It can be seen that $I_n$ is related to $\Gamma(n + \frac{1}{2})$ as
\[
\int_0^\infty x^n e^{-x^2}dx = \frac{1}{2} \int_0^\infty y^{n-\frac{1}{2}} e^{-y} dy = \frac{1}{2} \Gamma(n + \frac{1}{2}).
\]

Reference
1. N. Lord, An elementary single-variable proof of $\int_0^\infty e^{-x^2/2}dx = \sqrt{2\pi}$, 

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89.10 Convergents to $\sqrt{N}$ via the Farey mean?

Introduction
Let $a, b, c$ and $d$ be positive integers such that $\gcd(a, b) = \gcd(c, d) = 1$. The Farey mean (also known as the mediant) of the two rationals $a/b$ and $c/d$ is defined to be $(a + c)/(b + d)$. In his note [1], concerning some properties of the Farey mean, David Singmaster considers the following alternative to Newton's square root method for finding successively better rational approximations to $\sqrt{N}$. Starting with some rational estimate $a_1/b_1$ for $N$, we take the Farey mean (rather than the arithmetic mean used in Newton's method) of $a_1$ and $N/a_1 = Nb_1/a_1$ to give us an improved rational estimate $a_2$ for $\sqrt{N}$ (it is stated in [1] that $a_2$ will be equal to $(a_1 + Nb_1)/(a_1 + b_1)$, but we note here that this is not necessarily always the case since there is the possibility that $\gcd(Nb_1, a_1) \neq 1$). The next estimate $a_3$ is obtained by taking the Farey mean of $a_2$ and $N/a_2$, and so on.

Singmaster noticed that with $a_1 = 1$ and $N = 2$ this process seems to give the convergents to $\sqrt{2}$, and with $N = 3$ the convergents to $\sqrt{3}$ emerge. He also points out that for $N = 5$ we appear to get the convergents to $\sqrt{5}$, but at every third stage of the process, and that for other values of $N$ we obtain no convergents to $\sqrt{N}$ at all, although, by setting $a_1 = \lfloor \sqrt{N} \rfloor$, it is sometimes possible to get some beginning terms to be convergents. In this note we provide proofs of these conjectures, and go on to show that 2 and 3 are actually the only non-square integer values of $N$ for which the sequence of approximations obtained using this Farey mean process is the sequence of convergents to $\sqrt{N}$.

The cases $N = 2, 3$ and 5
Before proceeding with the proofs, let us define some notation that will be used throughout. Any positive irrational number $x$ can be expressed as an infinite continued fraction of the form
\[
c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \ldots}}}.
\]
where \( c_0 \) is a non-negative integer and \( c_1, c_2, c_3, \ldots \) are positive integers called the partial denominators of the continued fraction representation for \( x \). Adopting the usual notation for continued fractions, we may write \( x = \left[ c_0; c_1, c_2, c_3, \ldots \right] \). The rational number obtained from the above continued fraction by ignoring all its partial denominators from \( c_n \) onwards is called the \( n \)th convergent to \( x \), and is denoted \( p_n/q_n \). Finally, if, from some point on in a continued fraction representation of a number, there is a recurring string of partial denominators we may use an overbar to indicate this. So, for example, \( \left[ 4; 3, 1,1,2,1,1,2,1,1,2, \ldots \right] \).

The following relations may be found in [2, pp. 284-286]:

\[
p_{n+1} = c_n p_n + p_{n-1} \quad \text{and} \quad q_{n+1} = c_n q_n + q_{n-1} \quad \text{for} \quad n \geq 2.
\]

It is proved in [3], using these relations along with the fact that \( \sqrt{m^2 + 1} = \left[ m; \frac{2m}{2m} \right] \) that, if \( p_n/q_n \) denotes the \( n \)th convergent to \( \sqrt{m^2 + 1} \), where \( m \) is a positive integer, then, for \( n \geq 1 \),

\[
p_{n+1} = m p_n + (m^2 + 1) q_n \quad \text{and} \quad q_{n+1} = p_n + m q_n.
\]

Thus, with \( m = 1 \), we obtain the following relation for the convergents to \( \sqrt{2} \):

\[
\frac{p_{n+1}}{q_{n+1}} = \frac{p_n + 2q_n}{p_n + q_n} \quad \text{for} \quad n \geq 1.
\]

Then, since \( p_1/q_1 = 1 \) and \( \gcd(2, p_n) = 1 \) for \( n \geq 1 \), we see immediately that the Farey mean process will give rise to the convergents to \( \sqrt{2} \) when \( \alpha_1 = 1 \) and \( N = 2 \).

