega' \in \Omega_1$ gives

(v) $\omega', \omega \in \Omega_1$ are $D + D = 2D$-close to each other.

Since Ω_1 separates β from α, by assumption,

(vi) $|\Omega_1| \geq |\alpha|/M$.

We also claim

(vii) Some $\omega \in \Omega_1$ is not MM'-separated from α. For if all $\omega \in \Omega_1$ were so separated, we could combine at most C_1 such separating Γ, using Corollary 1.4, to find Γ, M-separating Ω_1 from α. But this Γ also separates β from α contradicting (vi).

Suppose the longest saddle $\omega_1 \in \Omega_1$ is D-close to α. Then by Lemma 1.1 applied to any $\omega \in \Omega_1$, ω_1, and α and (v) above we have for all $\omega \in \Omega_1$,

$$\omega \quad D' = 2DM + D\text{-close to } \alpha.$$

Choose by (vii) some ω not $M_1 = MM'$-separated from α. Then by (ii) β is D-close to ω and by (iv) is not C_1-separated from it. Our desired sequence is β, ω, α.

Suppose on the other hand this longest ω_1 is not D-close to α. We wish to construct Ω_2. Choose Ω_2' to have the same properties (i)–(iv) with respect to ω_1 and α that Ω_1 had with respect to β and α. In particular it properly separates ω_1 from α.

Case I. If Ω_2' separates all of Ω_1, not just ω_1, from α, let $\Omega_2 = \Omega_2'$. Since ω_1 is D-close to Ω_2, by Lemma 1.1 and (v) each $\omega \in \Omega_1$ is $D + 2D = 3D$-close to Ω_2.

Case II. If Ω_2' does not separate all of Ω_1 from α, let Ω_2'' be the combination of Ω_1 and Ω_2'. Since ω_1 is $2D$-close to Ω_1 and D-close to Ω_2 it is $3C_1D$-close to Ω_2'' by Proposition 1.3(i). Moreover $|\Omega_2''| \leq C_1(|\Omega_1| + |\Omega_2'|) \leq 2C_1|\Omega_1|$. These two facts again with Lemma 1.1 show each $\omega \in \Omega_1$ is $2D2C_1 + 3DC_1 = 7DC_1$-close to Ω_2''. Moreover it is clear any two $\omega \in \Omega_2''$ are D'-close to each other for some D' just as all $\omega \in \Omega_1$ are $2D$-close to each other. Let V be the component of the complement of Ω_2'' that

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contains α. Finally, let

$$\Omega_2 = \Omega_2'' \cap V.$$

In either case we have constructed Ω_2 such that each $\omega \in \Omega_1$ is D close to Ω_2 for some D. We need to show for each $\omega_2 \in \Omega_2 - \Omega_1$ there is a $\omega \in \Omega_1$ not M_1-separated from it for some M_1, M_1 universal.

In the first case where $\Omega_2 = \Omega_2'$ we have by construction property (iv): namely ω_1 is not C_1-separated from ω_2.

The second case is more complicated. One subcase is already taken care of. If $\omega_2 \in \Omega_2 - \Omega_1$ is already a saddle connection of Ω_2', then just as above, ω_1 cannot be C_1 separated from it. Therefore assume ω_2 is not a saddle connection of Ω_2'. The reason this case is more difficult is that since ω_2 is formed from pieces of Ω_1 and Ω_2' it may be much longer than Ω_2'. Thus a Γ separating ω_1 from it may also be longer than Ω_2' so the combination of Γ and Ω_2' does not give a contradiction to (iv).

Now ω_2 is homotopic to a finite alternating sequence $\sigma = \cdots \omega * \omega' \cdots$ of arcs ω of Ω_1 and ω' of Ω_2'. Together ω_2 and σ bound a region Z which is either simply connected or an annulus. Now suppose each $\omega \in \Omega_1$ which has a subarc appearing in σ is M_1-separated from ω_2 by some Γ. Then it is easy to see ω_2 is homotopic to a union of subarcs of $\Gamma \cap Z$ and subarcs of the $\omega' \cap Z$. Now ω_2 is a geodesic and thus shorter than this union. Thus

$$|\omega_2| \le \sum_{\omega' \in \Omega_2'} |\omega' \cap Z| + \sum_{\gamma \in \Gamma} |\gamma \cap Z| \le C_1(|\Omega_2'| + |\omega_2|/M_1).$$

For $M_1 \ge 2C_1$ this implies

$$|\omega_2| \le 2C_1|\Omega_2'|.$$

Next suppose some Γ' $2C_1$-separates ω_1 from ω_2. Then the above inequality gives $|\Gamma'| \le |\Omega_2'|$. We can't have Γ' separating ω_1 from Ω_2' for this would contradict the definition of Ω_2'. Nor can Γ' intersect the component of the complement of Ω_2' containing ω_1. For then the combination Γ of Γ' and Ω_2' would separate ω_1 from Ω_2' and still satisfy

$$|\Gamma| \le 2C_1|\Omega_2'|,$$

still a contradiction. Thus in fact Ω_2' must properly separate ω_1 from Γ'. Since Γ' separates ω_1 from ω_2 it must also separate Ω_2' from ω_2. Otherwise a path from Ω_2' to ω_2 missing Γ' could be connected to a path joining Ω_2' to ω_1 giving a path from ω_1 to ω_2 missing Γ'.

Then since Γ' separates Ω_2' from ω_2, $\Gamma' \cap Z$ separates $\Omega_2' \cap Z$ from ω_2 in Z. Recall previously we have subarcs of $\Gamma \cap Z$ together with subarcs of $\Omega_2' \cap Z$ homotopic to ω_2. Now we must have a union of subarcs of $\Gamma \cap Z$ and subarcs of $\Gamma' \cap Z$ homotopic to ω_2.

But the sum of lengths of subarcs of $\Gamma \cap Z \le C_1|\Gamma|$ and sum of length of subarcs of $\Gamma' \cap Z \le C_1|\Gamma'|$. Since ω_2 is a geodesic

$$|\omega_2| \le C_1|\Gamma| + C_1|\Gamma'| \le \frac{C_1|\omega_2|}{M_1} + \frac{C_1}{2C_1}|\omega_2|.$$

For $M_1 > 2C_1$ this is a contradiction. We have shown for each $\omega_2 \in \Omega_2$ there is some $\omega \in \Omega_1$ not M_1-separated from it. We now repeat the argument with Ω_2 in place of Ω_1, if necessary find Ω_3. After at most C_1 steps we have our desired sequence of Ω_i. □

The next construction associates to every short saddle connection a complex with boundary that is 'isolated' in a certain sense. This construction will be used to deal with the problem of saddle connections that are not ε-wide.

We first fix two additional constants for the rest of the paper. Others will be fixed as we go along. Fix $C > 1$ and $0 < \sigma < 1$. We require

$$C > \max\left(\frac{16}{\sigma}, 2C_1 - 1\right).$$

Also in the rest of the paper M with subscripts will also refer to absolute constants depending only on the above constants.

Definition. An ε' subcomplex is a triangulation of a subset of Y such that the vertices are zeroes of φ and the edges are saddle connections of length at most ε' and the faces are triangles without zeroes in their interior. We assume if three edges of a subcomplex bound a triangle, it is included in the subcomplex. A subcomplex has an interior if it contains a face.

Definition. A boundary edge is an edge which bounds less than two triangles in the complex.

LEMMA 1.8. *The maximal area of an ε' complex is*

$$A(\varepsilon') = \frac{3^{1/2}\varepsilon'^2}{6} C_1.$$

If $A(\varepsilon') < 1$, an ε' complex has a boundary edge.

Proof. The maximal area of a triangle whose sides are length $\leq \varepsilon'$ is $\varepsilon'^2 3^{1/2}/4$. There are at most $2C_1/3$ triangles in a complex.

Definition. β is (M, ε')-isolated if $|\beta| \leq M\varepsilon'$ and if any γ crossing β satisfies $|\gamma| \geq C \max(|\beta|, \varepsilon')$.

Definition. A system Γ is (M, ε')-isolated if $|\Gamma| \leq M\varepsilon'$ and for every γ crossing Γ, $|\gamma| \geq C \max(|\Gamma|, \varepsilon')$. Notice that isolated saddle connections cannot cross since $C > 1$. We now have the following basic construction.

LEMMA 1.9. [K–M–S]. *Suppose X is a connected complex and the boundary Γ is not $(1, \varepsilon')$ isolated. Then there is a connected $\varepsilon' + C\varepsilon'$ complex. $X_1 \supset X$ with more edges and triangles.*

We need a slightly more general construction. Let Γ be a system of saddle connections and U a complementary component.

Definition. An ε-extension of the pair (Γ, U) consists of a complex $X_1 \subset \bar{U}$ with nonempty interior such that the boundary Γ' satisfies

(i) $|\Gamma'| \leq (C+1)^d \varepsilon$

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(ii) area $X_1 \leq A(\varepsilon(C+1)^d)$ where d is the number of edges needed to triangulate U.

LEMMA 1.10. *Suppose X is an ε complex with boundary Γ as in Lemma 1.9. Then for some component of the complement U of X, $(X_1 - X) \cap U$ is an ε-extension of (Γ, U).*

Proof. Since X_1 is a $(C+1)\varepsilon$ complex, area $(X_1 - X) \leq A((C+1)\varepsilon)$. For some complementary U, $(X_1 - X) \cap \bar{U}$ has nonempty interior and its boundary has length $\leq (C+1)\varepsilon$.

