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THE HADAMARD CONJECTURE AND CIRCUITS OF LENGTH FOUR IN A COMPLETE BIPARTITE GRAPH

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Abstract

We show that the problem of settling the existence of an $n \times n$ Hadamard matrix, where n is divisible by 4, is equivalent to that of finding the cardinality of a smallest set T of 4-circuits in the complete bipartite graph $K_{n,n}$ such that T contains at least one circuit of each copy of $K_{2,3}$ in $K_{n,n}$.

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An Hadamard matrix is an $n \times n$ $(1, -1)$-matrix in which the rows are mutually orthogonal. The Hadamard conjecture asserts that there exists an Hadamard matrix of order n whenever n is divisible by 4. (See Wallis (1972) and the references found therein.) In Little and Thuente (1979), we restate the conjecture as a problem concerning the 1-factors of a complete bipartite graph. In the present paper, the conjecture is shown to be equivalent to one about the circuits of length 4 in a complete bipartite graph.

We begin with a lemma.

LEMMA 1. *Let S be a set with $|S| = n$ for some n divisible by 4. Suppose there exist subsets $T_1, T_2, \ldots, T_{n-1}$ of S, of cardinality $n/2$, such that $|T_i \cap T_j| = n/4$ whenever $i \neq j$. Then there exists an Hadamard matrix of order n.*

PROOF. Let $S = \{s_1, s_2, \ldots, s_n\}$. Define $H = (h_{ij})$, where $h_{1j} = 1$ for all $j \in \{1, 2, \ldots, n\}$ and, for all $i \in \{2, 3, \ldots, n\}$,

$$h_{ij} = \begin{cases} 1 & \text{if } s_j \in T_{i-1}, \\ -1 & \text{otherwise.} \end{cases}$$

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Since $|T_i| = n/2$ for all i, the first row is orthogonal to all the others. Furthermore since $|T_i| = n/2$, $|T_j| = n/2$ and $|T_i \cap T_j| = n/4$ for all $j \neq i$, we must have $|T_j - T_i| = |T_i - T_j| = n/4$, so that $|\bar{T}_i \cap \bar{T}_j| = n/4$ where $\bar{T}_i = S - T_i$ and $\bar{T}_j = S - T_j$. It follows that rows $i + 1$ and $j + 1$ are orthogonal. Hence H is an Hadamard matrix of order n.

As an example, let $S = \{s_1, s_2, s_3, s_4\}$, $T_1 = \{s_1, s_2\}$, $T_2 = \{s_1, s_3\}$, and $T_3 = \{s_1, s_4\}$. Then

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

The equivalence of the Hadamard conjecture with a problem on the 4-circuits of a complete bipartite graph is shown in the following theorem.

THEOREM. *Let S be the set of all 4-circuits of $K_{n,n}$ where n is even. Let $S_1, \ldots, S_k$ be the collection of all subsets S_i of S, of cardinality* 3, *such that the union of the three circuits of S_i is $K_{2,3}$. Let T be a smallest subset of S such that $T \cap S_i \neq \varnothing$ for each i. Then $|T| \geqslant \frac{1}{8}n^2(n-1)(n-2)$, and equality holds if and only if there exists an Hadamard matrix of order n.*

PROOF. Let A be an $n \times n$ $(1, -1)$-matrix (a_{ij}). Let $K_{n,n}$ be the complete bipartite graph with vertex set $\{v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n\}$, where v_i and w_j are adjacent for each i and j. Furthermore, for each i and j let the edge joining v_i to w_j be directed from v_i to w_j if $a_{ij} = 1$ and from w_j to v_i otherwise.

Note that a pair of rows and a pair of columns of A corresponds in an obvious way to an undirected 4-circuit in $K_{n,n}$. We say that this 4-circuit is clockwise even if the number of edges directed in the clockwise sense is even, and clockwise odd otherwise. Let C be a 4-circuit of $K_{n,n}$ with vertex set $\{v_h, v_i, w_j, w_k\}$. If $a_{hj} = a_{ij}$, then exactly one of the two edges of C incident on w_j is directed in the clockwise sense. If $a_{hj} \neq a_{ij}$, then those edges are directed in the same sense on C. Analogous results hold for a_{hk} and a_{ik}. It follows that C is clockwise odd if and only if exactly one of the equations $a_{hj} = a_{ij}$ and $a_{hk} = a_{ik}$ holds. This condition holds if and only if $a_{hj}a_{ij} + a_{hk}a_{ik} = 0$.