As 5 is also an integer of the form \( m^2 + 1 \), we next consider the Farey mean approximations to \( \sqrt{5} \). With \( m = 2 \), we have the following relation for the convergents to \( \sqrt{5} \):

\[
\frac{p_{n+1}}{q_{n+1}} = \frac{2p_n + 5q_n}{p_n + 2q_n} \quad \text{for} \quad n \geq 1.
\]

It is easily checked that \( \alpha_3 = 2/1 = p_1/q_1 \) so now assume that \( \alpha_{3k} = p_k/q_k \), for some positive integer \( k \). We then have (on noting that \( \gcd(5, p_n) = 1 \) for \( n \geq 1 \)):

\[
\alpha_{3k+1} = \frac{p_k + 5q_k}{p_k + q_k}, \quad \alpha_{3k+2} = \frac{3p_k + 5q_k}{p_k + 3q_k} \quad \text{and} \quad \alpha_{3k+3} = \frac{2p_k + 5q_k}{p_k + 2q_k} = \frac{p_{k+1}}{q_{k+1}},
\]

proving, by induction, that the convergents to \( \sqrt{5} \) appear as every third term in the sequence \( (\alpha_n) \).

Numbers of the form \( \sqrt{m^2 + 2} \), of which \( \sqrt{3} \) is one, have continued fraction representations given by \( \left[ m; \frac{2m}{m^2} \right] \). Using this fact, along with the previously mentioned relations from [2], we obtain, for \( n \geq 1 \), the following:

\[
p_{n+1} = m p_n + (m^2 + 2) q_n \quad \text{and} \quad q_{n+1} = p_n + m q_n \quad \text{when} \quad n \text{ is even,}
\]

and

\[
p_{n+1} = \frac{1}{2} \left( m p_n + (m^2 + 2) q_n \right) \quad \text{and} \quad q_{n+1} = \frac{1}{2} \left( p_n + m q_n \right) \quad \text{when} \quad n \text{ is odd,}
\]
giving us, with \( m = 1 \), the following relation for the convergents to \( \sqrt{3} \):

\[
\frac{p_{n+1}}{q_{n+1}} = \frac{p_n + 3q_n}{p_n + q_n} \quad \text{for } n \geq 1.
\]

Thus, since \( p_1/q_1 = 1 \) and \( \gcd(3, p_n) = 1 \) for \( n \geq 1 \), the Farey mean process will give rise to the convergents to \( \sqrt{3} \) when \( \alpha_1 = 1 \) and \( N = 3 \).

**The general case**

Let \( N \geq 5 \) be a non-square integer. Then we may write \( N = m^2 + k \) where \( m \) and \( k \) are integers such that \( m \geq 2 \) and \( 1 < k < 2m \). We are looking for possible values of \( N \) for which the sequence of Farey mean approximations for \( N \) is identical to the sequence of convergents to \( \sqrt{N} \). Since the first convergent is \( \lfloor \sqrt{N} \rfloor = m \), we must have \( \alpha_1 = m \) so that, with \( \gcd(m, k) = d \), our second approximation is

\[
\alpha_2 = \frac{m + ((m^2 + k)/d)}{1 + m/d} = m + \frac{k}{d + m}.
\]

On the other hand the second convergent is given by

\[
\frac{p_2}{q_2} = m + \frac{1}{\left\lfloor \frac{1}{\sqrt{m^2 + k} - m} \right\rfloor},
\]

so we require \( k \) to satisfy both \( 1 < k < 2m \) and \( \frac{m + d}{k} = \left\lfloor \frac{1}{\sqrt{m^2 + k} - m} \right\rfloor \).