The main construction now is the following:

PROPOSITION 1.11. *There are constants C_2, C_3, M such that for any $\delta > 0$ there is an $\varepsilon > 0$ such that for all systems Γ_0 dividing the surface into one or two components, if $\varepsilon_1 = |\Gamma_0| \leq \varepsilon$ and Γ_0 is not $(1, |\Gamma_0|)$ isolated, then for each complementary component U which has area $\geq \delta$ there is a domain $U_1 \not\subseteq U$ with boundary Γ_1 such that*

(a) *Γ_1 separates Γ_0 from U_1,*

(b) *$\Gamma_0 \not\subset \Gamma_1$,*

(c) *area $U_1 \geq g(\varepsilon, \delta) = (\delta/M) - f(\varepsilon)$ where $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$,*

(d) *$|\Gamma_1| \leq C_2 \varepsilon_1 \leq C_2 \varepsilon$,*

(e) *Γ_1 is (C_2, ε_1)-isolated,*

(f) *(Γ_1, U_1) cannot be $C_2\varepsilon_1$-extended,*

(g) *there is no Γ C_3-separating Γ_0 from Γ_1,*

(h) *Γ_1 is minimal with respect to* (a)–(f) *in the sense that there is no $\Gamma' \subset \bar{U}$ properly separating Γ_0 from Γ_1 also satisfying* (a)–(g).

Remark. By Lemma 1.10, (e) follows from (f) but we include it for emphasis. The constants C_2, C_3 will be fixed for the rest of the paper.

Proof. Let $\varepsilon_1 = |\Gamma_0|$. By assumption, Γ_0 is crossed by saddles of length $\leq C\varepsilon_1$. By Lemmas 1.9 and 1.10 there is an ε_1-extension X_1 of (Γ_0, U). Area $X_1 \leq (C+1)^{C_1}\varepsilon_1$. For ε sufficiently small X_1 is a proper subset of U. $U - X_1$ has at most C_1 components. Thus there is a component V_1 of $U - X_1$ with area $V_1 \geq (\delta - (C+1)^{C_1}\varepsilon_1)/C_1$. Moreover the boundary Γ^1 of V_1 satisfies $|\Gamma_1| \leq (C+1)^{C_1}\varepsilon_1$.

Let $\varepsilon_2 = (C_1+1)^{C_1}\varepsilon_1$. Again for ε sufficiently small, if (Γ^1, V_1) can be ε_2-extended to X_2, then X_2 is a proper subset of V_1 and some component V_2 of $V_1 - X_2$ has area

$$\geq \frac{\text{area } V_1 - (C+1)^{C_1}\varepsilon_2}{C_1}.$$

The boundary Γ^2 of V_2 satisfies $|\Gamma_2| \leq (C+1)^{C_1}\varepsilon_2$. Since there are at most $p \leq C_1$ steps in this process, there are constants M, C_2 and a function $f(\varepsilon)$ such that there must be $U_1 \subset U$ with boundary Γ_1 such that area

$$U_1 \geq \frac{\delta}{M} - f(\varepsilon_1),$$

$|\Gamma_1| \leq C_2\varepsilon_1$ and such that (Γ_1, U_1) cannot be $C_2\varepsilon_1$-extended. Thus by construction (a), (b) are satisfied as well as (c), (d), (f). As remarked before, (e) is satisfied by Lemma 1.10. Let $X(\Gamma_0) = X_1 \cup X_2 \cup \cdots \cup X_p$ the complex constructed. Since some (Γ_1, U_1) exists that satisfies (a)–(f) we can always find a minimal one. Now we

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claim for some C_3, (g) is satisfied as well. Suppose Γ C_3-separates Γ_0 from Γ_1. Let $\varepsilon' = |\Gamma| \le |\Gamma_1|/C_3$. Let $V \subset U$ be the component of the complement of Γ containing U_1 so area $V \ge$ area U_1.

If C_3 is sufficiently large, $\varepsilon' < \varepsilon_1$. If (Γ, V) cannot be ε'-extended we have already contradicted Γ_1 minimal. If (Γ, V) can be ε'-extended then the process described above for (Γ_0, U) applied instead to (Γ, V) will lead to Γ^*, $V^* \subset V$ and $X^*(\Gamma)$ such that (Γ^*, V^*) cannot be $C_2\varepsilon'$-extended, where again for C_3 sufficiently large,

$$\text{area } V^* \ge \frac{\delta}{M} - f(\varepsilon'), |\Gamma^*| \le C_2\varepsilon' \le C_2\varepsilon_1.$$

Now we claim Γ^* must separate Γ from Γ_1. To prove the claim, notice first Γ^* cannot intersect Γ_1 since Γ_1 is $C_2\varepsilon_1$-isolated. Thus Γ_1 and Γ^* are disjoint. Nor can Γ^* intersect U_1 for then $X^*(\Gamma) \cap U_1 \ne \varnothing$ and (Γ_1, U_1) could be $C_2\varepsilon_1$-extended to $X^*(\Gamma) \cap U_1$, a contradiction. Thus Γ^* must separate Γ from Γ_1, hence also Γ_0 from Γ_1, proving the claim. But then (Γ_1, U_1) is not minimal and we have a contradiction, proving the Proposition.

Now we will choose ε which will be fixed for the rest of the paper. For each ε let

$$g(\delta) = \frac{\delta}{M} - f(4C_2\varepsilon)$$

be the function given in Proposition 1.11(c). Given $\delta_0 = \frac{1}{2}$ let $\delta_1, \ldots, \delta_{C_1}$ be the values of the C_1 iterates of g evaluated at $\delta_0 = \frac{1}{2}$. Since $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$, ε can be chosen so $\delta_1, \ldots, \delta_{C_1}$ are bounded below away from zero. Choose ε sufficiently small so that for each δ_i, the value of $4C_2\varepsilon$ is such that Proposition 1.11 holds.

Now let $C_4 = 2C_3(2C_1)^{C_1}$. The constant C_4 will be fixed for the rest of the paper.

COROLLARY 1.12. *Suppose* Γ_0, Γ_1 *are as in Proposition* 1.11. *Then there is a* $\gamma_0 \in \Gamma_0$ *not* C_4*-separated from* Γ_1.

Proof. If every γ_0 in Γ_0 were C_4-separated from Γ_1, we could combine all separating Γ using Corollary 1.4 to find Γ separating all of Γ_0 from Γ_1. Such a Γ gives

$$|\Gamma| \le (2C_1)^{C_1}|\Gamma_1|/C_4 \le |\Gamma_1|/C_3,$$

a contradiction to the conclusion of Proposition 1.11.

2. *The number of isolated saddle connections*

In this section we will compute the number of systems of saddle connections that simultaneously become isolated within a certain minimal period of time. In particular this will include the set of saddle connections mentioned in the introduction. Recall now we have fixed ε, C_2. We will refer to (C_2, ε) isolated system simply as isolated. We will also simply write β is ε'-isolated to refer to β as $(1, \varepsilon')$-isolated.

Definition. A system Γ on Y is ε-wide if either

(i) $|\Gamma| \le 4C_2\varepsilon$ and Γ is isolated, or

(ii) $|\Gamma| \ge 4C_2\varepsilon$ and for some (θ, t) $|\Gamma| \le \varepsilon$ and is isolated with respect to $\varphi_{\theta,t}$ on $Y_{\theta,t}$ and $v_{\theta,t}(\Gamma) \ge \varepsilon/C_3$.

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Remark. This last condition in (ii) is essential for Propositions 2.1 and 2.2. The point is the following. A saddle connection may become short after a certain time if the vertical length becomes small and the horizontal length remains small. It may then remain small for a later time if the horizontal length remains small since the vertical length continues to decrease. The condition of $v_{\theta,t}(\Gamma)$ says that up to a factor C_3 the time t is as small as possible to make $|\Gamma| \le \varepsilon$.

Example. Suppose Γ is a system that is the boundary of an annulus of width $\ge C\varepsilon^2/|\Gamma|$ swept out by closed trajectories of length $|\Gamma|$. Let $\theta = \theta_\Gamma$ be the angle so $v_{\theta_\Gamma}(\gamma) = |\gamma|$ for $\gamma \in \Gamma$ and let $e^{t/2} = |\Gamma|/\varepsilon$. Then $|\Gamma|_{\theta,t} = \varepsilon$. Any curve crossing Γ crosses the annulus. The annulus has width $\ge C\varepsilon$ with respect to $\varphi_{\theta,t}$ so Γ is isolated at time t and thus ε-wide on the base surface.

Example. If the width $< C\varepsilon^2/|\Gamma|$, Γ may not be ε-wide for any (θ, t).

We now define several sets of saddle connections. Let

$$S(n) = \{\text{saddle connections on } X \text{ of length } \le n\}$$
$$S_\sigma(n) = \{\beta \in S(n) : \sigma n \le |\beta| \le n\}$$
$$S^\varepsilon(n) = \{\beta \in S(n) : \beta \text{ is } \varepsilon\text{-wide}\}.$$

Note. Theorem 1 is equivalent to showing card $S_\sigma(n)$ is $0(n^2)$.

