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CR Extension from Manifolds of Higher Type

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Abstract. This paper deals with the extension of CR functions from a manifold $M \subset \mathbb{C}^n$ into directions produced by higher order commutators of holomorphic and antiholomorphic vector fields. It uses the theory of complex "sectors" attached to real submanifolds introduced in recent joint work of the authors with D. Zaitsev. In addition, it develops a new technique of approximation of sectors by smooth discs.

1 Notations, Generalities, and Statements

Let M be a real submanifold of $\mathbb{C}^N$ of codimension l in a neighborhood of a point p_o. We assume throughout the paper that M is generic which means that its tangent plane $T_{p_o}M$ is not contained in any complex proper subspace of $\mathbb{C}^N$. A wedge W in $\mathbb{C}^N$ is a domain which, for an open cone Γ and a neighborhood B of p_o, satisfies,

$$((M \cap B) + \Gamma) \cap B \subset W. \tag{1.1}$$

The maximal cone such that (1.1) holds for every proper subcone Γ and suitable B is invariant under $T_{p_o}M$ and can therefore be identified with a cone Γ in the normal space $(T_M\mathbb{C}^N)_{p_o}$, the so called "directional" cone of W at p_o.

We deal with the space CR_M of continuous CR functions on M, that is, the solutions f of the equation $\bar\partial_M f = 0$ where $\bar\partial_M f$ denotes the component of $\bar\partial f$ tangential to M. (When f is not C^1 the equation $\bar\partial_M f = 0$ must be understood in the sense of currents.) A large class of CR functions is described as "topological" boundary values. Thus, if F is a holomorphic function on a wedge W with edge M, continuous up to M, then its boundary value $f = b(F)$ is a CR function on M due to $\bar\partial_M f = b(\bar\partial F)\,(= 0)$. Note that by the Ajrapetyan–Henkin edge of the wedge theorem [1], there is a maximal directional cone Γ for wedge extendibility of $f = b(F)$. In particular, if we denote by Γ^* the polar of this maximal cone, we can meaningfully define the analytic wave front set of $b(F)$ by $WF(b(F))_{p_o} = -\Gamma^*$. The notion of wave front set for CR functions more general than just boundary values requires heavy microlocal machinery [5] and goes beyond the purpose of this presentation. We write complex coordinates as $(z, w) \in \mathbb{C}^l \times \mathbb{C}^n = \mathbb{C}^N$, $z = x + iy$, and suppose that M is defined in a neighborhood of $p_o = 0$ by a system of equations $y_j = h_j$ $j = 1, \dots, l$ with $h(0) = 0$ and $\partial h(0) = 0$; we also write $r = (r_j)_j = (-y_j + h_j)_j$. Select one of the w-coordinates, say w_1, and define $\tilde M := M \cap (\mathbb{C}^l_z \times \mathbb{C}^1_{w_1} \times \{0\} \times \cdots)$. We decompose l as $l = l_1 + \cdots + l_r$, write $I_1 = (1, \dots, l_1), \dots, I_r =$

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$(\sum_{j\leq r-1} l_j, \dots, l)$, and decompose z as $z = (z_{I_1}, \dots, z_{I_r})$. For a set of integers $m_1 < \cdots < m_r$, where m_r is possibly $+\infty$, we define the notions of *weighted* homogeneity and vanishing order. For a function $g = g(x_{I_1}, \dots, x_{I_j}, w_1)$, with $j \leq r$, we say that g is homogeneous of weight m_j when $h(t^{m_1}x_{I_1}, \dots, t^{m_j}x_{I_j}, tw_1)$ is a homogeneous polynomial in t of degree m_j. We say that g is infinitesimal of weight m_j, and write $h = \mathcal{O}^{m_j}$, when $g(t^{m_1}x_{I_1}, \dots, t^{m_j}x_{I_j}, tw_1) = O(t^{m_j})$. A special definition is needed for $j = r$ and $m_r = +\infty$. In this case we say that g is infinitesimal of weight $+\infty$, and write $g = \mathcal{O}^{+\infty}$, when $g(t^{m_1}x_{I_1}, \dots, t^{m}x_{I_r}, tw_1) = O(t^m)$ for any m. In other words, g is divisible by some monomial in x_{I_r}. We recall the Bloom–Graham normal form for equations of $\tilde{M}$ [7]. Intrinsically associated with $\tilde{M}$ there are integers $m_1 < \cdots < m_r$, the so called Hörmander numbers, and $l_1, \dots, l_r$ with $\sum_j l_j = l$, their respective multiplicities. For $m_r < +\infty$, in suitable coordinates at p_o, $\tilde{M}$ is described by the following equations.

$$
\begin{aligned}
y_{I_1} &= P_{I_1}(w_1) + \mathcal{O}^{m_1+1},\\
y_{I_2} &= P_{I_2}(x_{I_1}, w_1) + \mathcal{O}^{m_2+1},\\
&\dots\\
y_{I_r} &= P_{I_r}(x_{I_1}, \dots, x_{I_{r-1}}, w_1) + \mathcal{O}^{m_r+1},
\end{aligned} \tag{1.2}
$$

with each P_{I_j} homogeneous of degree m_j and such that for any $\xi^o \in \mathbb{R}^{l_j}$, $\langle \xi^o, P_{I_j}\rangle$ is not $\tilde{M}$-pluriharmonic. (A homogeneous polynomial g of weight m_j is said to be $\tilde{M}$-pluriharmonic of weight m_j if there exists F holomorphic in $\mathbb{C}^{l+1}$ such that $g = \operatorname{Im} F|_{\tilde{M}} + \mathcal{O}^{m_j+1}$.) When $m_r = +\infty$, for any m there are coordinates such that (1.2) holds with the last equation replaced by $y_{I_r} = \mathcal{O}^m$. Then $\tilde{M}$ is said to be of *finite type* when $m_r < +\infty$, and $\tilde{M}$ is *semirigid* when each P_{I_j} is a function of w_1 only. The similar notions of finite type and semirigidity for M instead of $\tilde{M}$ apply when one deals with equations of type (1.2) involving all w-variables instead of w_1 only. We will see in §3 that finite type can be characterized by means of brackets instead of normal equations: iterated commutators of vector fields tangential to M, of $(1,0)$ and $(0,1)$ type, up to a certain finite number, the highest Hörmander number m_r, span the whole complexified tangent bundle $\mathbb{C} \otimes_{\mathbb{R}} TM$. Let us recall that if M is of finite type, then according to Tumanov [13], CR functions f are boundary values $f = b(F)$ of holomorphic functions F on a wedge W; in particular, in this situation, the notion of wave front set applies to any f.

Theorem 1.1 *Let M be a generic manifold of $\mathbb{C}^N$ of finite type, and, for a choice of a complex tangent direction w_1, let (1.2) be a normal system of equations for $\tilde{M} = M \cap (\mathbb{C}^l_z \times \mathbb{C}^1_{w_1} \times \{0\} \times \cdots)$. We assume that for some j, for $\xi^o \in \mathbb{R}^{l_j+\cdots+l_r}$ and with the notation $P := \langle \xi^o, P_{I_j}\rangle$ we have*

$$
\begin{cases} P = P(w_1) \text{ for a homogeneous polynomial } P \text{ of degree } m_j,\\ P(w_1) \geq 0 \text{ for } w_1 \text{ in a sector } \mathcal{S} \text{ of angle} > \frac{\pi}{m_j}. \end{cases} \tag{1.3}
$$

Then $\xi^o \notin WF(f)$ for all $f \in CR_M$.

The proof will follow in §2. The first of (1.3) is a sort of semirigidity in direction w_1 and codirection ξ^o. We will exhibit in §4 (Proposition 4.2 and Corollary 4.3) a large class of hypersurfaces M for which, when (1.3) is violated, we can find a barrier that is a holomorphic function F with $M \subset \{\operatorname{Im} F < 0\}$. In particular, for these M, there always exist CR functions $f \in CR_M$ such that $\xi^o \in WF(f)$ for $\xi^o = d(\operatorname{Im} F)$. This shows that the statement in Theorem 1.1 is sharp.

Remark 1.2. When $j = 1$ the first of (1.3) is automatically fulfilled. Also, since we are assuming that P_{I_1} is not $\tilde{M}$-harmonic, it is divisible by $|w_1|^2$ and therefore it has at most $2m_1 - 2$ zeroes on the unit circle $|w_1| = 1$. In particular, for either of $\pm P$ the second of (1.3) is satisfied.

Remark 1.3. There is a sort of hierarchy between the Hörmander numbers m_j whose geometric meaning will be fully clear from the proof in §2. According to it, (1.3) for $j > 1$ gives the control not of the whole $\operatorname{WF}(f)$, but only of its section $\operatorname{WF}(f) \cap \big(\{0\} \times \cdots \times \{0\} \times \mathbb{R}^{l_j+\cdots+l_r}\big)$. In fact, the proof of the theorem will consist in proving CR extension in some extra direction v close to the component normal to M of the disc attached to M over $\mathcal{S}$, and (1.3) does not give information for v itself but for $v_{I_j,\ldots,I_r}$.

When M is of finite type and semirigid (in the complex of its arguments w), our proof provides an alternative proof of the extension of any f to a wedge W. The first conclusion in this direction is due to [5], where a description of W is also given. We improve this description by specifying the vanishing order in a precise direction w_1. Also, the semirigidity in the first of (1.3) can be released, as well the hypothesis that the equations are in canonical form as in (1.2). What is indeed essential is the weighted vanishing order; non-$\tilde{M}$-harmonicity in the homogeneous terms is not needed. Thus, suppose that M is of finite type and that $\tilde{M} = M \cap (\mathbb{C}^l \times \mathbb{C} \times \{0\} \times \cdots)$ has equations in the (not necessarily normal) form $y_{I_j} = h_{I_j}$ with $h_{I_j} = \mathcal{O}^{m_j}$, $j = 1, \ldots, r$.

Theorem 1.4 *In the above situation suppose that for $j \le r$ with $m_j < +\infty$ and for some $\xi^o \in \mathbb{R}^{l_j+\cdots+l_r}$, we have for a suitable constant c*

(1.4) $\langle \xi^o, h_{I_j} \rangle > 0$ *for w_1 in a sector $\mathcal{S}$ of angle at least π/m_j and for $|x_{I_i}| < c|w_1|^{m_i}$.*

Then $\xi^o \notin WF(f)$ for all $f \in CR_M$.