If we let X_{hi} be the set of columns j of A for which $a_{hj} = a_{ij}$ and let Y_{hi} be the set of all the remaining columns of A, it follows from the above considerations that the number of clockwise odd 4-circuits containing v_h and v_i is $|X_{hi}|\,|Y_{hi}|$. This product is a maximum if $|X_{hi}| = |Y_{hi}|$, and this condition holds if and only if rows h and i of A are orthogonal. It follows that the number of clockwise odd 4-circuits of $K_{n,n}$ is maximised if A is an Hadamard matrix.

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Suppose therefore that A is an Hadamard matrix. Then $|X_{hi}| = |Y_{hi}| = \frac{1}{2}n$ for all h and i, so that there are $(\frac{1}{2}n)^2$ clockwise odd 4-circuits of $K_{n,n}$ containing v_h and v_i. Therefore $K_{n,n}$ has $\frac{1}{4}\binom{n}{2}n^2$ clockwise odd 4-circuits altogether, and therefore $\binom{n}{2}^2 - \frac{1}{4}\binom{n}{2}n^2 = \frac{1}{8}n^2(n-1)(n-2)$ clockwise even ones. Let T_0 be the set of all clockwise even 4-circuits of $K_{n,n}$.

The graph $K_{2,3}$ is drawn in Figure 1, where an orientation is given in which all three circuits are clockwise even. Since every edge of $K_{2,3}$ belongs to exactly two circuits of $K_{2,3}$, it follows that for any orientation of $K_{2,3}$ there are an odd number of clockwise even circuits. It follows that $T_0 \cap S_i \neq \varnothing$ for all i.

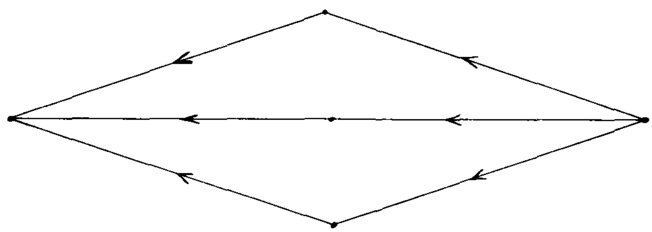

FIGURE 1

We have now proved that if there exists an Hadamard matrix of order n, then $|T| \leqslant \frac{1}{8}n^2(n-1)(n-2)$. We prove next that in fact $|T| \geqslant \frac{1}{8}n^2(n-1)(n-2)$. The existence of an $n \times n$ Hadamard matrix will then imply that $|T| = \frac{1}{8}n^2(n-1)(n-2)$. We will then prove the converse.

Suppose therefore that $T \cap S_i \neq \varnothing$ for all i. We consider first those copies of $K_{2,3}$ in $K_{n,n}$ which contain exactly three vertices of $\{v_1, \ldots, v_n\}$. We denote the complement of $K_{n,n}$ by $2K_n$, since it has exactly two components, C_1 and C_2, each isomorphic to K_n. Let C_1 be the component with vertex set $\{v_1, \ldots, v_n\}$. Then the complement (in K_5) of a copy of $K_{2,3}$ containing three vertices of $\{v_1, \ldots, v_n\}$ is $P_1 \cup P_2$, where P_1 is a triangle of C_1 and P_2 an edge of C_2. The complement (in K_4) of a circuit in $K_{2,3}$ is then the union of P_2 with an edge of P_1. Let us now temporarily fix P_2 and let P_1 run through all triangles in C_1. In order to contain at least one circuit in each of the corresponding copies of $K_{2,3}$, T must contain at least as many circuits as the cardinality of the smallest set of edges whose deletion from K_n yields a graph with no triangles, and furthermore each such circuit must contain both end-vertices of the edge P_2. By a well known theorem of Turán (see Turán (1941) or Harary (1969) p. 17), the largest subgraph of K_n having no triangles is $K_{n/2,n/2}$, since n is even. Since K_n has $\binom{n}{2}$ edges and $K_{n/2,n/2}$ has $\frac{1}{4}n^2$ edges, T must contain at least $\binom{n}{2} - \frac{1}{4}n^2$ circuits which include the end-vertices of P_2. Since there are $\binom{n}{2}$ choices for P_2, it follows that $|T| \geqslant \binom{n}{2}[\binom{n}{2} - \frac{1}{4}n^2] = \frac{1}{8}n^2(n-1)(n-2)$.