Now suppose α is a saddle connection on Y. We need to count saddle connections disjoint from α. Let

$$T(n, \alpha) = \{\text{saddle connections } \beta \text{ disjoint from } \alpha : |\beta| \le n|\alpha|\}.$$
$$T(n, \alpha, M) = \{\beta \in T(n, \alpha) : \beta \text{ is not } M\text{-separated from } \alpha\}.$$
$$T^\varepsilon(n, \alpha, D) = \{\beta \in T(n, \alpha) : \beta \text{ is } D\text{-close to } \alpha \text{ and } \varepsilon\text{-wide}\}.$$
$$T(n, \alpha, M, D) = \{\beta \in T(n, \alpha, M) : \beta \text{ is } D\text{-close to } \alpha\}.$$

Our main result about these latter sets is

THEOREM 4.1. card $T(n, \alpha, M)$ *is* $0(n(\log n)^k)$ *for some k, and bound* 0 *depending on M,* C_1 *but not n or* $|\alpha|$.

Remark. The importance of this result, proved in § 4, is that the cardinality depends only on the ratio n, of the length of β to the length of α, and is less than quadratic in that ratio.

We first give the computations for ε-wide saddle connections.

PROPOSITION 2.1. card $S^\varepsilon(n)$ *is* $0(n^2)$.

PROPOSITION 2.2. card $T^\varepsilon(n, \alpha, D)$ *is* $0(n)$.

Remark. The bounds depend on C_1, ε and in Proposition 2.2 on D as well but not on $|\alpha|$ or n or on the quadratic differential in either case.

The propositions are based on two simple computations which we state as lemmas. The first says that if $|\beta|_{\theta,t}$ is small, θ is near the vertical angle θ_β. The second says that if two saddle connections are isolated at a certain time and they cross, their vertical angles can not be too close to each other. For the lemmas assume

$$|\beta| \ge 4C_2\varepsilon \text{ and } |\beta|_{\theta,t} \le \varepsilon' \le C_2\varepsilon.$$

LEMMA 2.3. $|\theta - \theta_\beta|/2 \le 2\varepsilon' e^{-t/2}/|\beta|$.

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Proof. We have

$$\sin\frac{|\theta-\theta_\beta|}{2}=\frac{h_\theta(\beta)}{|\beta|}=\frac{e^{-t/2}h_{\theta,t}(\beta)}{|\beta|}\le\frac{e^{-t/2}\varepsilon'}{|\beta|}. \tag{2.1}$$

However since $h_{\theta,t}(\beta)\le\varepsilon'$ and $t>0$, $h_\theta(\beta)\le\varepsilon'\le|\beta|/4$. Thus $v_\theta(\beta)\ge 3|\beta|/4$ and we have

$$\frac{3}{4}|\beta|e^{-t/2}\le e^{-t/2}v_\theta(\beta)\le\varepsilon' \text{ so } e^{-t/2}\le\frac{4}{3}\frac{\varepsilon'}{|\beta|} \tag{2.2}$$

From (2.1) and (2.2),

$$\sin\frac{|\theta-\theta_\beta|}{2}\le\frac{4}{3}\frac{(\varepsilon')^2}{|\beta|^2}\le\frac{1}{2}.$$

Thus

$$\frac{|\theta-\theta_\beta|}{2}\le 2\sin\frac{|\theta-\theta_\beta|}{2}$$

which together with (2.1) gives the result.

LEMMA 2.4. *Further assume γ is another saddle connection which crosses β and $|\beta|\ge|\gamma|$. Further assume β is ε'-isolated with respect to $\varphi_{\theta,t}$. Then*

$$|\theta-\theta_\gamma|\ge\frac{C\varepsilon' e^{-t/2}}{2|\gamma|}.$$

Proof. Since γ crosses β and β is ε'-isolated, $|\gamma|_{\theta,t}\ge C\varepsilon'$. Now

$$v_{\theta,t}(\gamma)\le e^{-t/2}|\gamma|\le e^{-t/2}|\beta|\le\tfrac{4}{3}\varepsilon'\le\frac{C}{2}\varepsilon'$$

by (2.2) and the definition of C. Therefore $h_{\theta,t}(\gamma)\ge\frac{1}{2}C\varepsilon'$ which implies $h_\theta(\gamma)\ge \frac{1}{2}C\,e^{-t/2}\varepsilon'$ and therefore

$$\frac{|\theta-\theta_\gamma|}{2}\ge\sin\frac{|\theta-\theta_\gamma|}{2}=\frac{h_\theta(\gamma)}{|\gamma|}\ge\frac{C}{2}\frac{e^{-t/2}\varepsilon'}{|\gamma|}. \qquad \square$$

Proof of Proposition 2.1. Isolated curves with respect to a given metric can not cross and there are at most C_1 disjoint saddle connections. Thus we may assume all β satisfy $|\beta|\ge 4C_2\varepsilon$ and are ε-wide. Now suppose $\beta,\gamma\in S_\sigma(n)$, they cross and $|\beta|\ge|\gamma|$. Let (θ, t) be such that $|\beta|\le\varepsilon$ and is isolated with respect to $\varphi_{\theta,t}$ and $v_{\theta,t}(\beta)\ge\varepsilon/C_3$. By Lemma 2.4,

$$\frac{|\theta-\theta_\gamma|}{2}\ge\frac{C}{2}\frac{e^{-t/2}\varepsilon}{|\gamma|}.$$

Since Lemma 2.3 gives $|\theta-\theta_\beta|\le 4e^{-t/2}\varepsilon/|\beta|$ and $C>16$,

$$|\theta_\beta-\theta_\gamma|\ge\frac{C}{4}\frac{e^{-t/2}}{|\gamma|}.$$

Now

$$\frac{\varepsilon}{C_3}\le e^{-t/2}v_\theta(\beta)\le e^{-t/2}|\beta| \text{ so } e^{-t/2}\ge\frac{\varepsilon}{|\beta|C_3}.$$

This gives

$$|\theta_\beta-\theta_\gamma|\ge M/|\beta||\gamma| \tag{2.3}$$

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for M depending only on C, C_3, ε, σ. Since both β, γ have lengths in the interval $[\sigma n, n]$, $|\theta_\beta-\theta_\gamma|\geq M'/n^2$ for some M'. Since there are at most C_1 saddle connections that do not cross, this inequality shows there are $0(n^2)$ saddle connections in $S_\sigma(n)\cap S^\varepsilon(n)$.

Proof of Proposition 2.2. Again we may assume $|\beta|\geq 4C_2\varepsilon$. By rotation we can assume $\theta_\alpha=0$. Then β D-close to α means

$$\sin\frac{|\theta_\beta|}{2}=\frac{h_{\theta_\alpha}(\beta)}{|\beta|}\leq\frac{D}{|\alpha||\beta|}. \tag{2.4}$$

We first show there are $0(1)$ β such that $|\beta|\leq|\alpha|$, the bound independent of $|\alpha|$. Suppose two such β_i cross and $|\beta_1|\geq|\beta_2|$. We may assume $|\theta_{\beta_1}-\theta_{\beta_2}|\leq\pi/2$. Let L be such that $|\alpha|\geq L|\beta_1|$. Then (2.4) gives

$$\sin\frac{|\theta_{\beta_i}|}{2}\leq\frac{D}{L|\beta_i||\beta_1|},\ i=1,2.$$

By (2.3) we have

$$|\theta_{\beta_1}-\theta_{\beta_2}|\geq\frac{M}{|\beta_1||\beta_2|}$$

since β_1, β_2 are ε-wide. Then for L sufficiently large, depending only on D and M we have

$$|\theta_{\beta_1}-\theta_{\beta_2}|\geq 4\left(\sin\frac{|\theta_{\beta_1}|}{2}+\sin\frac{|\theta_{\beta_2}|}{2}\right)$$

which is impossible. Thus for some L there are at most C_1 saddle connections β with $|\beta|\leq|\alpha|/L$. Now for crossing β_i with $|\alpha|\geq|\beta_i|\geq|\alpha|/L$, (2.3) shows $|\theta_{\beta_1}-\theta_{\beta_2}|\geq L'/|\alpha|^2$ for some L'. For $|\alpha|$ sufficiently large depending only on D, L (2.4) shows θ_{β_i} is restricted to an interval of width $(2D/|\alpha|^2)L$ about $\theta_\alpha=0$. Since $|\theta_{\beta_1}-\theta_{\beta_2}|\geq L'/|\alpha|^2$, there can only be $0(1)$ such β_i. For $|\alpha|$ smaller than this fixed number, again $|\theta_{\beta_1}-\theta_{\beta_2}|\geq L'/|\alpha|^2$ shows there are $0(1)$ β.

Next consider crossing β_i that satisfy $|\alpha|\leq\sigma n|\alpha|\leq|\beta_i|\leq n|\alpha|$. Now (2.3) gives $|\theta_{\beta_1}-\theta_{\beta_2}|\geq M/n^2|\alpha|^2$ for some M. If $\frac{1}{2}\sigma n|\alpha|^2\leq D$ there are only $0(n)$ such β_i. Thus we can assume $\frac{1}{2}\sigma n|\alpha|^2\geq D$. Then (2.4) gives

$$\sin\frac{|\theta_{\beta_i}|}{2}\leq\frac{D}{|\alpha||\beta_i|}\leq\frac{D}{|\alpha|^2\sigma n}\leq\frac{1}{2}\ \text{so}\ \frac{|\theta_{\beta_i}|}{2}\leq 2\sin\frac{|\theta_{\beta_i}|}{2}\leq\frac{2D}{|\alpha|^2\sigma n}.$$

Thus the θ_{β_i} are restricted to an angle of width $0(1/n|\alpha|^2)$ about 0. Since they are $M/n^2|\alpha|^2$ apart there are at most $0(n)$ in the interval $[\sigma n, n]$. This proves the proposition.