Now, letting \( m = sd \) and \( k = td \) for some integers \( s \) and \( t \), and using the result

\[
\left\lfloor \frac{1}{\sqrt{m^2 + k} - m} \right\rfloor = \left\lfloor \frac{2m + \sqrt{m^2 + k} + m}{k} \right\rfloor = \left\lfloor \frac{2m}{k} \right\rfloor + \left\lfloor \frac{1}{\sqrt{m^2 + k} + m} \right\rfloor = \left\lfloor \frac{2m}{k} \right\rfloor,
\]

(since \( 0 < h < k - 1 \) and \( 0 < \frac{1}{\sqrt{m^2 + k} + m} < \frac{1}{2m} \)), we have that \( s + 1 \leq \frac{2s}{t} \).

This tells us that \( t \) divides \( s + 1 \) and \( \frac{s + 1}{t} \leq \frac{2s}{t} < \frac{s + 1}{t} + 1 \). The inequality simplifies to \( 1 < s < t + 1 \) so \( s + 1 < t + 2 \). Thus \( s + 1 < t \) (so \( s + 1 = 0 \)), or \( s + 1 = t \), or \( s + 1 = t + 1 \) (so \( t = 1 = s \)).

If \( s = t = 1 \) then \( k = m \). Using the fact that \( \sqrt{m^2 + m} = \left[ m; 2, 2m \right] \) we obtain \( \frac{p_3}{q_3} = \frac{4m^2 + 3m}{4m + 1} \). Comparing this to \( \alpha_3 = \frac{2m^2 + 4m + 1}{2m + 3} \) it is clear that \( \alpha_3 \neq p_3/q_3 \) for \( m \geq 2 \) (we have already dealt with \( \sqrt{2} \), the case corresponding to \( m = 1 \)).
If, on the other hand, \( t = s + 1 \) then \( k = m + d \). Numbers of the form \( \sqrt{m^2 + m + d} \) (remembering that \( 1 < d < m \)) have continued fraction representations of the form \([m; 1, c_2, \ldots]\) so that

\[
\frac{p_1}{q_1} = m, \quad \frac{p_2}{q_2} = m + 1 \quad \text{and} \quad \frac{p_3}{q_3} = \frac{m + c_2(m + 1)}{1 + c_2} = m + \frac{c_2}{1 + c_2}.
\]

Since \( d \) is a factor of \( m \) (and thus \( \gcd(m^2 + m + d, m + 1) = 1 \)) we also have, in this case, that

\[
\alpha_1 = m, \quad \alpha_2 = m + 1 \quad \text{and} \quad \alpha_3 = \frac{m + 1 + (m^2 + m + d)}{1 + (m + 1)} = m + \frac{d + 1}{m + 2}.
\]

Therefore, if \( \alpha_3 \) is to equal \( p_3/q_3 \), we require \( \frac{d + 1}{sd + 2} \) to be equal to \( \frac{c_2}{1 + c_2} \).

We see that, with this condition, either \( s = 1 \) or \( s = 2 \).

If \( s = 1 \) then \( m = d = c_2 - 1 \). In this case, therefore, we just need to consider numbers of the form \( \sqrt{m^2 + 2m} = [m; 1, 2m] \). It is clear here that \( m \neq c_2 - 1 \) when \( m > 2 \) (we have already dealt with \( \sqrt{3} \), the case corresponding to \( m = 1 \)).

If, however, \( s = 2 \) then \( d = m/2 \) and \( c_2 = 1 \), so we now only need to consider numbers of the form \( \sqrt{4k^2 + 3k} \). It is easy to show that, for \( k > 2 \), such numbers have \( c_2 \geq 2 \). When \( k = 1 \) then \( N = 7 \) and, in this case, a simple check reveals that \( \alpha_4 \neq p_4/q_4 \), showing finally that, for \( N > 5 \), there exist no non-square integer values of \( N \) for which the sequence of Farey mean approximations is the sequence of convergents to \( \sqrt{N} \).

References


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89.11 Relations between Euler’s constant, Riemann’s zeta function and Bernoulli numbers

First, we give three definitions:

1. Euler’s constant \( \gamma \) is defined by \( \gamma = \lim_{N \to \infty} \left( \sum_{k=1}^{N-1} \frac{1}{k} - \ln N \right) \).

2. Riemann’s zeta function \( \zeta(s) \) is defined by \( \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, s > 1 \).

3. The Bernoulli numbers \( B_n \) may be defined by the generating function (see [1])

\[
\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \text{ for } |t| < 2\pi.
\]