(If $m_r = +\infty$ in (1.4), the condition $|x_{I_r}| < |w_1|^{m_r}$ means $|x_{I_r}| < |w_1|^m$ for all m.) The proof follows in §2. If $h_{I_j} = P_{I_j} + \mathcal{O}^{m_j+1}$, then clearly the components of h_{I_j} have the same sign as those of P_{I_j} under the constraint $|x_{I_i}| \le c|w_1|^{m_i}$; hence the second of (1.3) implies (1.4). This shows that Theorem 1.1 is a particular case of Theorem 1.4.

There are two main streams of CR extension: unspecified extension through *minimality*; extension in Levi or higher type directions. As for the first, it was completely solved by Tumanov [13] (see Trepreau [12] in case M is a hypersurface). He introduced the notion of *minimality* of M as the absence of proper submanifolds $S \subset M$

with the same CR structure as M, that is, $T^{\mathbb{C}}S = T^{\mathbb{C}}M|_S$. Note that if M is of finite type, then it is minimal. (First, finite type, in the sense that $m_r < +\infty$ in a system of normal equations, is equivalent to finite bracket type according to the subsequent discussion of §3. But then the presence of S as above would force all brackets to belong to $\mathbb{C} \otimes_{\mathbb{R}} TS$.) He then proved that if M is minimal, then there exist arbitrarily small discs of defect 0 and hence endowed with infinitesimal deformations which span all normal directions to M. Collecting all these directions by the edge of the wedge theorem of [1], he got a common wedge W to which all CR functions are forced to extend. As for the necessity of minimality for such an extension, this is a simpler result (and even elementary if M is a hypersurface). However, a precise description of W has not yet been found. Our paper aims at this attempt and deals with extension in directions produced by higher type commutators. Let us briefly recall the related literature. The first theorems go back to Ajrapetyan–Henkin [1] and Boggess–Polking [10] and state extension in directions of the Levi cone. Next, Boggess–Pitts [9] proved extension in directions shown by iterated brackets up to the first Hörmander number. More recently, in collaboration with Zaitsev, the authors obtained generalizations to the case of CR functions not defined on the whole M but, instead, on a subwedge $V \subset M$. Let us point out the main novelties of the present paper. In (1.2) the weighted homogeneity degrees $m_1 < m_2 < \cdots$ are calculated with respect to w_1 and not to the complex of the variables. Also, m_j is not the smallest among the m_i's. On the contrary, most of the other classical CR extension criteria concern the first Hörmander number in all w directions as in [9], or at least in one selected direction as in [5, Theorem 11]. (Let us point out that it seems that the method of [9] also can be adapted to treat this second situation though this is absent from their statements.) The paper by Baouendi and Rothschild, whose conclusions are the closest to ours, does indeed give an extension also related to further Hörmander numbers [5, Theorem 8]. But in this case its method, founded on Fourier calculus, requires an assumption of semirigidity in the complex of the equations and of the arguments w. To explain the difference, let us consider, for instance, the manifold M in $\mathbb{C}^4$, with coordinates (z_1, z_2, w_1, w_2) defined by

$$y_1 = |w_1|^2 + |w_2|^2 + f_1(x_1, x_2, w_1, w_2), \quad y_2 = |w_2|^4 + x_1|w_1|^2 + f_2(x_1, x_2, w_1, w_2),$$

where $f_j = \mathcal{O}^{2j+1}$. Extension in direction $v^1 \sim (1, 0)$ is clear according to all authors. For extension in directions with non-trivial y_2-component, we notice that the method in [5] fails because of the lack of semirigidity. (Also, [9] and [5, Theorem 11] cannot be applied because w_2 appears in the second equation in a higher homogeneity degree than in the first.) However our Theorem 1.1 applies for sectors in the w_2-plane, and yields extension in direction $v^2 = (1 + \sigma(\eta), \eta^2 + o(\eta^2))$ with $\sigma(\eta)$ infinitesimal with η. Our generalization goes also in another direction (though this was already achieved in [6]). We are able to obtain extension in more general situations and to a larger set of directions. Thus, for instance, let M be the manifold in $\mathbb{C}^3$ defined by ($y_1 = |w|^4 + a|w|^2 \operatorname{Re} w^2$, $y_2 = |w|^4$). Boggess–Pitts [9] gives extension for $a > 2$ in directions which satisfy $y_1 > -|y_2|(\frac{a}{2} - 1)$. On the other hand, by the proof of our Theorems 1.1 and 1.4, we have extension when $a > \sqrt{2}$ in directions satisfying $y_1 > -|y_2|(\frac{a}{\sqrt{2}} - 1)$. We refer to §4 for a complete proof of these claims.

2 Proof of Theorems 1.1 and 1.4

2.1 Preliminaries on $\mathcal{F}^\alpha$ Spaces

Let $0 < \alpha < 1$ and denote by $\tau = re^{i\theta}$ the variable in the standard disc Δ. Let us recall [15–17] some basics about attaching analytic discs to M in the subclasses $\mathcal{F}^\alpha$ of the Hölder classes C^α. These are the spaces of real continuous functions $\sigma(\theta)$, $\theta \in [-\pi, \pi]$, that are $C^{1,\alpha}$ out of 0 and for which the following norm is finite:

$$\|\sigma\|_{\mathcal{F}^\alpha} := \|\sigma\|_{C^0} + \|\theta\sigma^{(1)}\|_{C^\alpha}.$$

(Here $\sigma^{(1)}$ denotes the first derivative of σ.) We remark that for $\sigma \in \mathcal{F}^\alpha$ we must have $\theta\sigma^{(1)}|_{\theta=0} = 0$, for otherwise $\theta\sigma^{(1)} \to c \neq 0$, which implies $|\sigma| \geq \log\frac{|c|}{2} + \log|\theta|$, contradicting the boundedness of σ. This shows that $\mathcal{F}^\alpha$ is continuously embedded into C^α. It is easy to check that $\mathcal{F}^\alpha$ is a Banach algebra. Also, if $\sigma_i \in \mathcal{F}^{\alpha_i}$ $i = 1, 2$, then $\sigma_1 \cdot \sigma_2 \in \mathcal{F}^{\alpha_1+\alpha_2}$ for $\alpha_1 + \alpha_2 < 1$, resp. $\sigma_1 \cdot \sigma_2 \in C^{1,\beta}$ with $\beta := (\alpha_1 + \alpha_2) - 1$ for $\alpha_1 + \alpha_2 > 1$. In both cases the multiplication is continuous with values in the respective spaces.

Let T_1 denote the Hilbert transform normalized by the condition $T_1(\cdot)(1) = 0$; it is easy to see that T_1 is a bounded operator in $\mathcal{F}^\alpha$. We come back to our manifold M. We write coordinates in $\mathbb{C}^N \simeq \mathbb{C}^l \times \mathbb{C}^{N-l}$ as (z, w) with $z = x + iy$, choose a distinguished direction, say w_1, and describe $\tilde{M} := M \cap (\mathbb{C}^l_z \times \mathbb{C}^1_{w_1} \times \{0\} \times \cdots)$ by the system of equations $y_{I_i} = h_{I_i}(x, w)$ with $h_{I_i} = \mathcal{O}^{m_i}$. (The Bloom–Graham normal form is not needed.) We consider in $\mathbb{C}^N$ analytic discs $A(\tau) = (z(\tau), w(\tau))$, $\tau \in \Delta$, (the standard disc in $\mathbb{C}$) *attached* to $\tilde{M}$, that is, satisfying $A(\partial\Delta) \subset \tilde{M}$. If we prescribe an analytic function $w_1(\tau)$ $\tau \in \Delta$, the so called *CR component*, and a point $p = (z, w_1)$ with $y = h(x, w_1)$, and look for an analytic completion $z(\tau)$ for $A(\tau) = (z(\tau), w_1(\tau))$ with $A(1) = p$, we are led to Bishop's equation

$$u(\tau) = -T_1 h(u(\tau) + x, w_1(\tau)) \quad \tau \in \partial\Delta. \tag{2.1}$$

In fact if $u(\tau)$ solves (2.1), then if we set $z(\tau) = u(\tau) + iv(\tau) + z$, we obtain that $A(\tau) = (z(\tau), w_1(\tau))$ is holomorphic, $v(\tau) = h(u(\tau), w_1(\tau))$ for all $\tau \in \partial\Delta$, and finally $A(1) = p$. We consider equation (2.1) in the spaces $\mathcal{F}^\alpha$, $\mathcal{F}^{m_i\alpha}$, and $C^{1,\beta}$ for which T_1 is bounded. We also use the composition properties of h_{I_i} for $i \geq j$ with functions in the above classes as stated in [6]. To take advantage of this composition we assume $m_j\alpha > 1$ (and, to be sharp, $\alpha(m_j - 1) < 1$). Here is our main technical tool.

Proposition 2.1 *Let h_{I_i} be of class C^{m_i+3} and satisfy $h_{I_i} = \mathcal{O}^{m_i}$. Then for any ϵ there is δ such that if $\|h_{I_i}\|_{C^{1,\alpha}} < \delta$, $\|w_1\|_{\mathcal{F}^\alpha} < \delta$, $|x| < \delta$, then equation* (2.1) *has a unique solution $u \in \mathcal{F}^\alpha$ with $\|u\|_{\mathcal{F}^\alpha} < \epsilon$. Moreover, $u_{I_1} \in \mathcal{F}^{m_1\alpha}, \dots, u_{I_{j-1}} \in \mathcal{F}^{m_{j-1}\alpha}$, and $(u_{I_j}, \dots, u_{I_r}) \in C^{1,\beta}$ for $\beta = m_j\alpha - 1$. If w_1 depends on some parameter $\lambda \in \mathbb{R}^d$ so that $\lambda \mapsto w_{1\lambda}$, $\mathbb{R}^d \to \mathcal{F}^\alpha$ is C^k for $k \leq m_i$, then also $\lambda, x \mapsto (u_{I_i})_{\lambda x}$, $\mathbb{R}^{d+l} \to C^{1,\beta}$ is C^k. In particular, there exist mixed derivatives in λ, x, and r up to order k and* 1 *respectively, and they commute, that is, $\partial_r\partial^{k'}_{\lambda x}u = \partial^{k'}_{\lambda x}\partial_r u$ for all $k' \leq k$.*

Proof One first solves Bishop's equation (2.1) in the $\mathcal{F}^\alpha$-spaces by the aid of the implicit function theorem. To this end one considers the mapping $F\colon (\lambda, x, w_1, u) \mapsto u - T_1 h(u + x, w_1)$, $\mathbb{R}^d \times \mathbb{R}^l \times \mathcal{F}^\alpha \times \mathcal{F}^\alpha \to \mathcal{F}^\alpha$. Then for the partial Jacobian $\partial_u F$ with respect to u, one has $\partial_u F\colon \dot{u} \mapsto \dot{u} - T_1 \partial_x h \dot{u}$. In particular, if we evaluate at $(\lambda, x, w_1, u) = (0, 0, 0, 0)$, then this is invertible since $\partial_x h|_0 = 0$. The differentiability with respect to the parameters in the space $\mathcal{F}^\alpha$ is also clear in view of [16, Proposition 11].