Recall we need to consider disjoint systems of curves that simultaneously become short. We will need to know the number of such systems that contain a given saddle connection. In what follows $\Gamma=(\gamma_1,\ldots,\gamma_p)$ will be a system such that $\sigma^{l_i+1}m\leq|\gamma_i|\leq\sigma^{l_i}m$ (respectively

$$\sigma^{l_i+1}m|\alpha|\leq|\gamma_i|\leq\sigma^{l_i}m|\alpha|).$$

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We will assume their lengths are in decreasing order with $0 = l_1 \le l_2 \le \cdots \le l_p$. We will also assume t satisfies

$$\frac{3}{4}\frac{\sigma n}{\varepsilon} \le e^{t/2} \le \frac{2n}{\varepsilon}$$

(respectively $\frac{3}{4}\sigma n|\alpha| \le e^{t/2} \le 2\varepsilon^{-1} n|\alpha|$) for some $n \ge m$. Next let

$$U = U(m, l_1, \ldots, l_p) = \{\Gamma = (\gamma_1, \ldots, \gamma_p):$$

for some (θ, t), t in the given range, $\Gamma_{\theta,t}$ is isolated with respect to $\varphi_{\theta,t}$.$\}$ Let

$$U_i = U_i(m, l_i, n) = \{\gamma_i : \gamma_i \text{ is an element of a } p\text{-tuple in } U(m, l_1, \ldots, l_p, n).\}$$

PROPOSITION 2.5. Card $U = O(\sigma^{-l_i})$ card U_i *where the bound does not depend on n*, m, l_i, $|\alpha|$.

Proof. We prove this for $\sigma^{l_i+1} m \le |\gamma_i| \le \sigma^{l_i} m$; the case with $|\alpha|$ is identical. Let γ_j belong to a p-tuple with associated (θ, t). By (2.1) with $\varepsilon' = |\gamma_j|_{\theta,t} \le C_2 \varepsilon$

$$\sin \frac{|\theta - \theta_{\gamma_j}|}{2} \le \frac{\varepsilon' e^{-t/2}}{|\gamma_j|} \le \frac{4}{3}\frac{\varepsilon}{\sigma n}\frac{\varepsilon'}{|\Gamma_j|} \le \frac{4}{3}\frac{\varepsilon^2 C_2}{\sigma n}. \tag{2.5}$$

If γ_j' crosses γ_j, then γ_j isolated at time t means $e^{t/2} h_\theta(\gamma_j') \ge \frac{1}{2} C \max(C\varepsilon, C\varepsilon')$ which implies

$$\sin \left|\frac{(\theta - \theta_{\gamma_j'})}{2}\right| \ge \frac{C}{2}\frac{\varepsilon}{2n}\frac{\max(C\varepsilon, C\varepsilon')}{|\gamma_j'|}. \tag{2.6}$$

Now since γ_j and γ_j' cross they cannot belong to the same p-tuple. Moreover if $\sigma^{l_j+1} m \le |\gamma_j'| \le \sigma^{l_j} m$ as well, then 2.5, 2.6 and the fact that $C \ge 16/\sigma$ means the angles θ about θ_{γ_j} and $\theta_{\gamma_j'}$ are disjoint. In particular for some M_1 depending only on C, ε, C_2 and σ,

$$|\theta_{\gamma_j} - \theta_{\gamma_j'}| \ge M_1 / nm\sigma^{l_j}. \tag{2.7}$$

Now for $j > i$, $\sigma^{-l_j} \ge \sigma^{-l_i}$. Thus (2.5) and (2.7) mean that the θ_{γ_j} are further apart than the size of the interval of angles θ about θ_{γ_i}. The fact that the intervals of θ about γ_j and γ_j' are disjoint means for a given γ_i there are $0(1)$ γ_j such that γ_i, γ_j belong to the same p-tuple. For $j = i - 1$ the lower bound (2.7) and the upper bound (2.5) on the size of the interval of θ about θ_{γ_i} show there are at most $0(\sigma^{l_{i-1} - l_i})\gamma_{i-1}$ for each γ_i. Continuing, we find there are $0(\sigma^{l_{i-2} - l_{i-1}})\gamma_{i-2}$ for each pair γ_{i-1}, γ_i; that is $0(\sigma^{l_{i-2} - l_i})$ triples. We continue in this manner and since $l_1 = 0$, there are $0(\sigma^{-l_i})$ p-tuples $(\gamma_1, \ldots, \gamma_p)$ for each γ_i, proving the proposition.

To motivate the next proposition we recall the plan. For each β which is not ε-wide we will associate a new Riemann surface Y on which β will have length ε. On Y we will find a subcomplex with isolated boundary with β in its interior. Consider this subcomplex simply as a topological surface with boundary without its metric structure. Other β may determine the same topological surface considered as a subcomplex of a different Riemann surface with a different metric. We wish to consider one single image Riemann surface with a quadratic differential for the purposes of computing the total number of β determining that topological surface (Theorem 4.1). A saddle connection β which has length ε with respect to one metric

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of course need not have length ε with respect to another. The following computes the length of a curve with respect to an image metric when it is short with respect to another, provided both metrics make some other curve short.

LEMMA 2.6. *Suppose β_1, β_2 are saddle connections that satisfy $4C_2\varepsilon \le \sigma n \le |\beta_i| \le n$ (respectively $4C_2\varepsilon \le \sigma n|\alpha| \le |\beta_i| \le n|\alpha|$) and (θ_i, t_i) satisfy $|\beta_i|_{\theta_i,t_i} \le \varepsilon$ and $v_{\theta_i,t_i}(\beta_i) \ge \varepsilon/C_3$. Suppose further γ is disjoint from each, $|\gamma| \ge 4C_2\varepsilon$ and $|\gamma|_{\theta_i,t_i} = \varepsilon_i \le C_2\varepsilon$. Further suppose $\varepsilon_2 \ge \varepsilon_1$. Then for some M depending on ε, σ but not on $|\beta_i|$, $|\gamma|$ or ε_i*

$$\frac{|\beta_1|_{\theta_2,t_2}}{|\gamma|_{\theta_2,t_2}} \le \frac{M|\beta_1|}{|\gamma|}.$$

Proof. Exactly as in Lemma 2.3

$$\frac{1}{2}|\theta_i - \theta_\gamma| \le \frac{2\varepsilon_i\, e^{-t_i/2}}{|\gamma|} \le \frac{2\varepsilon_2\, e^{-t_i/2}}{|\gamma|}. \tag{2.8}$$

By the conditions on $v_{\theta_i,t_i}(\beta_i)$ and the fact that the $|\beta_i|$ are bounded in terms of each other, the times t_1, t_2 are also. By (2.8)

$$\sin\frac{|\theta_1-\theta_2|}{2} \le \frac{|\theta_1-\theta_2|}{2} \le \frac{M'\varepsilon_2\, e^{-t/2}}{|\gamma|}$$

for some M'. Thus

$$\frac{h_{\theta_2,t_2}(\beta_1)}{|\gamma|_{\theta_2,y_2}} = \frac{e^{t_2/2}h_{\theta_2}(\beta_1)}{\varepsilon_2} = \frac{e^{t_2/2}|\beta_1|}{\varepsilon_2}\sin\frac{|\theta_2-\theta_2|}{2} \le \frac{M'|\beta_1|}{|\gamma|}.$$

On the other hand,

$$\begin{aligned}\frac{v_{\theta_2,t_2}(\beta_1)}{|\gamma|_{\theta_2,t_2}} &\le \frac{2e^{-t_2/2}v_{\theta_2}(\beta_1)}{e^{-t_2/2}v_{\theta_2}(\gamma)+e^{t_2/2}h_{\theta_2}(\gamma)}\\ &\le \frac{2v_{\theta_2}(\beta_1)}{v_{\theta_2}(\gamma)+e^{t_2/2}h_{\theta_2}(\gamma)} \le \frac{2|\beta_1|}{|\gamma|}.\end{aligned}$$

Together this gives $|\beta_1|_{\theta_2,t_2}/|\gamma|_{\theta_2,t_2} \le M|\beta_1|/|\gamma|$. □

3. *Complexes with isolated boundaries*

The idea in this section is as follows. We start with β on X or on Y and in that case disjoint from a saddle connection α on Y. We suppose β is not ε-wide. We rotate so β is vertical and contract the length to ε using the Teichmüller map. By Proposition 1.11 we build an isolated separting system Γ disjoint from β. We would like to count, using Propositions 2.1 (or Proposition 2.2) and 2.5, the number of Γ that occur in this process. However the vertical length of Γ may be $<\varepsilon/C_3$ at this time so Γ may not be ε-wide. Thus Propositions 2.1 and 2.5 will not apply. If that is the case we go 'back' in time until the vertical length of Γ is essentially ε and then ask if it is still isolated. If so Γ contains an ε-wide saddle connection γ on X (respectively $Y-\alpha$). We will be able to count the number of such γ using Propositions 2.1 or 2.2 and then use Proposition 2.5 to count the Γ that contain γ.