We continue the argument under the assumption that

$$|T| = \tfrac{1}{8}n^2(n-1)(n-2)$$

and prove the existence of an $n \times n$ Hadamard matrix. We now consider the copies of $K_{2,3}$ in $K_{n,n}$ which have only two vertices of $\{v_1, \ldots, v_n\}$. The complement (in K_5) of such a copy of $K_{2,3}$ is $P_1 \cup P_2$ where P_1 is an edge of C_1 and P_2 a triangle of C_2. The complement (in K_4) of any circuit in such a copy Z of $K_{2,3}$ is the union of P_1 with an edge e of P_2. We have already seen that in order to include at least one circuit of each copy of $K_{2,3}$ that includes the end-vertices of e and three vertices of $\{v_1, \ldots, v_n\}$, T must contain all the 4-circuits whose complements in K_4 are pairs of edges where one edge of the pair is e and the other is chosen from the complement, $2K_{n/2}$, in C_1 of a fixed copy of $K_{n/2,n/2}$. In order to ensure that T contains a circuit of Z, the copies of $K_{n/2,n/2}$ in C_1 corresponding to the edges of P_2 must be chosen in such a way that the edge P_1 appears in the complement of at least one of them. Since P_1 is any edge of C_1, we find that C_1 must be the union of three copies of $2K_{n/2}$, each copy being the complement in C_1 of a copy of $K_{n/2,n/2}$ chosen to correspond to an edge of P_2. Since P_2 is any triangle of C_2, we see that to each edge of C_2 there corresponds a subgraph $2K_{n/2}$ of C_1 in such a way that for any triangle of C_2 the union of the corresponding subgraphs of C_1 is C_1 itself. For any edge e of C_2, let us denote by $V_1(e)$ and $V_2(e)$ the vertex sets of the copies of $K_{n/2}$ in the subgraph $2K_{n/2}$ of C_1 corresponding to e. Thus $|V_1(e)| = |V_2(e)| = \frac{1}{2}n$ for each e.

Let us now consider a triangle of C_2 with edge set $\{e_1, e_2, e_3\}$. Since C_1 is the union of the corresponding copies of $2K_{n/2}$, each pair of vertices of C_1 must be contained in at least one of the sets $V_i(e_j)$ where $e \in \{1, 2\}$ and $j \in \{1, 2, 3\}$. It follows that

$$\{V_1(e_3), V_2(e_3)\} = \{[V_1(e_1) \cap V_1(e_2)] \cup [V_2(e_1) \cap V_2(e_2)],$$
$$[V_1(e_1) \cap V_2(e_2)] \cup [V_2(e_1) \cap V_1(e_2)]\}.$$

Note that $|V_1(e_1) \cap V_1(e_2)| = |V_2(e_1) \cap V_2(e_2)|$, since

$$|V_1(e_1)| = |V_2(e_2)|, |V_1(e_1)| = |V_1(e_1) \cap V_1(e_2)| + |V_1(e_1) \cap V_2(e_2)|$$

and $|V_2(e_2)| = |V_1(e_1) \cap V_2(e_2)| + |V_2(e_1) \cap V_2(e_2)|$. Since

$$|V_1(e_1) \cap V_1(e_2)| + |V_2(e_1) \cap V_2(e_2)| = |V_1(e_3)| = |V_2(e_3)| = \tfrac{1}{2}n,$$

it follows that n is divisible by 4 and $|V_1(e_1) \cap V_1(e_2)| = \frac{1}{4}n$.

Finally we consider a subgraph $K_{1,n-1}$ of C_2. Any pair of the $n-1$ edges $f_1, \ldots, f_{n-1}$ in this subgraph form two sides of a triangle in C_2. It is now immediate that the sets $V_1(f_1), \ldots, V_1(f_{n-1})$ satisfy the conditions of Lemma 1. The existence of an Hadamard matrix of order n follows.

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