We show that the components u_{I_i}, $i \geq j$, of the solution to Bishop's equation, as well as their harmonic conjugates v_{I_i}, are in fact in $\mathcal{F}^{m_i \alpha}$ for $i < j$ (resp. $C^{1,\beta}$ for $i \geq j$) with $\beta := m_j \alpha - 1$. We also prove differentiability in the parameters with values in this space. The key point is that the composition $\varphi((1 - \tau)^\alpha)$, and in bigger generality $\varphi(w_1)$ for $w_1 \in \mathcal{F}^\alpha$, $w_1(1) = 0$, with $\varphi = O^{m_i}$, belongs to $\mathcal{F}^{m_i \alpha}$ for $i < j$ (resp. $C^{1,\beta}$ for $i \geq j$). We put $z(\tau) = u(\tau) + iv(\tau) + z$ with $v = T_1 u$ and $z = x + ih(x, w)$, and also write $\tau = e^{i\theta}$ on $\partial\Delta$. We can check that if $z_{I_i}(\tau) \in \mathcal{F}^{k\alpha}$ for $k \leq m_i - 2$, then in fact $z(\tau) \in \mathcal{F}^{(k+1)\alpha}$. In fact, v gains regularity at each step because $h_{I_i} = \mathcal{O}^{m_i}$ together with the fact that if $\sigma(\theta) \in \mathcal{F}^{k\alpha}$ and $\sigma(0) = 0$, then $|\theta|^\alpha \sigma(\theta) \in \mathcal{F}^{(k+1)\alpha}$ due to

$$\left| (|\theta|^\alpha \sigma(\theta))^{(1)} \right| = \left| |\theta|^{\alpha - 1} \sigma(\theta) + |\theta|^\alpha \sigma^{(1)}(\theta) \right| \leq c |\theta|^{(k+1)\alpha - 1}.$$

But the Hilbert transform interchanges the $\mathcal{F}^{(k+1)\alpha}$ regularity from v to u and thus $z(e^{i\theta}) \in \mathcal{F}^{(k+1)\alpha}$. This completes the proof when $i < j$. On the other hand, when $i \geq j$, in order to pass from $\mathcal{F}^{(m_i - 1)\alpha}$ to $C^{1,\beta}$ we must prove that $(\theta^\alpha u)^{(1)} = \theta^{\alpha - 1} u + \theta^{\alpha - 1}(\theta u^{(1)})$ belongs to C^β. But in fact, since both u and $\theta u^{(1)}$ are in $C^{(m_i - 1)\alpha}$ and are 0 at $\theta = 0$, then their product by $\theta^{\alpha - 1}$ is in C^β as one can easily check by the Hardy–Littlewood Lemma. It follows that $(\theta^\alpha u)^{(1)} \in C^\beta$ and hence $\theta^\alpha u \in C^{1,\beta}$. Thus $u(e^{i\theta})$, and hence $z(e^{i\theta})$ itself, is in $C^{1,\beta}$. As for the differentiability on x and on the parameters, it is a variant of [6, Proposition 15] by the same feed-back argument as above. ∎

We can think of the family of discs produced by the above statement as a *deformation* of the disc $A(\tau) \equiv 0$ which is a trivial solution to Bishop's equation. By the next statement we show how it is possible to make infinitesimal deformations of discs which are no longer assumed to be small.

Proposition 2.2 *Let $h_{I_i} \in C^{m_i + 3}$ satisfy $h_{I_i} = \mathcal{O}^{m_i}$, let $\tilde{w}_1(\tau) \in C^{1,\beta}$, $\tilde{w}_1(1) = 0$ be small in $\mathcal{F}^\alpha$ (not necessarily in $C^{1,\beta}$), and let $\tilde{u}(\tau) \in \mathcal{F}^\alpha$ be a solution of Bishop's equation $\tilde{u} = -T_1 h(\tilde{u}, \tilde{w})$; in particular $\tilde{u}_{I_i} \in C^{1,\beta}$ for any $i \geq j$ according to Proposition* 2.1*. Then for any $w_1(\tau)$ with $\|w_1 - \tilde{w}_1\|_{C^{1,\beta}} < \delta$, $|x| < \delta$, there is a unique solution $u \in \mathcal{F}^\alpha$ with $u_{I_i}(\tau) \in C^{1,\beta}$ for all $i \geq j$ of Bishop's equation with $\|u_{I_i} - \tilde{u}_{I_i}\|_{C^{1,\beta}} < \epsilon$ for all $i \leq j$. Moreover, if $\lambda \mapsto (w_1)_\lambda$ is C^k, $k \leq m_i$, then also $\lambda, x \mapsto (u_{I_i})_\lambda$ is C^k.*

Proof In the present situation we define $F\colon \mathbb{R}^d \times \mathbb{R}^l \times C^{1,\beta}$ similarly as in the proof of Proposition 2.1 and wish to prove that $\partial_u F$ is still invertible. For this purpose it is enough to show that $\partial_u h_{I_i}(\tilde{u}, \tilde{w})$ is small in $C^{1,\beta}$-norm. But in fact recall that $|\partial_x h_{I_i}(u, w)| = O(|w|^2)$ and therefore $\|\partial_x h_{I_i}(\tilde{u}, \tilde{w})^{(1)}\|_{C^\beta} \leq c \|\tilde{w}_1\|_{C^\beta} \|\tilde{w}_1^{(1)}\|_{C^\beta} \leq \epsilon$. ∎

2.2 Construction of a Singular Disc Attached to M with Controlled Normal Component

Let us suppose that (1.4) is fulfilled. It is not restrictive to assume that the sector $\mathbb{C}_{w_1}$ where $g \geq 0$ contains $(1-\tau)^\alpha i e_{l+1}$, $\tau \in \Delta$. (Here e_{l+1} is the unit vector of the w_1-plane.) Let α satisfy $\alpha m_j > 1$, $\alpha(m_j - 1) < 1$. For a small real parameter $\eta > 0$ we define $w_1(\tau) = (w_1)_\eta(\tau) := \eta(1-\tau)^\alpha i e_{l+1}$. We attach to $\tilde{M}$ a family of $\mathcal{F}^\alpha$-discs $A(\tau) = A_\eta(\tau)$ whose w_1-component is $w_1(\tau)$. We recall from Section 2.1 that for $i \geq j$, the function $\mathbb{R} \to C^{1,\beta}$, $\eta \mapsto (z_{I_i})_\eta(\tau)$, is C^{m_i}. We also write $z_{I_i}(\tau)$ instead of $(z_{I_i})_\eta(\tau)$, $z_{I_i}(\tau) = u_{I_i}(\tau) + iT_1 v_{I_i}(\tau)$, and finally $A(\tau) = (z(\tau), w(\tau))$. We note that we have $\partial^s_\eta v_{I_i}|_{\eta=0} \equiv 0$, and $\partial^s_\eta u_{I_i}|_{\eta=0} \equiv 0$, for all $s \leq m_i - 1$. This is clear for $s = 0, 1$. If it is true for any $s \leq m_i - 2$, then it is also true for $s = m_i - 1$, due to $h_{I_i} = \mathcal{O}^{m_i}$ by a feed-back procedure. If we then Taylor-expand $\partial_r v_{I_i}$ at $\eta = 0$, we get

$$\partial_r v_{I_i} = \frac{\partial^{m_i}_\eta \partial_r v_{I_i}|_{\eta=0}}{m_i!}\eta^{m_i} + o(\eta^{m_i}). \tag{2.2}$$

By a similar argument we can also prove that

$$|v_{I_i}| \leq c|w_1|^{m_i}, \quad |u_{I_i}| \leq c|w_1|^{m_i}. \tag{2.3}$$

In fact, in the classes $\mathcal{F}^{k\alpha}$ regularity and vanishing order are coincident. Thus the equation $v_{I_i} = h_{I_i}$ gives control of the vanishing order of v_{I_i} which is transferred as regularity to u_{I_i} through the Hilbert transform, and again as vanishing order to v_{I_i}. In this way we can prove that each v_{I_i} and u_{I_i} belongs to $\mathcal{F}^{m_i\alpha}$ (and also to $C^{[m_i\alpha],\{m_i\alpha\}}$) where $[m_i\alpha]$, resp. $\{m_i\alpha\}$, is the integer, resp. fractional, part of $m_i\alpha$. Recall that if ξ_o is, say, the unit vector in the $l' := l_{1+\cdots+l_{j-1}+1}$-direction, we have $\langle \xi_o, h\rangle \geq 0$ if w_1 is in a sector $\mathcal{S}$ of angle greater than $\frac{m_j}{\pi}$ and $|x_{I_i}| \leq c|w_1|^{m_i}$. We first observe that this latter condition $|x_{I_i}| \leq c|w_1|^{m_i}$ is automatically fulfilled by the components $x_{I_i} = u_{I_i}$ of our discs $A(\tau)$, due to (2.3). We show now that $\partial_r v_{l'} < 0$. In fact we have in this situation

$$\langle \xi_o, \partial^{m_j}_\eta v_{l'}\rangle|_{\eta=0} \geq 0 \quad \text{for all } \tau \in \bar{\Delta}.$$

Hence (1.4) yields, through Hopf's Lemma,

$$\langle \xi_o, \partial_r \partial^{m_j}_\eta v_{l'}\rangle|_{\tau=1 \eta=0} = -c < 0. \tag{2.4}$$

By (2.2) we conclude $\langle \xi_o, \partial_r v_{l'}\rangle|_{\tau=1} = -c'\eta^{m_j} < 0$, for any η sufficiently small. We fix such a small η and, by rescaling, we even suppose $\eta = 1$, and we define $v_o = \partial_r v|_{\tau=1}$. According to (2.4) we have $\langle v_o, \xi_o\rangle < -\frac{c}{2}$.