If Γ is not isolated, again using Proposition 1.11 we build a further Γ' disjoint from Γ and continue. By Proposition 1.11 this construction must end after C_1 steps

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with some ε-wide Γ. The following Proposition makes this precise. We are going to state it more generally with Γ_0, a system, instead of β simply a saddle connection. The condition is there should be some θ, t such $|\Gamma|_{\theta,t}$ is short. This is clearly satisfied for β a saddle connection with $\theta = \theta_\beta$. For systems that are not singletons there is also the added condition that at a later time the system is isolated. The added generality of considering systems is needed in the proof of the Theorem. Again $|\ |$ without subscripts refers to lengths on the base surface X (respectively $X - \alpha$). Also the angle θ will be suppressed as a subscript.

PROPOSITION 3.1. *Suppose Γ_0 is either a saddle connection or a system of disjoint saddles that separates the surface into two components. Suppose either $|\Gamma_0| \le 4C_2\varepsilon$ or for some (θ, t), $t > 0$, $\varepsilon/2 \le v_{\theta,t}(\Gamma_0) \le |\Gamma_0|_{\theta,t} \le \varepsilon$. Assume further in the second case, that if Γ_0 is not a singleton, there is a $s \ge t$ such that $|\Gamma_0|_{\theta,s} \le \varepsilon$ and $(\Gamma_0)_{\theta,s}$ cannot be ε-extended. Then there is a number M independent of $|\Gamma_0|$, a sequence $\Gamma_0, \Gamma_1, \ldots, \Gamma_p$, $p \le C_1$, of mutually disjoint systems on X, (respectively $Y - \alpha$), a sequence of times $t_1 \ge t_2 \ge \cdots \ge t_p \ge t_{p+1} \ge 0$ such that*

(0) *If Γ_0 is ε-wide, $p = 0$, $t_1 = 0$,*
(i) *$|\Gamma_j| \ge |\Gamma_{j+1}|/M$,*
(ii) *for each $j \ge 1$, Γ_j divides X (respectively $Y - \alpha$) into components U_j and V_j where $U_j \supsetneqq U_{j+1}$ and $\Gamma_0 \subset V_1$,*
(iii) *for $j \ge 1$, $\varepsilon' = |\Gamma_j|_{t_j} \le 4C_2^2\varepsilon$ and Γ_j is $(1, \varepsilon')$-isolated with respect to the metric defined by φ_{t_j},*
(iv) *if $|\Gamma_0| \le 4C_2\varepsilon$ and is not ε-wide then $p = 1$, $t_1 = t_2 = 0$; $\varepsilon' \le |\Gamma_1| \le 4C_2^2\varepsilon$ and Γ_1 is $(1, \varepsilon')$-isolated,*
(v) *if neither (0) or (iv) holds, then $t_1 = t > 0$ and for $1 \le j < p$, $v_{t_j}(\Gamma_j) < \varepsilon/C_3$ and $t_{j+1} < t_j$.*
  (a) *If in addition $|\Gamma_j| \le 4C_2\varepsilon$ then $p = j+1$, $t_{p+1} = t_p = 0$ and $\varepsilon' = |\Gamma_p| \le 4C_2^2\varepsilon$ and Γ_p is $(1, \varepsilon')$-isolated.*
  (b) *If on the other hand $|\Gamma_j| > 4C_2\varepsilon$, then $\varepsilon/2 \le v_{t_{j+1}}(\Gamma_j) \le |\Gamma_j|_{t_{j+1}} \le \varepsilon$ so $t_{j+1} > 0$.*
(vi) *If (v)(a) does not hold then either $v_{t_p}(\Gamma_p) \ge \varepsilon/C_3$ and $t_{p+1} = t_p$ or $v_{t_p}(\Gamma_p) < \varepsilon/C_3$. In that case either $|\Gamma_p| \le 4C_2\varepsilon$ and $t_{p+1} = 0$ or $|\Gamma_p| > 4C_2\varepsilon$ and $\varepsilon/2 \le v_{t_{p+1}}(\Gamma_p) \le |\Gamma_p|_{t_{p+1}} \le \varepsilon$. In all cases Γ_p is ε-wide.*
(vii) *For each $0 \le j \le p-1$ there is a saddle connection $\gamma_j \in \Gamma_j - \Gamma_{j+1}$ such that*
  (a) *$|\gamma_j|_{t_{j+1}} \ge |\Gamma_{j+1} - \Gamma_j|/C_4$*
  (b) *there is no Γ $C_5 = 8C_4$-separating γ_j from Γ_{j+1} at time t_{j+1}.*
  (c) *$vt_{j+1}(\gamma_j) \ge \frac{1}{2}h_{t_{j+1}}(\gamma_j)$ if $t_{j+1} > 0$.*

Remarks. We will use either (iv), (v)(a), or (vi) and Propositions 2.1 or 2.2, and Proposition 2.5 to calculate the number of Γ_p that occur. (vii) is designed so that with Theorem 4.1 for each Γ_{j+1} we can compute the number of γ_j and therefore by Proposition 2.5 and (iii) the number of Γ_j that occur. In particular (vii)(c) is necessary to control the length of γ_j on X.

Proof. To simplify notation let $\bar{\varepsilon} = 4C_2\varepsilon$. Γ_0 has a complementary component U with area $\ge \frac{1}{2}$. If Γ_0 is already ε-wide there is no construction; (0) holds. If $|\Gamma_0| \le \bar{\varepsilon}$ and is not ε-wide form the system Γ_1 and complementary component U_1 provided

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by Proposition 1.11. Area $U_1 \geq g(\frac{1}{2})$. We set $p=1$, $t_1=t_2=0$ in this case. Now (i)-(iv) are automatically satisfied by construction. (vii)(a) and (b) follow from Corollary 1.12. We are done in this case. If $|\Gamma_0| > \bar{\varepsilon}$ let $t=t_1$. On the surface X_{t_1} (respectively $Y_{t_1}-\alpha$) we have $\varepsilon/2 \leq v_{t_1}(\Gamma_0) \leq |\Gamma_0|_{t_1} \leq \varepsilon$ and Γ_0 not isolated. Let (Γ_1, U_1) denote the system with complementary component given by Proposition 1.11; Area $U_1 \geq g(\frac{1}{2})$ and $|\Gamma_1|_{t_1} \leq C_2\varepsilon$. If $v_{t_1}(\Gamma_1) \geq \varepsilon/C_3$ we set $p=1$; $t_1=t_2$ and the construction ends. Now we have to check (ii) and (vii). Here (i), (iii), (vi) are satisfied by definition and (iv) and (v) are vacuous. To see (ii), the fact that $|\Gamma_0|_{t_1} \geq M|\Gamma_1|t_1$ for some M and the fact that at time t_1 the ratio of vertical to horizontal length of Γ_0 is bounded away from zero means the inequality between $|\Gamma_0|_{t_1}$ and $|\Gamma_1|_{t_1}$ persists up to another universal factor at an earlier time $t=0$.

Checking (vii)(c) is more complicated if Γ_0 is not a singleton saddle connection. The point is the γ_0' provided by Corollary 1.12 is not necessarily the saddle connection γ_0 that satisfies $\varepsilon/2 \leq v_{t_1}(\gamma_0) \leq |\Gamma_0|_{t_1} \leq \varepsilon$. We argue therefore as follows. If there is no Γ C_5-separating γ_0 from Γ_1, then γ_0 satisfies (vii). If there is such a Γ then $|\Gamma_1|_{t_1} \geq C_5|\Gamma|_{t_1}$. Then $|\gamma_0'|_{t_1} \geq |\Gamma_1|_{t_1}/C_4 \geq (C_5/C_4)|\Gamma|_{t_1}$. At the later time $s \geq t_1$ we claim $|\Gamma|_s \geq |\Gamma_0|_s$. To prove the claim assume otherwise and consider the combination Γ' of Γ and Γ_0 at time s. Then

$$|\Gamma'|_s \leq C(|\Gamma_0|_s + |\Gamma|_s) \leq 2C_1|\Gamma_0|_s \leq (C+1)|\Gamma_0|_s,$$

the last inequality by the definition of C. Also Γ' separates Γ_0 from Γ_1. Now we consider the region X' bounded by Γ' and Γ_0 inside the complex X_1 bounded by Γ_0 and Γ_1. At time t_1, X_1 has area $\leq A(C_2\varepsilon)$ and since area is preserved under the Teichmüller flow, this is true at time s as well. Therefore X_2 has area $\leq A(C_2\varepsilon)$ and since

$$|\Gamma'|_s \leq (C+1)|\Gamma_0|_s \leq (C+1)\varepsilon,$$

X_2 is a $C_2\varepsilon$-extension of Γ_0 at time s, a contradiction to the assumption proving the claim. Thus as time goes from t_1 to s, $|\gamma_0'|$ decreases from $\geq 8|\Gamma|_{t_1}$ to a number $\leq |\Gamma_0|_s \leq |\Gamma|_s$. If the ratio in length of two saddle connections goes from at least eight to less than one in positive time, the first cannot be nearly horizontal. A definite estimate gives (vii)(c). We have found γ_0' and we are done in this case.