2.3 Polynomial Approximation of $(1-\tau)^\alpha$ in $\mathcal{F}^\gamma(\bar{\Delta})$ for $\gamma < \alpha$

We have the Taylor expansion

$$\begin{aligned}(1-\tau)^\alpha &= 1 - \alpha\tau - \frac{\alpha(1-\alpha)}{2!}\tau^2 - \frac{\alpha(1-\alpha)(2-\alpha)}{3!}\tau^3 + \cdots \\ &= 1 - \sum_{n=1}^{+\infty}\left|\binom{\alpha}{n}\right|\tau^n.\end{aligned} \tag{2.5}$$

We call $S_N = S_N(\tau)$ the partial sum of the series (2.5) for $1 \le n \le N$. Our goal is to prove the following.

Theorem 2.3 *We have $S_N(\tau) \to (1-\tau)^\alpha$ in $\mathcal{F}^\gamma(\bar{\Delta})$ for any $\gamma < \alpha$.*

Before giving the proof of Theorem 2.3, let us recall that

$$\|\sigma\|_{\mathcal{F}^\gamma} = \|\sigma\|_{C^0} + \|(1-\tau)\sigma'\|_{C^\gamma}.$$

Hence we must prove that

$$S_N \to (1-\tau)^\alpha \text{ in } C^0(\bar{\Delta}), \quad (1-\tau)S'_N \to -\alpha(1-\tau)^\alpha \text{ in } C^\gamma(\bar{\Delta}). \tag{2.6}$$

To prove the first of (2.6) we note that since

$$|S'_N(\tau)| \le \sum_{n=1}^N \left|\binom{\alpha}{n}\right| n|\tau|^{n-1} \to \alpha(1-|\tau|)^{\alpha-1},$$

then in particular the partial sums $|S'_N(\tau)|$ are bounded on Δ, uniformly over N, by $\alpha(1-|\tau|)^{\alpha-1}$. In particular, the sequence of the S_N's is uniformly continuous in $\bar{\Delta}$, which yields at once the first of (2.6). As for the second of (2.6) we note that

$$|S''_N| \le \sum_n \left|\binom{\alpha}{n}\right| n(n-1)|\tau|^{n-2} \to \alpha|\alpha-1|(1-|\tau|)^{\alpha-2}.$$

It follows that

$$\begin{aligned}|((1-\tau)S'_N)'| &\le |S'_N| + |1-\tau||S''_N| \\ &\le \alpha(1-|\tau|)^{\alpha-1} + \alpha|\alpha-1|(1-|\tau|)^{\alpha-1} = c(1-|\tau|)^{\alpha-1}.\end{aligned}$$

To conclude the proof of Theorem 2.3 it suffices to use the following one real variable lemma.

Lemma 2.4 *Let $\{f_N\}$ be a sequence of real functions such that for any ϵ, $f_N \to 0$ in $C^0([0,1-\epsilon])$, and*

$$|f'_N| \le c(1-t)^{\alpha-1} \text{ in } [0,1), \tag{2.7}$$

with c independent of ϵ. Then $f_N \to 0$ in $C^\gamma([0,1])$ for any $\gamma < \alpha$.

Proof We have by integration $|f_N(x) - f_N(y)| \le c|x-y|^\alpha$ (for a different c). It follows that for any ϵ and for suitable $\delta = \delta_\epsilon$ we have, when $|x-y| < \delta$

$$\frac{|f_N(x) - f_N(y)|}{|x-y|^\alpha}|x-y|^{\alpha-\gamma} \le |x-y|^{\alpha-\gamma} < \epsilon.$$

On the other hand, if $|x-y| \ge \delta$, then

$$\frac{|f_N(x) - f_N(y)|}{|x-y|^\gamma} \le \delta^{-\gamma}|f_N(x) - f_N(y)| \le \delta^{-\gamma}(|f_N(x)| + |f_N(y)|).$$

Hence it suffices to prove that $f_N \to 0$ in $C^0([0,1])$. By (2.7) $\{f_N\}$ is equicontinuous. Given ϵ, we thus have $|f_N(x) - f_N(\xi)| \le \epsilon$ uniformly on N for any ξ such that $|x-\xi| \le \delta$, in addition to $\sup_{[0,1-\delta]} |f_N| < \epsilon$ for any $N \ge N_\epsilon$. In conclusion, given x, we take $\xi \in [0, 1-\delta]$ with $|x - \xi| < \delta$ and then get for any $N \ge N_\epsilon$

$$|f_N(x)| \le |f_N(x) - f_N(\xi)| + |f_N(\xi)| < \epsilon.$$

This conclude the proof of the lemma. The proof of Theorem 2.3 is also complete. ∎

2.4 Construction of a Smooth Disc Transversal to M and of Its Infinitesimal Deformation

We put $w_N(\tau) = S_N^\alpha(\tau) - S_N(1)$, let u_N be the solution in $\mathcal{F}^\gamma$ to Bishop's equation $u_N = -T_1 h(u_N, w_N)$, and let $z_N = u_N + iv_N$ for $v_N = T_1 u_N$. Let u be the solution to $u = -T_1 h(u, (1-\tau)^\alpha)$, and set $z = u + iv$ for $v = T_1 u$. Since

$$w_N(\tau) \to i(1-\tau)^\alpha \text{ in } \mathcal{F}^\gamma(\bar{\Delta}),$$

and since $w_N(1) \equiv 0$ for all N, $(z_{I_j,\dots,I_r})_N(\tau) \to z_{I_j,\dots,I_r}(\tau)$ in $C^{1,\beta'}(\bar{\Delta})$ by Proposition 2.1. (Clearly we are supposing γ close enough to α so that $\beta' := m_j\gamma - 1 > 0$.) In particular for any ϵ and for large N the discs $A_N = (z_N, w_N)$ are in $C^{1,\beta'}$ and satisfy

$$\partial_r (v_{I_j,\dots,I_r})_N(1)) = v'_o \quad \text{for } |v'_o - v_o| < \epsilon,$$

uniformly in N. Let $\tilde{A} = (\tilde{z}, \tilde{w})$ be one of these discs. We are ready to construct a half-space M_1^+ in a manifold M_1 which contains M and gains one more direction by a deformation of the disc $\tilde{A}$ such that CR functions extend to M^+. For this we consider Bishop's equation $u = -T_1 h(u + x, w + \tilde{w})$, for $x \in \mathbb{R}^l$, $w \in \mathbb{C}^n$ with $|x| < \delta$, $|w| < \delta$. According to Proposition 2.2, for any ϵ and for suitable $\delta = \delta_\epsilon$ there is a unique solution u which satisfies $\|u - \tilde{u}\|_{C^{1,\beta'}} < \epsilon$ for $\beta' < \beta := k\alpha - 1$. We write $p = x + ih(x,w), w)$ with $v = T_1 u$, and define $A_p(\tau) = p + (u(\tau) + iv(\tau), \tilde{w}(\tau))$. We also write $I_p = A_p|_{[-1,+1]}$ and define $M_1^+ = \bigcup_p I_p([1-\epsilon, 1])$.

Proposition 2.5 *M_1^+ is a half space in a manifold M_1 of codimension $l-1$ with boundary M and inward conormal v'_o for v'_o close to v_o.*

Proof We consider the mapping

$$\Phi\colon \mathbb{C}^n \times \mathbb{R}^l \times [1-\epsilon, 1] \to V', \ (w, x, r) \to I_p(r) \text{ for } p = (x + ih(x,w), w).$$

By Proposition 2.1, Φ is $C^{1,\beta'}$ in the complex of its arguments (w, x) and r up to $r = 1$, and we have

$$\Phi'_{(0,0,1)}(\mathbb{C}^n \times \mathbb{R}^l \times [1-\epsilon, 1]) = T_pM + \mathbb{R}^+ v'_o.$$

In particular, Φ extends as a $C^{1,\beta'}$ mapping to $\mathbb{C}^n \times \mathbb{R}^l \times [1-\epsilon, 1+\epsilon]$ whose image defines a manifold $M_1 = \Phi(\mathbb{C}^n \times \mathbb{R}^l \times [1-\epsilon, 1+\epsilon])$ which contains M_1^+ and satisfies $T_pM_1^+ = T_pM + \mathbb{R}^+ v'_o$. ∎

2.5 End of Proof of Theorems 1.1 and 1.4

First, we recall again that it suffices to prove Theorem 1.4. In fact, for $h_{I_j} = P_{I_j} + \mathcal{O}^{m_j+1}$ we have that $\langle \xi^o, P_{I_j} \rangle > 0$ for $w_1 \in \mathcal{S}$ implies $\langle \xi^o, h_{I_j} \rangle > 0$ for $w_1 \in \mathcal{S}$ and $|x_{I_i}| \leq c|w_1|^{m_i}$. Hence (1.4) is a consequence of (1.3). Thus, let f be a CR function on M. By the celebrated Baouendi–Trèves approximation theorem [4], f is the uniform limit of polynomials on compact subsets of M. By the maximum principle it extends to all analytic discs whose boundary is contained in this compact set. In particular, it extends to the half-space M_1^+ of §2.4, for this is defined as the union of discs attached to M. On the other end, by [13] it extends to a wedge W with edge M and directional cone, say Γ, since we are assuming that M is of finite type. Thus by [1] it extends to a larger wedge $\widehat{W}$ whose directional cone $\widehat{\Gamma}$ is the convex hull of Γ and v_o' with $\langle \xi^o, v_o' \rangle > 0$. In particular, for any F holomorphic in $\widehat{W}$, we have $\xi^o \notin WF(b(F))$. This completes the proof of Theorem 1.4 and hence also that of Theorem 1.1.

We discuss some complements of our Theorems 1.1 and 1.4. We keep our choice of the w_1-direction, select an index i, suppose $P_{I_i} = P_{I_i}(w_1)$ and (1.3), or suppose (1.4), and define $\Gamma_{w_1,i} = \text{convex hull}\{v_o'\}$, where v_o' ranges through the family of directions produced by Theorem 1.1 or Theorem 1.4 for different directions ξ_o and sectors $\mathcal{S}$. Now we use the Ajrapetyan–Henkin edge of the wedge theorem. In our setting it allows us to state that all different directions of extension produced by Theorem 1.1 or Theorem 1.4, and even those obtained as their convex combinations, can be collected to generate the directional cone of a wedge of extension. Precisely, for any ϵ there is a wedge V' with edge M and directional cone Γ'_{w_1} satisfying $\Gamma'_{w_1} \subset (\Gamma_{w_1})_\epsilon$ and $\Gamma_{w_1} \subset (\Gamma'_{w_1})_\epsilon$ such that CR functions extend from M to V'. (Here $(\,\cdot\,)_\epsilon$ denotes the ϵ conical neighborhood of $(\,\cdot\,)$. Also, in the above situation we will say that the cones Γ_{w_1} and Γ'_{w_1} are ϵ-close.) We can also play with different directions of the w-plane, say w_k. Thus, if we have equations of type $y_{w_k,I_i} = h_{w_k,I_i}$ with $h_{w_k,I_i} = \mathcal{O}^{m_i}$, then through Theorems 1.1 and 1.4 we get directions $v'_{w_k,i}$, which we collect in a cone

$$\Gamma := \sum_{k,i} \Gamma_{w_k,i}.$$

For this cone Γ we have the following.

Proposition 2.6 *For any ϵ there is a wedge V' with edge M and directional cone Γ' which is ϵ-close to Γ, such that CR functions extend from M to V'.*

As already mentioned, the proof is an immediate consequence of Theorems 1.1 and 1.4 with the aid of the Ajrapetyan–Henkin edge of the wedge theorem. Now we want to discuss the dimensions of Γ_{w_k} and Γ. Since we are dealing with various directions w_k, we write m_{i,w_k}, l_{i,w_k} from now on. We have the following.

Proposition 2.7 *Assume that the equations $y_{w_k,I_i} = h_{w_k,I_i}$ of $\tilde{M} = \mathbb{C}^l \times \mathbb{C}^1_{w_k}$ satisfy $h_{w_k,I_i} = \mathcal{O}^{m_i}$ and that h_{w_k,I_i} is not $\tilde{M}$-harmonic. Then* $\dim(\Gamma_{w_k}) = \sum_i l_{w_k,i}$.

Proof We first prove that $\dim(\Gamma_{w_k,1}) = l_{w_k,1}$. We write $h_{w_k,1} = P_{w_k,1}(x, w_k) + \mathcal{O}^{m_{w_k},i}+1$ and know from the hypotheses that for any $\xi_o \in \mathbb{R}^{l_{w_k,1}}$, $\langle \xi_o, h_{w_k,I_1}(\tau w_k, x) \rangle$ is non-harmonic. In particular it is divisible by $|\tau|^2$ and hence, being of degree $m_{w_k,1}$, it has at most $2(m_{w_k,1}-2)$ zeroes on the unit circle $|\tau|=1$. In particular, there is a sector of angle at least $\pi/(m_{w_k,1}-1)$ where it keeps constant sign and thus gives rise to a direction v^o such that $\langle \xi_o, v^o\rangle \neq 0$. If we play with all ξ_o and all corresponding sectors, we conclude that these directions v^o cannot be contained in any proper plane of $\mathbb{R}^{l_{w_k,1}}$.

Now we prove the statement in full generality. For any i we take a system of $l_{w_k,i}$ independent vectors $\xi \in \mathbb{R}^{l_{w_k,i}}$ and of corresponding sectors

$$\mathcal{S}_\xi = \eta_i e^{i\theta_\xi}(1-\tau)^{\alpha_1} w_k, \text{ for } \tau \in \Delta, \text{ with } \alpha_1 \text{ satisfying } \frac{1}{m_{w_k,i}-1} > \alpha_1 > \frac{1}{m_{w_k,i}}.$$

We assume $\eta_1 \ll \eta_2 \ll \cdots \ll 1$. This gives rise to a set of extension directions $v' = v'_{w_k,i,\xi,\mathcal{S}_\xi}$ of the type $v' = (\eta_i^{m_1} v'_{I_1}, \eta_i^{m_2} v'_{I_2}, \dots, \eta_i^{m_i} v'_{I_i}, \eta)$, $\eta \ll \eta_i$ for all i, with the property that for each fixed i:

$$\dim\big(\operatorname{Span}_{\xi,\mathcal{S}_\xi}\{v'_{w_k,i,\xi,\mathcal{S}_\xi}\}\big) = l_{w_1,i}.$$

It is also clear, taking all i and playing with different η_i, that

$$\dim(\operatorname{Span}_{i,\xi,\mathcal{S}_\xi} v'_{i,\xi,\mathcal{S}_\xi}) = \sum_i l_{w_k,i}. \qquad \blacksquare$$

Again, if we play with different directions w_k we have the similar result as Proposition 2.7, that is

$$\dim\Big(\sum_{k,i}\Gamma_{w_k,i}\Big) = \sum_i (\operatorname{rank}\{v'_{w_k,i}\}_k). \tag{2.8}$$

(In this context the assumption that M is of finite type, that is, $m_r < +\infty$ for a system of equations in Bloom–Graham normal form for the whole M, and not just for its $(l+1)$-dimensional sections $\tilde{M}$, because Proposition 2.7 and formula (2.8) mean precisely that $\dim\Gamma = l$.)

3 Hörmander's Numbers of Submanifolds of $\mathbb{C}^N$

Let $T^{1,0}M$ and $T^{0,1}M$ denote the bundles of vector fields tangent to M which are holomorphic and antiholomorphic respectively. Let $T^{\mathbb{C}}M = TM \cap iTM$ be the complex tangent bundle to M; note that its complexification verifies $\mathbb{C}\otimes_{\mathbb{R}} T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$. Note that $\mathbb{C}\otimes_{\mathbb{R}} TM$ is integrable, that is, closed under Lie brackets, but $\mathbb{C}\otimes_{\mathbb{R}} T^{\mathbb{C}}M$ is not, in general. We introduce a finite interpolation between $\mathbb{C}\otimes_{\mathbb{R}} T^{\mathbb{C}}M$ and $\mathbb{C}\otimes_{\mathbb{R}} TM$. We set $\mathcal{L}^1 = \mathbb{C}\otimes_{\mathbb{R}} T^{\mathbb{C}}M$ and denote by $\mathcal{L}^j$ the distribution of vector spaces spanned by Lie brackets of holomorphic and antiholomorphic vector fields of length $\leq j$. Suppose that for an integer $m_1 \geq 2$ we have

$$\mathcal{L}^j_{p_o} = T^{1,0}_{p_o}M \oplus T^{0,1}_{p_o}M \quad \text{for all } j \leq m_1 - 1,\ \mathcal{L}^{m_1}_{p_o} \supsetneqq T^{0,1}_{p_o}M \oplus T^{1,0}_{p_o}M. \tag{3.1}$$

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Let $\dim \mathcal{L}^{m_1}_{p_o}/\mathcal{L}^1_{p_o} = l_1$; in this situation it is usual to refer to m_1 as the *first Hörmander number* of M at p_o, and to l_1 as its *multiplicity*. In case $\mathcal{L}^j = \mathcal{L}^1$ for any j, we set $m_1 = +\infty$ with multiplicity $l_1 = l$. Next we look for $m_2 > m_1$ such that

$$\mathcal{L}^j_{p_o} = \mathcal{L}^{m_1}_{p_o} \text{ for all } j < m_2, \quad \mathcal{L}^{m_2}_{p_o} \neq \mathcal{L}^{m_1}_{p_o},$$

and set $l_2 = \dim\left(\mathcal{L}^{m_2}_{p_o}/\mathcal{L}^{m_1}_{p_o}\right)$; again m_2 is possibly $+\infty$. We continue the above process. We will call M of finite type when commutators span the full $\mathbb{C}\otimes_{\mathbb{R}} T^{\mathbb{C}}_{p_o}M$. Thus the above chain will end with a number $m_r < +\infty$ or $m_r = +\infty$ according to whether or not the type is finite. Now we want to discuss in greater detail the first Hörmander number. By the properties of linearity of commutators, one obtains easily the equivalence of (3.1) to

$$\begin{aligned}&[X_1, [X_2, \dots, [X_{j-1}, X_j] \cdots]] \in T^{1,0}M \oplus T^{0,1}M \\ &\qquad\qquad\qquad \text{for all } X_i \in T^{1,0}M \oplus T^{0,1}M, \text{ for all } j \leq m_1 - 1\end{aligned}$$

$$\begin{aligned}&[X_o^{\epsilon_1}, [X_o^{\epsilon_2}, \dots, [X_o, \bar{X}_o] \cdots]] \notin T^{1,0}M \oplus T^{0,1}M \\ &\qquad\qquad\qquad \text{for some } X_o \text{ and some choice of } X_o^{\epsilon_i} = X_o \text{ or } \bar{X}_o.\end{aligned} \tag{3.2}$$

One proves that commutators $[X_1[X_2, \dots, [X_{j-1}, X_j] \cdots]_{p_o}$ only depend, modulo $\mathbb{C}\otimes_{\mathbb{R}} T^{\mathbb{C}}M$, on the initial values $X_1(p_o), X_2(p_o)\dots$ and not on the choice of the extended sections. This property is referred to as *tensoriality* of the iterated brackets of vector fields. We take a basis of equations $y_j = h_j$, $j = 1, \dots, l$, for M at $z_o = 0$ with $h(0) = 0$ and $\partial h(0) = 0$, and also set $r_j = -y_j + h_j$ and $r = (r_j)$. We identify $\frac{TM}{T^{\mathbb{C}}M} \xrightarrow{\sim} T_M\mathbb{C}^N$ by the complex structure J, and $T_M\mathbb{C}^N \xrightarrow{\sim} \mathbb{R}^l$ by the dual basis to ∂r_j. We look closely at X_o in (3.2); assume, say, $X_o(p_o) = w_o\partial_w$, and denote by p'_o the projection of p_o on the plane of (x, w). We denote by $n-1$ (resp. $m-1$) the occurrences of $X^{\epsilon_j} = X_o$ (resp. $X_o^{\epsilon_j} = \bar{X}_o$) in (3.2). We can prove that

$$\begin{cases} \mathcal{J}\frac{1}{2i}[X_o^{\epsilon_1}, \dots, X^{\epsilon_j}, [X_o, \bar{X}_o], \dots](h)(p'_o) = 0 \quad \text{for all } j < m_1 - 2 \\ \mathcal{J}\frac{1}{2i}[X_o^{\epsilon_1}, \dots, X^{\epsilon_{m_1-2}}, [X_o, \bar{X}_o \dots](h)(p'_o) = -\partial^n_{w_o}\bar{\partial}^m_{w_o} h(p'_o). \end{cases} \tag{3.3}$$