If $v_{t_1}(\Gamma_1) < \varepsilon/C_3$ and $|\Gamma_1| \leq \bar{\varepsilon}$ set $t_2=0$. If Γ_1 is isolated at $t_2=0$ (on X or $Y-\alpha$) set $p=1$. Again the construction ends and (i)-(vii) are satisfied just as in the last case. If Γ_1 is not isolated we find Γ_2 given by Proposition 1.11. The existence is guaranteed by the fact that area $U_1 \geq \delta_1 = g(\frac{1}{2})$. Then we set $p=2$ and $t_3=0$. Now (vii)(c) is satisfied at $t_2=0$ because just as in the argument above there is a later time, namely t_1, where Γ_1 cannot be $C_2\varepsilon$-extended.

What remains is the case $v_{t_1}(\Gamma_1) < \varepsilon/C_3$ and $|\Gamma_1| \geq \bar{\varepsilon}$. Choose γ_1 so $v_{t_1}(\gamma_1) = v_{t_1}(\Gamma_1)$. Find $t_2 < t_1$ so $v_{t_2}(\gamma_1) = \varepsilon/\sqrt{2}$. That is, go back in time, increasing the vertical length of γ_1 until it is $\varepsilon/\sqrt{2}$. Now

$$v_{t_1}(\gamma_1) < \varepsilon/C_3 \text{ implies } e^{(t_1-t_2)/2} \geq C_3/\sqrt{2}.$$

Also $|\Gamma_1| \geq \bar{\varepsilon}$ and $h_{t_1}(\Gamma_1) \leq C_2\varepsilon$ imply $t_2 \geq 0$. For any $\gamma \in \Gamma_1$, $h_{t_1}(\gamma) \leq C_2\varepsilon$ and

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$e^{(t_1-t_2)}/2 \geq C_3/\sqrt{2}$ imply $h_{t_2}(\gamma) < \varepsilon\sqrt{2}/4$ and therefore

$$|\gamma|_{t_2} \leq \left(\left(\frac{\varepsilon\sqrt{2}}{4}\right)^2 + \left(\frac{\varepsilon}{\sqrt{2}}\right)^2\right)^{1/2} \leq \varepsilon.$$

If Γ_1 is isolated at $t = t_2$, we set $p = 1$, the construction ends; (i)-(vii) are satisfied as before. If not, construct (Γ_2, U_2) by Proposition 1.11. $U_2 \subset U_1$, area $U_2 \geq \delta_2 = g(\delta_1)$. Once again (i)-(vii) are satisfied by the same arguments. We continue this process. After $p \leq C_1$ steps it must stop. Since $\delta_{C_1} > 0$ it must end with some (Γ_p, U_p) for which there is no Γ_{p+1}. This means either $v_{t_p}(\Gamma_p) \geq \varepsilon/C_3$ and it is isolated, or if not, either $|\Gamma_p| \leq 4C_2\varepsilon$ and Γ_p is isolated at $t_{p+1} = 0$ or for some $t_{p+1} < t_p$, $\varepsilon/2 \leq v_{t_{p+1}}(\Gamma_p) \leq |\Gamma|_{p+1} \leq \varepsilon$ and Γ_p is isolated there. For if it were not isolated at these three possibilities we could build a further Γ_{p+1}. □

Suppose now the following situation occurs as in the conclusion to Proposition 3.1. There is a saddle connection τ, a pair $(\theta, t) = (\theta(\tau), t(\tau))$ and constants M_1, M_2 independent of τ such that

(i) $|\tau|_{\theta,t} \leq \varepsilon' \leq \varepsilon$, $v_t(\tau) \geq \varepsilon'/C_3$.

(ii) there is a system Γ disjoint from τ such that $|\Gamma|_{\theta,t} \leq M_1|\tau|_{\theta,t}$.

(iii) τ is not M_2-separated from Γ at time t.

Under these circumstances we will say Γ is ε'-associated to τ. Let $B_L(\Gamma) = \{\tau : \Gamma$ is ε-associated to τ, $\sigma L \leq |\tau| \leq L$, and $|\Gamma| \geq 4C_2\varepsilon\}$. Let $\varepsilon_0 = \max_{\tau \in B_L(\Gamma)} |\Gamma|_{\theta(\tau),t(\tau)}$, the maximum taken on for some τ_0 and $(\theta_0, t_0) = (\theta_0(\tau_0), t_0(\tau_0))$. By Lemma 2.6, for any $\tau \in B_L(\Gamma)$,

$$\frac{|\tau|_{\theta_0,t_0}}{\varepsilon_0} \text{ is } 0\left(\frac{|\tau|}{|\Gamma|}\right) = 0\left(\frac{L}{|\Gamma|}\right).$$

By assumption τ is not M_2-separated from Γ on the surface $Y_{\theta(\tau),t(\tau)}$ but it may be so separated on Y_{θ_0,t_0}. We will wish to compute card $B_L(\Gamma)$ in Theorem 4.1 making the computations on the surface Y_{θ_0,t_0} where the separating hypothesis is necessary for that theorem.

The purpose of the following lemma is to show there are a bounded number of $\tau_i \in B_L(\Gamma)$ and a number M_3 such that for each τ there is a τ_i such that τ is not M_3-separated on $Y_{\theta(\tau_i),t(\tau_i)}$.

LEMMA 3.2. *There are sets $B_i \subset B(\Gamma)$, $i \leq C_1$, and $M_3 = M_3(M_1, M_2)$ such that*

(i) $B_L(\Gamma) = U_iB_i$,

(ii) *for each i there is a $\tau_i \in B_i$ and $(\theta_i, t_i) = (\theta_i(\tau_i), t(\tau_i))$ such that for all $\tau \in B_i$,*

$$\frac{|\tau|_{\theta_i,t_i}}{|\Gamma|_{\theta_i,t_i}} \text{ is } 0\left(\frac{L}{|\Gamma|}\right)$$

(iii) *$\tau \in B_i$ is not M_3-separated from Γ on $Y_i = Y_{\theta_i,t_i}$.*

Proof. Let $B_0 = \{\tau \in B_L(\Gamma) : \tau$ is not $(2C_1)^{C_1}M_2$-separated from Γ on $Y_0 = Y_{\theta_0,t_0}\}$. Suppose $B_0 \neq B_L(\Gamma)$. By taking at most C_1 combinations of various Γ' that do $(2C_1)^{C_1}M_2$-separate various $\tau \in B_L(\Gamma)$ from Γ and using Corollary 1.4 we find a Γ' that M_2-separates every $\tau \in B_L(\Gamma) - B_0$ from Γ on Y_0. Let U' the component of $Y_0 - \Gamma'$ containing such τ and V' the complementary component. Now consider

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$\max_{\tau\in B_L(\Gamma)-B_0}|\Gamma|_{\theta,t}$. Suppose the maximum is taken on by τ_1 at (θ_1, t_1) with corresponding Y_1. By assumption $|\Gamma'|_{\theta_1,t_1}\geq|\Gamma|_{\theta_1,t_1}/M_2$ since τ_1 is not M_2-separated from Γ on Y_1. Now let $B_1=\{\tau\in B_L(\Gamma)-B_0: \tau$ is not $(2C_1)^{C_1}M_2$-separated from Γ on $Y_1\}$. Again (ii) is satisfied for $\tau\in B_1$ by Lemma 2.6. Again it is possible $B_L(\Gamma)-B_0-B_1\neq\emptyset$. That is, there is a $\tau\notin B_0\cup B_1$ which is $(2C_1)^{C_1}M_2$-separated by Γ'' from Γ on Y_1. Such a Γ'' must intersect U' for otherwise τ_1 itself would be so separated. As before taking at most C_1 combinations of such Γ'' and then the combination of the resulting system with Γ' we find Γ'' separating all $\tau\in B_L(\Gamma)-B_1-B_2$ from Γ. Now $\Gamma''\subset U'$. Let U'' be the complementary component containing all such τ.

We repeat the maximizing procedure to find a new τ_2 and a new B_2. Since the U'' are decreasing, the process ends after at most C_1 steps and

$$B_L(\Gamma)=\bigcup_{i=0}^{p} B_i. \qquad \square$$

4. *Proof of the theorems*

We collect our results in this section and prove both the preliminary Theorem 4.1 and the main theorem.

THEOREM 4.1. *There exists* $k=k(\varepsilon, M, C_1)$ *such that* card $T(m,\alpha,M)$ *is* $0(m(\log m)^k)$.

We will adopt the following terminology. Suppose Ω, Γ are disjoint systems; by card $\{\Omega|\Gamma\}$ will refer to the number of Ω disjoint from Γ for a given Γ.

Proof of Theorem 4.1. We start by remarking that $|\beta|\geq|\alpha|/M$ since β is not M-separated from α.

The proof is by induction on the number, $r\leq C_1$, of disjoint segments that can be added to $Y-\alpha$. At each stage of the induction the exponent k can increase by a fixed amount. Since the induction is of length $\leq C_1$ the final exponent will still be bounded. At each stage we will denote this exponent as k *even as it changes.*

If $r=1$ then either Y is simply connected or is an annulus. If Y is simply connected it is either a quadrilateral and α is on the boundary or has five sides and α crosses the domain. In either case card $T(m,\alpha)=2$.