This is a special case of Proposition 3.3. The above relation, together with the fact that harmonic terms can be removed by change of coordinates, makes (3.3) equivalent, in suitable coordinates, to

$$\begin{gathered} \partial^\alpha_w \bar{\partial}^\beta_w h(p'_o) = 0, \text{ for all } |\alpha| + |\beta| \leq m_1 - 1, \\ \partial^\alpha_w h(p'_o) = 0, \quad \bar{\partial}^\alpha_w h(p'_o) = 0, \text{ for all } |\alpha| \leq m_1, \\ \partial^n_{w_o}\bar{\partial}^m_{w_o} h(p'_o) \neq 0 \text{ for } X_o(p_o) = w_o\partial_w \text{ and suitable } n + m = m_1. \end{gathered} \tag{3.4}$$

We also write ∂_{w_o} instead of $w_o\partial_w$ and consider the homogeneous term of lowest degree in the Taylor expansion of h in the w_o-plane:

$$g(\tau w_o) = \sum_{\substack{m+n=k \\ m\geq 1\, n\geq 1}} \partial^m_{w_o}\bar{\partial}^n_{w_o} h(p'_o)\tau^m\bar{\tau}^n.$$

The above polynomial is real homogeneous and has some non-null coefficient on account of the third of (3.4). Hence it has only a discrete set of zeroes for $|\tau| = 1$, that is, for all $\theta \in [0, 2\pi]$ but a discrete set, we have $\sum \partial^m_{w_o} \bar{\partial}^n_{w_o} h(p'_o) e^{i(m-n)\theta} \neq 0$. Sometimes we prefer to use the notation $\tilde{w}_o = e^{i\theta} w_o$ and then write in this notation

$$\sum_{\substack{m+n=m_1\\ m\geq 1\, n\geq 1}} \partial^m_{\tilde{w}_o} \bar{\partial}^n_{\tilde{w}_o} h(p'_o) \neq 0. \tag{3.5}$$

We also denote by v^o the vector in (3.5). We remark that if $\xi_o \in \mathbb{R}^l$ verifies $\langle \xi_o, v^o \rangle \neq 0$, then

$$\langle \xi_o, g(w_o) \rangle \gtrless 0 \text{ in a sector of the plane } \mathbb{C}_{w_o} \text{ of width} \geq \frac{\pi}{m_1 - 2}.$$

In fact, each $g_i(\tau w_o)$ is divisible by $|\tau|^2$, and hence $|\tau|^{-2}\langle \xi, g(\tau w_o)\rangle$ has at most $m_1 - 2$ zeroes for $|\tau| = 1$. Hence (3.3), or its equivalent version (3.4), implies our condition (1.2).

To go further with our discussion, we need to fix better our notations. We fix numbers $m_1 < \cdots < m_r$ (perhaps $m_r = +\infty$) and multiplicities l_i with $\sum_i l_i = l$. We take multiindices $I_1 = (1, \dots, l_1), \dots, I_r = (\sum_{i<r} l_i, \dots, l)$, give *weight* m_i to the x_{I_i} variables, and define the *weighted* vanishing order for a function $f = f(\dots, x_{I_i}, \dots, w)$ by putting $f = \mathcal{O}^{+\infty}$ when $m_r = +\infty$ and f contains some monomial in the x_{I_r}, and, otherwise, putting $f = \mathcal{O}^m$ when $f(\dots, t^{m_i} x_{I_i}, \dots, tw) = \mathcal{O}(t^m)$. We then suppose that the equations of M are presented according to increasing vanishing orders

$$\begin{aligned} y_{I_1} &= h_{I_1},\\ &\vdots\\ y_{I_r} &= h_{I_r}, \end{aligned} \tag{3.6}$$

with $h_{I_i} = \mathcal{O}^{m_i}$ for any i. We point out that this is not necessarily the normal form in the Bloom–Graham sense. In fact we are not assuming that each h_{I_i} is in the form $h_{I_i} = P_{I_i}(x_{I_1}, \dots, x_{I_{i-1}}, w)$ with $\langle \xi, P_{I_i} \rangle$ non M-pluriharmonic for any i and any $\xi \in \mathbb{R}^{l_i}$. (In this situation, *weighted homogeneity* does not serve any purpose.) To carry on our discussion, we need a description of a basis $\{X_j\}$ of vector fields for $T^{1,0}M$. We put $r_{I_i} = -y_{I_i} + h_{I_i}$, $r = {}^t(r_1, \dots, r_l)$, define an $(N-l) \times l$ matrix $A = (a_{jh})$ by $A = -{}^t(\partial_w r)\, {}^t(\partial_z r)^{-1}$, and set $X_j = \sum_{h=1}^l a_{jh}\partial_{z_h} + \partial_{w_j}$. We have

$$\sum_h a_{jh} \partial_{z_h}(r_{I_i}) + \partial_{w_j}(r_{I_i}) = 0 \text{ for all } i = 1, \dots, r. \tag{3.7}$$

Derivation of (3.7) yields

$$\begin{cases} \partial^\beta_{w\bar{w}} \partial^{\alpha_1}_{x_{I_1}} \cdots \partial^{\alpha_{i-1}}_{x_{I_{i-1}}}(a_{j,I_i}) = 0 \quad \text{for } |\beta| + \sum\limits_{j\leq i-1} m_j|\alpha_j| \leq m_i - 2,\\ \sum\limits_h \partial^\beta_{w\bar{w}} \partial^{\alpha_1}_{x_{I_1}} \cdots \partial^{\alpha_{i-1}}_{x_{I_{i-1}}}(a_{j,h}) = -2i\partial^\beta_{w\bar{w}} \partial^{\alpha_1}_{x_{I_1}} \cdots \partial^{\alpha_{i-1}}_{x_{I_{i-1}}} \partial_{w_j}(r_{I_i})\\ \qquad\qquad\qquad \text{for } |\beta| + \sum\limits_{j\leq i-1} m_j|\alpha_j| \leq m_i - 1. \end{cases} \tag{3.8}$$

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Once the equations are ordered as in (3.6), we can introduce for any $i \leq r$ a diagram

$$
\begin{tikzcd}
\frac{TM}{T^{\mathbb{C}}M} \arrow[r, "\varphi_1"] \arrow[d] & \mathbb{R}^l \arrow[d] \\
\frac{TM}{\mathcal{L}^{m_i-1}} \arrow[r, "\varphi_2"] & \mathbb{R}^{l_i+\cdots+l_r},
\end{tikzcd}
\tag{3.9}
$$

where φ_1 is defined by $[v] \mapsto J(v)(\partial r)$ and φ_2 by $[v] \mapsto (Jv)^t(\partial r_{I_i}, \dots, \partial r_{I_r})$. We have to show that φ_2 is well defined (in which case the diagram (3.9) is commutative). To see this, we preliminarily remark that, just by the vanishing condition in (3.6), we have $\{\partial r_{I_i}, \dots, \partial r_{I_r}\}^{\perp_{\mathbb{C}}} = \operatorname{Span}_{\mathbb{R}}\{\partial_w, \bar\partial_w, \partial_{I_1}, \dots, \partial_{I_{i-1}}\}$ (normal form being unessential for this conclusion). Thus our claim is a consequence of the following.

Proposition 3.1 *We have* $\mathcal{L}^{m_{i-1}} \subset \operatorname{Span}\{\partial_w, \bar\partial_w, \partial_{x_{I_1}}, \dots, \partial_{x_{I_{i-1}}}\}$.

Proof We must show that

$$J[X_o^{\epsilon_1}, \dots, [X_o^{\epsilon_{m_i-3}}, [X_o, \bar X_o]\cdots]](r_{I_j})(p_o) = 0 \quad \text{for all } j \geq i \text{ and for any } \epsilon.$$

We recall (3.7) and (3.8) and fix $j = i$. We use the notation $[\,\cdot\,,\,\cdot\,]^k$ to denote brackets of X_o or $\bar X_o$ performed $k-1$ times. We assume, for instance, $X_o^{\epsilon_1} = \sum_h a_{1h}\partial_{z_h} + \partial_{w_1}$, and begin by remarking that

$$
\begin{aligned}
[\,\cdot\,,\,\cdot\,]^{m_i-1} &= \Big[\sum_h a_{1h}\partial_{z_h} + \partial_{w_1}, [\,\cdot\,,\,\cdot\,]^{m_i-2}\Big] \\
&= [\partial_{w_1}, [\,\cdot\,,\,\cdot\,]^{m_i-2}],
\end{aligned}
$$

due to $a_{1h}(p_o) = 0$ and $[\,\cdot\,,\,\cdot\,]^{m_i-2}(a_{1h}) = 0$. Continuing in this way we end up with

$$
\begin{aligned}
&\langle[\partial^\beta_{w_1\bar w_1}, [\sum_h a_{1h}\partial_{z_h} + \partial_{w_1}, \sum_h \bar a_{1h}\bar\partial_{z_h} + \bar\partial_{w_1}]], \bar\partial r_{I_i}\rangle \\
&\quad= \partial^\beta_{w_1\bar w_1}\Big(\sum_h a_{1h}\partial_{z_h}(\bar a_{1I_i}) - \frac{i}{2}\partial_{w_1}(\bar a_{1I_i}) + \sum_h a_{1h}\partial_{z_h}\bar\partial_{w_1}(r_{I_i}) + \partial_{w_1}\bar\partial_{w_1}(r_{I_i})\Big) + \cdots
\end{aligned}
\tag{3.10}
$$

where β is a biindex of length $|\beta| = m_i - 3$ and the dots denote similar terms as the four in the right-hand side of (3.10). Now

$$
\begin{aligned}
\partial^\beta_{w_1\bar w_1}\partial_{w_1}(\bar a_{1I_i}) &= 0 \qquad \text{(by (3.8))}, \\
\partial^\beta_{w_1,\bar w_1}\Big(\sum_h a_{1h}\partial_{z_h}\bar\partial_{w_1}(r_{I_i})\Big) &= \sum_{\gamma+\delta=\beta}\sum_h \partial^\gamma_{w_1,\bar w_1}(a_{1h})\partial^\delta_{w_1,\bar w_1}\partial_{z_h}\bar\partial_{w_1}(r_{I_i}).
\end{aligned}
$$

Thus, if $h \in I_j$ for $j \geq i$, the above term is clearly 0. Otherwise, either $|\gamma| \leq m_h - 2$ and hence $\partial^{\gamma}_{w_1\bar{w}_1}(a_{1h}) = 0$, or else $|\delta| \leq m_i - 2 - m_h$ and hence $\partial^{\delta}_{w_1\bar{w}_1}\partial_{z_h}\bar{\partial}_{w_1}(r_{I_i}) = 0$. By the same reason, we have for the remaining term in (3.10):

$$\partial^{\beta}_{w_1\bar{w}_1}\Big(\sum_h a_{1h}\partial_{z_h}\Big)(\bar{a}_{1I_i}) = 0.$$

Finally, $\partial^{\beta}_{w_1\bar{w}_1}\partial_{w_1}\bar{\partial}_{w_1}(r_{I_i})$ is also 0, again by (3.8). The proof is complete. ∎

Remark 3.2. Note that φ_2 is an isomorphism precisely when we have equality in Proposition 3.1. But this is equivalent to asking that the equations (3.6) be in normal form.

Let us choose a vector field $X_o \in T^{1,0}M$ with $X_o(p_o) = w_o\partial_w$; we will also use the notation ∂_{w_o} instead of $w_o\partial_w$. We have the following.

Proposition 3.3

$$J[X_o^{\epsilon_1}, \dots, [X_o^{\epsilon_{m_i-2}}, [X_o, \bar{X}_o] \cdots]](r_{I_j})(p_o) = 0 \textit{ for all } j > i \textit{ and any } \epsilon.$$

If moreover $\langle \xi_o, h_{I_i}\rangle$, restricted to $\mathbb{C}_{w_o} \times \mathbb{R}^l_x$, is in the form $P + \mathcal{O}^{m_i+1}$ for $P = P(w_o)$ homogeneous of degree m_i with $m_i < +\infty$, then

$$J[X_o^{\epsilon_1}, \dots, [X_o^{\epsilon_{m_1-2}}, [X_o, \bar{X}_o] \cdots]]\langle \xi_o, r_{I_i}\rangle(p_o) = -2\partial^n_{w_o}\bar{\partial}^m_{w_o}(\langle \xi_o, h_{I_i}\rangle)(p'_o).$$

Proof The first statement is a variant of Proposition 3.1. As for the second, in the same way as in the proof of Proposition 3.1, we get for a suitable $|\beta| = m_i - 2$

(3.11)

$$\begin{aligned}
&\langle [X_o^{\epsilon_1}, \dots, [X_o^{\epsilon_{m_1-2}}, [X_o, \bar{X}_o] \dots], \bar{\partial} r_{I_i}\rangle \\
&\quad = \langle \big[\partial^{\beta}_{w_1\bar{w}_1}, \big[\sum_h a_{1h}\partial_{z_h} + \partial_{w_1}, \sum_h \bar{a}_{1h}\bar{\partial}_{z_h} + \bar{\partial}_{w_1}\big]\big], \bar{\partial} r_{I_i}\rangle \\
&\quad = \partial^{\beta}_{w_1\bar{w}_1}\Big(\sum_h a_{1h}\partial_{z_h}(\bar{a}_{I_i}) - \frac{i}{2}\partial_{w_1}(\bar{a}_{I_i}) + \sum_h a_{1h}\partial_{z_h}\bar{\partial}_{w_1}(r_{I_i}) + \partial_{w_1}\bar{\partial}_{w_1}(r_{I_i})\Big) + \cdots,
\end{aligned}$$

where the dots denote similar terms. Now the fourth term disappears by elimination with the terms in the dots (where it appears with opposite sign). The first and third terms are not 0, in general. However, they vanish if we apply vector fields not to the whole r_{I_i} but just to $\langle \xi_o, r_{I_i}\rangle$, because of the hypothesis of *semirigidity* contained in the second statement of the proposition. Thus, for the third term, we have

$$\partial^{\beta}_{w_1\bar{w}_1}\Big(\sum_h a_{1h}\partial_{z_h}\bar{\partial}_{w_1}\Big)\langle \xi_o, r_{I_i}\rangle = \sum_{\gamma+\delta=\beta}\partial^{\gamma}_{w_1\bar{w}_1}\Big(\sum_h a_{1h}\partial^{\delta}_{w_1\bar{w}_1}\partial_{z_h}\bar{\partial}_{z_h}\bar{\partial}_{w_1}\Big)\langle \xi_o, r_{I_i}\rangle.$$

Again, if $|\gamma| \leq m_h - 2$, then $\partial^{\gamma}_{w_1\bar{w}_1}a_{ih} = 0$. If, instead, $|\delta| \leq m_i - 1 - m_h$, then $\partial^{\delta}_{w_1\bar{w}_1}\partial_{z_h}\bar{\partial}_{w_1}\langle \xi_o, r_{I_i}\rangle = \partial^{\delta}_{w_1\bar{w}_1}\partial_{z_h}\bar{\partial}_{w_1}(P + \mathcal{O}^{m_i+1}) = 0$. In the same way one proves that

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the first term in the second line of (3.11) is 0. The only term which survives is the second (which also appears with the same sign in the dots terms). We thus have

$$\begin{aligned} J[X_o^{\epsilon_1}, \dots, [X_o^{\epsilon_{m_1-2}}, [X_o, \bar{X}_o] \cdots]]\langle \xi_o, r_{I_i} \rangle &= -\frac{i}{2}(\partial^\beta_{w_1\bar{w}_1}\partial_{w_1}\langle \xi_o, \bar{a}_{1I_i}\rangle + \cdots) \\ &= -\frac{i}{2}(\partial^\beta_{w_1\bar{w}_1}\partial_{w_1}\bar{\partial}_{w_1}\langle \xi_o, r_{I_i}\rangle \frac{2}{i} + \cdots) \\ &= -2\partial^\beta_{w_1\bar{w}_1}\partial_{w_1}\bar{\partial}_{w_1}\langle \xi_o, r_{I_i}\rangle. \end{aligned}$$

This completes the proof of the proposition. ■

We assume now that for some vector field X_o with $X_o(p_o) = \partial_{w_o}$, for some $\epsilon = (\epsilon_1, \dots, \epsilon_{m_i-2})$, and for some $\xi_o \in \mathbb{R}^{l_i}$, we have

$$[X_o^{\epsilon_1}, \dots, [X^{\epsilon_{m_i-2}}, [X_o, \bar{X}_o] \cdots]] \notin \operatorname{Span}\{\partial_w, \partial_{\bar{w}}, \partial_{x_{I_1}}, \cdot, \cdot, \partial_{x_{I_{i-1}}}\} \tag{3.12}$$

and

$$\langle \xi_o, h_{I_i}\rangle|_{\mathbb{C}_{w_o}\times\mathbb{R}^l_x} = P(w_o) + \mathcal{O}^{m_i+1}. \tag{3.13}$$

It follows that $P(w_o) = |w_o|^2 Q(w_o)$ with Q real homogeneous of degree $m_i - 2$. Since Q has at most $m_i - 2$ zeroes on the circle $|w_o| = 1$,

$$P \gtrless 0 \text{ for } w_o \text{ in a sector of width } > \frac{\pi}{m_i - 2}.$$

Hence we enter in the hypotheses of Theorem 1.4 and conclude that CR functions on M extend to a new direction v^o satisfying $\langle \xi_o, v^o\rangle \gtrless 0$. Note that in that theorem normal equations as in (1.3) are not needed. What is really needed for equations such as (3.6), is to assume $\langle \xi_o, h_{I_i}\rangle = P(w_o) + \mathcal{O}^{m_i+1}$ and $P \geq 0$ (or $P \leq 0$) in a sector $> \frac{\pi}{m_i}$.

Naturally, if the equations are normal, we have the significant simplification that $\mathcal{L}^{m_i} = \operatorname{Span}\{\partial_w, \partial_{\bar{w}}, \partial_{x_{I_i}}, \dots, \partial_{I_{m_{i-1}}}\}$. Thus vector fields X_o which satisfy (3.12) do exist. If for one of them with, say, $X_o(p_o) = \partial_{w_o}$, and for some $\xi_o \in \mathbb{R}^{l_i}$, (3.13) is also satisfied, then Proposition 2.6 yields CR extension to some v^o with $\langle \xi_o, v^o\rangle \gtrless 0$.

4 Comparison with Boggess–Pitts

Let M be a manifold of class C^{k+2} which satisfies (1.1) with g homogeneous of degree k and non M-harmonic (in particular whose first Hörmander number is $m_1 = k$. Remember that in this situation (1.2) is also satisfied. Let v be the direction normal to M given by the formula

$$v = \sum_\epsilon C_\epsilon J[X_o^{\epsilon_1}, X_o^{\epsilon_2}, \dots, [X_o, \bar{X}_o] \cdots]](r)(p_o) \text{ where } C_\epsilon := \frac{1}{\epsilon^+!\epsilon^-!}$$

with ϵ^+ and ϵ^- denoting the occurrences $X_o^{\epsilon_i} = X_o$ and $X_o^{\epsilon_i} = \bar{X}_o$, respectively. Note that the last two occurrences are fixed as $X_o^{\epsilon_{k-1}} = X_o$ and $X_o^{\epsilon_k} = \bar{X}_o$. Let $X_o(p_o) =$

∂_{w_o}. By tensoriality of brackets and by the combinatorial remark that the number of choices of ϵ's which give rise to the same pair of occurrences m, n is $\binom{k-2}{m-1}$, one gets

$$v = \sum_{\substack{m+n=k \\ m\geq 1\, n\geq 1}} \binom{k-2}{m-1} \frac{1}{m!n!} \partial^m_{w_o} \bar{\partial}^n_{w_o} h(p'_o). \tag{4.1}$$