If Y is an annulus, α crosses from one boundary to the other. Then α, an edge of the boundary of Y, and β bound a triangle so β and α are 2-close. If $|\beta|\leq 4C_2\epsilon$ it is isolated in $Y-\alpha$ since there can be no ε complex disjoint from α containing β. Similarly if $|\beta|>4C_2\varepsilon$ it is ε-wide. Thus card $T(m,\alpha)=$ card $T^\varepsilon(m,\alpha,2)=0(m)$ by Proposition 2.2.

Now suppose the theorem is true whenever fewer than r_0 curves can be added and r_0 trajectories can be added to $Y-\alpha$. This is the induction hypothesis in place for the rest of the proof. There are several cases to consider.

Case I. $|\beta|\leq 4C_2\varepsilon$. Form the complex containing β with isolated boundary Γ_1 with longest curve γ_1. There are $0(1)$ such Γ_1 since it is isolated on Y. If Γ_1 does not separate β from α then β and α are contained in a smaller complex $Y-\Gamma_1$ to which

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fewer than r_0 curves can be added. For each such Γ_1 apply the induction hypothesis. If Γ_1 does separate, then by assumption $|\gamma_1| = |\Gamma_1| \geq |\alpha|/M$. Consider the complex Z containing β bounded by Γ_1 where the quadratic differential is renormalized so that Z has unit area. Fewer than r_0 trajectories can be added. We apply the induction hypothesis to β and γ_1. Since $|\gamma_1|/|\alpha| \geq M^{-1}$ and there are $0(1)$ such Γ_1 we are done. There are $0(m(\log m)^k)$ such β.

Case II. $|\beta| \geq 4C_2\varepsilon$. Consider the sequence $\Gamma_1, \ldots, \Gamma_p$ and times $t_1, \ldots, t_{p+1}$ constructed in Proposition 3.1.

Recall either Γ_p is ε-wide or isolated on $Y - \alpha$. Recall also $|\Gamma_i|_{t_{i+1}} \geq |\Gamma_{i+1}|_{t_{i+1}}/M$ for some constant M. The first possibility is $|\Gamma_p| < 4C_2\varepsilon$. The cardinality of such Γ_p is $0(1)$ since Γ_p is isolated. Then whether or not Γ_p separates β from α we argue exactly as in the previous two paragraphs when considering $|\beta| \leq 4C_2\varepsilon$. Thus we may assume $|\Gamma_p| \geq 4C_2\varepsilon$.

We now make the additional assumption, to be removed later, *that β is D-close to α*. Then by Lemma 1.2 and induction each Γ_i is D'-close to α for D' depending on D.

If $|\Gamma_p| \leq |\alpha|$ then by Proposition 2.2 and Proposition 2.5 there are $0(1)$ such Γ_p. Again whether or not Γ_p divides we apply the induction hypothesis to β and $Y - \Gamma_p$. Again note if Γ_p does divide, $|\Gamma_p| \geq |\alpha|/M$. Thus we may assume $|\Gamma_p| \geq |\alpha|$. Now let $V(m, \alpha)$ be the set of systems Γ such that

(a) Γ is either a saddle connection or divides $Y - \alpha$ into two components,

(b) $|\Gamma| \leq m|\alpha|$,

(c) Γ is D'-close to α, a universal D',

(d) if $|\Gamma| \geq 4C_2^2\varepsilon$ there is some $t > 0$ such that $\varepsilon/2 \leq v_t(\Gamma) \leq |\Gamma|_t \leq \varepsilon$ and if Γ is not a singleton, some $t' \geq t$ such that $|\Gamma|_{t'} \leq C_2^2\varepsilon$ and $\Gamma_{t'}$ cannot be $C_2\varepsilon$-extended. if $4C_2\varepsilon \leq |\Gamma| \leq 4C_2\varepsilon$ then either Γ is isolated or there is some t as above, and if Γ is not a singleton, some t' as above.

if $|\Gamma| \leq 4C_2\varepsilon$ and Γ is not a singleton, there is some $t' \geq 0$ as above.

(e) for each Γ determining a sequence $\Gamma = \Gamma_1, \ldots, \Gamma_p$ by Proposition 1.11, $|\Gamma_p| \geq |\alpha|$.

Claim. card $V(m, \alpha) = 0(m(\log m)^k)$.

To prove the claim, for any $\Gamma_0 \in V(m, \alpha)$ consider the sequence $\Gamma_0, \ldots, \Gamma_p$ constructed in Proposition 3.1. The proof is by induction on the length of the sequence. Let γ_i be the longest curve on Γ_i. If $p = 0$, Γ_0 is already isolated at time t. Then card $\{\gamma_0\}$ is $0(m)$ by Proposition 2.1 and Card $\{\Gamma_0\}$ is $0(m)$ by Proposition 2.5.

The induction hypothesis is card $\{\Gamma_0 \in V(m)\}$ is $0(m(\log m)^k)$ for the set of Γ_0 that determine sequences $\Gamma_0, \ldots, \Gamma_j$, $j \leq p - 1$ of length $\leq p$. Now suppose Γ_0 determines a sequence $\Gamma_0, \ldots, \Gamma_p$ of length $p+1$. Then $\sigma^{l_1+1} m|\alpha| \leq |\Gamma_1| \leq \sigma^{l_1} m|\alpha|$ where $M_1 \leq l_1 \leq M_2 \log m_1$ the last inequality by the assumption on $|\Gamma_p|$ and Proposition 3.1(i).

Now $\Gamma_1 \in V(\sigma^{l_1} m, \alpha)$ by Proposition 3.1. Now Γ_1 determines a sequence $\Gamma_1, \ldots, \Gamma_p$ of length p. By the induction hypothesis

$$\text{card}\,\{\Gamma_1\} = 0(\sigma^{l_1} m(\log(\sigma^{l_1} m))^k).$$

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By Proposition 3.1(vii) there is a $\gamma_0' \in \Gamma_0$ such that γ_0' is not C_5-separated from Γ_1 at time t_1. If $t_1 > 0$, $v_{t_1}(\gamma_0') \geq \frac{1}{2} h_{t_1}(\gamma_0')$ as well. This implies for $t_1 > 0$

$$|\gamma_0'| \geq e^{t_1/2} v_{t_1}(\gamma_0') \geq M\, e^{t_1/2} |\gamma_0'|_{t_1} \geq M\, e^{t_1/2} |\Gamma_1|_{t_1} / C_5$$
$$\geq M\, e^{t_1/2} (h_{t_1}(\Gamma_1) + v_{t_1}(\Gamma_1)) \geq M |\Gamma_1|$$

for a set of different constants M. If $t_1 = 0$, $|\gamma_0'| \geq M|\Gamma_1|$ by Proposition 1.11 directly. Thus $\sigma^{j_0+1} m|\alpha| \leq |\gamma_0'| \leq \sigma^{j_0} m|\alpha|$ where $M_4 \leq j_0 \leq M_3 l_1$. Moreover if $t_1 > 0$, $v_{t_1}(\gamma_0') \geq \frac{1}{2} h_{t_2}(\gamma_0')$ implies $v_{t_1}(\gamma_0') \geq |\gamma_0'| t_1 / C_3$. We are now in a position to apply first Lemma 3.2 and then the induction hypothesis on the number of curves.

First by Lemma 3.2 there are $s \leq C_1$ pairs (θ_i, t_i) such that for each such γ_0' there is a (θ_i, t_i) such that

$$\frac{|\gamma_0'|_{\theta_i, t_i}}{|\Gamma_1|_{\theta_i, t_i}} \text{ is } 0\left(\frac{|\gamma_0'|}{|\Gamma_1|}\right) = 0(\sigma^{j_0 - l_1})$$

and γ_0' is not M_7-separated from Γ_1 on Y_{θ_i, t_i}. Renormalize each such Y_{θ_i, t_i} to have unit area. Fewer than r_0 saddle connections can be added to Y_{θ_i, t_i}. By the induction hypothesis on the number of curves,

$$\text{card}\{\gamma_0' | \Gamma_1\} \text{ is } 0\left(\frac{|\gamma_0'|_{\theta_i, t_i}}{|\Gamma_1|_{\theta_i, t_i}}\left(\log \frac{|\gamma_0'|_{\theta_i, t_i}}{|\Gamma_1|_{\theta_i, t_i}}\right)^k\right) = 0(\sigma^{j_0 - l_1}(l_1 - j_0)^k).$$

Now by Proposition 2.5,

$$\text{card}\{\Gamma_0 | \Gamma_1\} = 0(\sigma^{-j_0}) 0(\sigma^{j_0 - l_1})(l_1)^k = 0(\sigma^{-l_1} l_1^k).$$

To find card $\{\Gamma_0\}$ we multiply this quantity by card $\{\Gamma_1\}$ and sum over the possible lengths for Γ_1 and γ_0'. That is,

$$\text{card}\{\Gamma_0\} = \sum_{l_1 = M_1}^{M_2 \log m} \sum_{j_0 = M_3}^{M_4 \log l_1} 0(\sigma^{-l_1} l_1^k) \sigma^{l_1} m (\log \sigma^{l_1} m)^k$$
$$= 0(m(\log m)^k)(\text{different } k).$$

This proves the claim.