Again, once the complex plane of $X_o(p_o)$ is fixed, in our case the w_o-plane, there might be many vectors $v = v_\varphi$ produced through (4.1) just by replacing w_o by $e^{i\varphi} w_o$. The result by Boggess–Pitts [9] is that for each of these vectors v, one obtains CR extension from M to M' where M' points to a direction v' close to v. We first discuss this extension in case M is a hypersurface of $\mathbb{C}^N$ defined, in coordinates $(z, w) \in \mathbb{C}^1 \times \mathbb{C}^n$, $w = (w_1, w')$, for a pair of even integers k and p with $p \leq k-2$, for a choice of a coefficient $a \geq 0$, and with the notation $w_o = (1, 0, \dots)$, by an equation

$$y_1 = |w_1|^k + a|w|^{k-p} \operatorname{Re} w_1^p + O(|x_1|^2 + |w_1|^{k+1} + |x_1||w_1| + |w||w'|). \tag{4.2}$$

We denote by $g = g(w_1)$ the homogeneous polynomial in the right side of (4.2). With $p_o = 0$ and $X_o = \partial_{w_1}$ and with the notation $k - 2 = p + 2q$, we have extension in directions $v_\varphi = (ic_\varphi, 0, \dots)$ for

$$c_\varphi = \binom{k-2}{\frac{k}{2}-1} + a\cos(p\varphi)\binom{k-2}{p+q}.$$

In particular, if we look for extension *down*, that is for v_φ with negative first component, we have to require $k \geq 4$, $p \geq 2$. Then $v_\varphi < 0$ will occur exactly for $\varphi = \frac{\pi}{p}$ (which yields $\cos(p\varphi) = -1$) and

$$a \geq \frac{(p+q)!q!}{(\frac{k}{2}-1)!(k-1-\frac{k}{2})!}.$$

We compare the above condition with that which is given by sector property. We consider the restriction of g on the unit circle $w_1 = e^{i\theta}$ given by $g(e^{i\theta}) = 1 + a\cos(p\theta)$. It is clear that for any choice of a we have $g \geq 0$ in a sector of width bigger than $\frac{\pi}{p}$ which is in turn bigger than $\frac{\pi}{k}$. Hence by Theorem 1.1 we get holomorphic extension *up*.

If we search, instead, for extension *down*, we can use the following result which generalizes similar conclusions by Baouendi–Trèves [3] concerning the case $k = 4$.

Proposition 4.1 *We have $g < 0$ in a sector of width $> \frac{\pi}{k}$ if and only if*

$$a > \frac{1}{\cos\left(\frac{p\pi}{2k}\right)}.$$

Proof Let $a > 0$; it is clear that $1 + a\cos(p\theta)$ attains its minimum at $\theta = \frac{\pi}{p}$. It is also clear that in order that the sector where $g < 0$ have an angle bigger than $\frac{\pi}{k}$, it is necessary and sufficient that $a\cos(\pi + \frac{p\pi}{2k}) < -1$, which is equivalent to the condition in the statement of the proposition. ∎

We also have the following statement, which shows necessity of sector property for holomorphic extendibility.

Proposition 4.2 *Let p divide k and $a \leq 1/\cos(\frac{p\pi}{2k})$. Then for $b = \frac{p}{k} tg(\frac{p\pi}{2k})$ we have that the trigonometric polynomial $g_1 = 1 - a\cos(p\theta) + b\cos(k\theta)$ verifies $g_1 \geq 0$ for all θ. In particular, if in addition the plane of the w variables has dimension 1, by adding another harmonic term $\epsilon \sin(k\theta)$ we can achieve $g_1(w_1) \geq c_1|w_1|^k$ for $c_1 > 0$.*

Proof Since $a \leq 1/\cos(\frac{p\pi}{2k}t)$, for $g_1 \geq 0$ it will suffice that

$$b\cos(k\theta) \geq \frac{1}{\cos(\frac{p\pi}{2k})}\cos(p\theta) - 1. \tag{4.3}$$

It is clear that it will suffice to take b such that

$$\overset{\cdot}{\overbrace{b\cos(k\theta)}}\big|_{-\frac{\pi}{2k}} = \frac{1}{\cos\left(\frac{p\pi}{2k}\right)}\overset{\cdot}{\overbrace{\cos(p\theta)}}\big|_{-\frac{\pi}{2k}}. \tag{4.4}$$

In fact this choice of b will imply that the derivative on the left of (4.4) dominates (respectively is dominated by) the one on the right in the interval $[-\frac{\pi}{2k}, 0]$ (respectively in $[-\frac{\pi}{k}, -\frac{\pi}{2k}]$). Hence (4.3) holds in the interval $[-\frac{\pi}{k}, 0]$ and also, by symmetry, in the whole interval $[-\frac{\pi}{k}, +\frac{\pi}{k}]$. It also holds trivially in the remaining part of $[-\frac{\pi}{p}, +\frac{\pi}{p}]$. On the other hand, this is a complete cycle of the trigonometric function $1 - a\cos(p\theta) + b\cos(k\theta)$, due to the assumption that p divides k. ∎

Corollary 4.3 *Let M be a hypersurface in $\mathbb{C}^N$ defined by (4.2), and assume that p divides k. If $a \leq 1/\cos(\frac{p\pi}{2k})$, then there are CR functions $f \in CR_M$ which do not extend down.*

Proof In new complex coordinates we can arrange that $M \subset \{y_1 > 0\}$. Since $\{y_1 > 0\}$ is pseudoconvex, the conclusion follows. ∎

The comparison between the conditions of [9] and the sector property is expressed by the following.

Lemma 4.4 *Let $k - 2 = p + 2q$. Then*

$$\frac{(p+q)!q!}{(\frac{k}{2}-1)!(\frac{k}{2}-1)!} > \frac{1}{\cos(\frac{p\pi}{2k})}. \tag{4.5}$$

Proof The most delicate case is when $p = 2$. In this case (4.5) becomes

$$\frac{\frac{k}{2}!(\frac{k}{2}-2)!}{(\frac{k}{2}-1)!(\frac{k}{2}-1)!} > \frac{1}{\cos(\frac{\pi}{k})},$$

or else

$$\frac{\frac{k}{2}}{\frac{k}{2}-1} > \frac{1}{\cos(\frac{\pi}{k})}. \quad ∎$$

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Hence the method of sectors is sharper than that of [9]. In particular it yields extension *down* for an extra range of values of a that is for

$$\frac{1}{\cos(\frac{p\pi}{2k})}\leq a<\frac{(p+q)!q!}{(\frac{k}{2}-1)!(\frac{k}{2}-1)!}.$$

The above conclusions are generalizations of former results by Baouendi–Trèves [3].

We pass now to the higher codimensional case. We discuss CR-extension for $M\subset\mathbb{C}^3$ defined in coordinates (z_1,z_2,w) by the system

$$\begin{aligned}y_1&=|w|^k+a|w|^2\operatorname{Re}w^p+O(|x|^2+|w|^{k+1}+|x||w|),\\ y_2&=|w|^k+O(|x|^2+|w|^{k+1}+|x||w|).\end{aligned}\tag{4.6}$$

We also denote by $g=(g_j)_j$, $j=1,2$ the vector with polynomial entries on the right of (4.6) and, for $\xi\in\mathbb{R}^2$, we use the notation $g_\xi=\langle\xi,g\rangle$. We can express the extension directions v_φ by [9] as

$$v_\varphi=\left(\binom{k-2}{\frac{k}{2}-1}+a\cos(p)\binom{k-2}{p+q},\binom{k-2}{\frac{k}{2}-1}\right)^t,$$

where $(\,\cdot\,)^t$ denotes transposition. Let us search for v_φ whose first component is <0. The first occurrence, which takes place for $\varphi=\frac{\pi}{p}$ is when $a>\frac{(p+q)!q!}{(\frac{k}{2}-1)!(\frac{k}{2}-1)!}$. In this case extension to directions arbitrarily close to

$$v=\left(\binom{k-2}{\frac{k}{2}-1}-a\binom{k-2}{p+q},\binom{k-2}{\frac{k}{2}-1}\right)^t$$

holds according to [9]. If we look, instead, to our sector property and search for v whose first component is <0 and the second is, say, >0, we are led to the sector property of g_ξ for suitable ξ with $\xi_1<0$ and $\xi_2<0$. The condition reads in this case

$$g_\xi(\theta)=\xi_1(1+a\cos(p\theta))+\xi_2>0\text{ in a sector of angle }>\frac{\pi}{k},$$

that is

$$1+a\frac{\xi_1}{\xi_1+\xi_2}\cos(p\theta)<0\text{ in a sector with angle }>\frac{\pi}{k}.$$

We write $a_\xi=a\frac{\xi_1}{\xi_1+\xi_2}$. Then the sector where $g_\xi>0$ is centered at $\theta=\frac{\pi}{p}$ and its angle is $>\frac{\pi}{k}$ if and only if $a_\xi>1/\cos(\frac{p\pi}{2k})$. Now the first such occurrence is for $a>1/\cos(\frac{p\pi}{2k})$ and for ξ close to $(-1,1-a\cos(\frac{p\pi}{2k}))^t$. Hence we get extension to vectors v with $\langle v,\xi\rangle>0$ according to Theorem 1.1. Direct inspection of the second equation of M shows that $v_2>0$. Also, extension to directions of a conic neighborhood of the diagonal is evident. In conclusion, using also the edge of the wedge theorem, we get extension to all intermediate directions, among which some are close to $v = (-a\cos(\frac{p\pi}{2k}) + 1, 1)^t$. We now write

$$a_1 = a\frac{(\frac{k}{2}-1)!(k-1-\frac{k}{2})!}{(p+q)!q!}$$

and $a_2 = a\cos(\frac{p\pi}{2k})$. According to Lemma 4.4 we always have $a_1 < a_2$. (In the simplest cases we have $a_2/a_1 = 2/\sqrt{2}$ when $k = 4$, $p = 2$ and $a_2/a_1 = 3$ for $k = 6$, $p = 4$.)

Summarizing, we get:

(1) Extension for an extra range of values of a, that is, $a_1 < a \leq a_2$ which were not taken care of by [9]. For this purpose the higher codimension is not really needed; the examples by Rea and Baouendi–Trèves would suffice as well.

(2) Extension to a wedge V' with bigger directional cone Γ' even for common values of $a > a_1$. (Here codimension > 1 is really essential.) In fact in [9] the cone is

$$\Gamma = \{(y_1, y_2) : y_1 > -|y_2|(a_1 - 1)\},$$

whereas in our case it is

$$\Gamma = \{(y_1, y_2) :\ y_1 > -|y_2|(a_2 - 1)\}.$$

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