To calculate $\{\beta\}$ we proceed in much the same way. For each β consider the sequence $\beta = \Gamma_0, \ldots, \Gamma_p$ determined by Proposition 3.1. We have $\sigma^{l_1+1} m|\alpha| \leq |\Gamma_1| \leq \sigma^{l_1} m|\alpha|$ where $m_1 \leq l_1 \leq M_2 \log m$ and $\Gamma_1 \in V(\sigma^{l_1} m, \alpha)$. By the claim

$$\text{card}\{\Gamma_1\} = 0(\sigma^{l_1} m (\log \sigma^{l_1} m)^k).$$

To calculate card $\{\beta | \Gamma_1\}$ we proceed in exactly the same way. On the surface bounded by Γ_1 containing β renormalize so the area is 1. By Lemma 3.2 and Proposition 3.1 there are $s \leq C_1$ pairs (θ_i, t_i) such that for each β there is an (θ_i, t_i) such that

$$\frac{|\beta|_{\theta_i, t_i}}{|\Gamma_1|_{\theta_i, t_i}} \text{ is } 0\left(\frac{|\beta|}{|\Gamma_1|}\right) = 0(\sigma^{-l_1})$$

and β is not M_6-separated from Γ_1 with respect to φ_{θ_i, t_i}. Fewer than r_0 curves can be added to the subcomplex bounded by Γ_1. Thus

$$\text{card}\{\beta | \Gamma_1\} = 0(\sigma^{-l_1}(\log \sigma^{-l_1})^k) = 0(\sigma^{-l_1} l_1^k).$$

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Thus

$$\text{card}\,\{\beta\} = \sum_{l_1=M_1}^{M_2 \log m} 0(\sigma^{-l_1} l_1^k \sigma^{l_1} m(\log \sigma^{l_1} m)^k) = 0(m(\log m)^k)$$

for a different k. This completes the proof under the assumption β is D-close to α.

To complete the proof of Theorem 4.1 we need to consider β not D-close to α. For any such β we apply Proposition 1.6 to find a sequence $\beta = \beta_0, \ldots, \beta_n = \alpha$, $n \le C_1$, with the property that β_i is not M_1-separated from β_{i+1} and is D-close to it.

The proof is now by induction on n. If $n = 1$, β is D-close to α for which we have the result. Assume the result is true for sequences of length $n \le n_0 + 1$. Then $\beta_1, \ldots, \beta_{n_0} = \alpha$ is a sequence of length n_0. Suppose $\sigma^{l+1} m|\alpha| \le |\beta_1| \le \sigma^l m|\alpha|$.

Since $|\beta_i| \ge |\beta_{i+1}|/M_1$ for each i, we have $|\beta_1| \ge |\alpha|/M_2$ for some M_2. Thus $M_4 \le l \le M_3 \log m$ for universal M_3, M_4. By the induction hypothesis on the length of the sequence,

$$\text{card}\,\{\beta_1\} = 0(\sigma^l m(\log \sigma^l m)^k)$$

and since β_0 D-close to β_1 and not M_1-separated from it

$$\text{card}\,\{\beta_0/\beta_1\} = 0(\sigma^{-l}(\log \sigma^{-1})^k)$$

since we have proved the theorem in that case. Thus

$$\text{card}\,\{\beta_0\} = \text{card}\,\{\beta_0/\beta_1\}\ \text{card}\,\{\beta_1\} = \sum_{l=M_4}^{M_3 \log m} 0(\sigma^{-l} l^k \sigma^l m(\log \sigma^l m)^k) = 0(m(\log m)^k.$$

The proof is complete.

Proof of Theorem 1. The number of saddle connections on X of length $\le 4C_2\varepsilon$ is finite. Thus we may assume $|\beta| \ge 4C_2\varepsilon$. Similarly in what follows we need only consider systems $\ge 4C_2\varepsilon$ in length. Motivated by Proposition 3.1, for any m and just as in the proof of Theorem 4.1 let $V(m)$ be the set of systems Γ such that

(a) Γ is either a saddle connection or divides X surface into two components,

(b) $4C_2\varepsilon \le |\Gamma| \le m$,

(c) for some (θ, t) $\varepsilon/2 \le v_{\theta,t}(\Gamma) \le |\Gamma|_{\theta,t} \le \varepsilon$,

(d) for some $t' \ge t$, if Γ is not a singleton, $|\Gamma|_{\theta,t'} \le C_2\varepsilon$ and cannot be $C_2\varepsilon$-extended.

Claim. card $V(m) = 0(m^2)$. To prove the claim, for any $\Gamma_0 \in V(m)$ consider the sequence $\Gamma_0, \Gamma_1, \ldots, \Gamma_p$ constructed in Proposition 3.1. The proof is by induction on the length of this sequence.

Let γ_i be the longest curve on Γ_i. If $p = 0$, γ_0 is already isolated at time t. Then card $\{\gamma_0\}$ is $0(m^2)$ by Proposition 2.1 and card $\{\Gamma_0\}$ is $0(m^2)$ by Proposition 2.5.

The induction hypothesis is now: card $\{\Gamma_0 \in V(m)\}$ is $0(m^2)$ for the set of Γ_0 that determine sequences $\Gamma_0, \ldots, \Gamma_j$, $j \le p-1$. Now suppose $\Gamma_0 \in V(m)$ by Proposition 3.1 defines a sequence $\Gamma_0, \ldots, \Gamma_p$ of length $p+1$. Then $\sigma^{l_1+1} m \le |\Gamma_1| \le \sigma^{l_1} m$ where $M_2 \le l_1 \le M_1 \log m$.

Now either $|\Gamma_1| \le 4C_2\varepsilon$ or $|\Gamma_1| > 4C_2\varepsilon$ in which case $\Gamma_1 \in V(\sigma^{l_1} m)$ and Γ_1 defines a sequence $\Gamma_1, \ldots, \Gamma_p$ of length p. In the first case card $\{\Gamma_1\}$ is $0(1)$. In the second the induction hypothesis applies to give

$$\text{card}\,\{\Gamma_1\} = 0(\sigma^{2l_1} m^2).$$

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By Proposition 3.1 there is a $\gamma_0 \in \Gamma_0$ not C_5-separated from Γ_1 at time $t = t_1$ and satisfying $v_{t_1}(\gamma_0) \geq \frac{1}{2} h_{t_1}(\gamma_0)$. Suppose

$$\sigma^{l_0+1} m \leq |\gamma_0| \leq \sigma^{l_0} m.$$

As in the proof of Theorem 4.1 we have $0 \leq l_0 \leq M_3 l_1$. We conclude by Lemma 3.2 that there are $s \leq C_1$, pairs (θ_i, t_i) such that for any γ_0 there is a pair (θ_i, t_i) such that

$$\frac{|\gamma_0|_{\theta_i, t_i}}{|\Gamma_1|_{\theta_i, t_i}} \text{ is } 0\left(\frac{|\gamma_0|}{|\Gamma_1|}\right) = 0(\sigma^{l_0 - l_1}).$$

On the surface Y_{θ_i, t_i} renormalize the complex with boundary Γ_1 so it has area 1. Then by Theorem 4.1,

$$\text{card}\,\{\gamma_0 | \Gamma_1\} = 0(\sigma^{l_0 - l_1}(l_1 - l_0)^k)$$

and so

$$\text{card}\,\{\Gamma_0 | \Gamma_1\} = 0(\sigma^{l_0 - l_1}(l_1 - l_0)^k \sigma^{-l_0}) = 0(\sigma^{-l_1}(l_1 - l_0)^k)$$

by Proposition 2.5. Thus

$$\begin{aligned} \text{card}\, V(m) &= \sum_{l_1 = M_2}^{M_1 \log m} \sum_{l_0 = 0}^{M_3 l_1} \text{card}\,\{\Gamma_0 | \Gamma_1\}\,\text{card}\,\{\Gamma_1\} \\ &= \sum_{l_1 = M_2}^{M_1 \log m} \sum_{l_0 = 0}^{M_3 l_1} 0(\sigma^{l_1} m^2 (l_1 - l_0)^k) \\ &\leq 0\left(m^2 \sum_{l_1 = M_2}^{M_1 \log m} \sigma^{l_1} l_1^k\right) = 0(m^2) \end{aligned}$$

since $\sum_{l = M_2}^{\infty} \sigma^l l^k$ converges for fixed k, M_2. This proves the claim.

Finally we compute card $S_\sigma(n)$. Card $S_\sigma^\varepsilon(n)$ is $0(n^2)$ so we may assume $\beta \in S_\sigma(n)$ is not ε-wide. We may assume $|\beta| \geq 4C_2\varepsilon$. For each β we form the sequence $\beta = \Gamma_0, \Gamma_1, \ldots, \Gamma_p$ given by Proposition 3.1. Then $\Gamma_1 \in V(\sigma^{l_1} n)$ where $M_2 \leq l_1 \leq M_1 \log n$. By the claim, card $V(\sigma^{l_1} n)$ is $0(\sigma^{2l_1} n^2)$. Then by Lemma 3.2 and Theorem 4.1 just as in the proof of the claim,

$$\text{card}\,\{\beta | \Gamma_1\} \text{ is } 0(\sigma^{-l_1}) l_1^k.$$

Then card $\{\beta\}$ is $\sum_{l_1 = M_2}^{M_1 \log n} \sigma^{-l_1} l_1^k 0(\sigma^{2l_1} n^2)$ which is $0(n^2)$ since $\sum_{l_1 = 1}^{\infty} \sigma^{l_1} l_1^k < \infty$. □

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