A NOTE ON JACOBSON'S CONJECTURE FOR RIGHT NOETHERIAN RINGS

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In 1956, Jacobson asked whether the intersection of the powers of the Jacobson radical, $J(R)$, of a right Noetherian ring R, must always be zero [**4**, p. 200]. His question was answered in the negative by I. N. Herstein [**3**], who noted that $R_1 = \begin{pmatrix} \mathbf{Z}_{(2)} & \mathbf{Q} \\ 0 & \mathbf{Q} \end{pmatrix}$, where $\mathbf{Z}_{(2)}$ denotes the ring of rational numbers with denominator prime to 2, affords a counterexample. In contrast, the ring $R_2 = \begin{pmatrix} k & k[[x]] \\ 0 & k[[x]] \end{pmatrix}$, though similar in appearance to R_1, satisfies $\bigcap_1^\infty J^n(R_2) = 0$. (Here, k denotes a field.)

An explanation for the differing behaviour of these rings is provided by the following.

THEOREM. *Let R be a right Noetherian, right fully bounded ring, all of whose simple modules are finitely generated over a central subring C of R. Then $\bigcap_1^\infty J^n(R) = 0$.*

A prime ring is *right bounded* if each of its essential right ideals contains a non-zero two sided ideal. A ring is *right fully bounded* if each of its prime factor rings is right bounded. For example, rings satisfying a polynomial identity are right fully bounded [**5**, Ch. II, 5.2 and 5.7]; in particular, this is true of the rings R_1 and R_2.

All rings are assumed to have an identity, and all modules are unital, and are right modules unless otherwise described. The injective hull of a module M over the ring R is denoted by $E_R(M)$. We shall need the following lemmas.

LEMMA 1 [**6**, Lemma 6]. *If R is a ring and $\mathcal{S}$ is a set of representatives of isomorphism classes of simple R-modules then $E \equiv \bigoplus_{S \in \mathcal{S}} E_R(S)$ is a faithful R-module.*

LEMMA 2 [**1**, Lemma 3.4]. *Let M be a finitely generated faithful uniform module over the prime, right fully bounded, right Noetherian ring R. Then every non-zero submodule of M is faithful.*

Let M be an R-R-bimodule. We write M_R (resp. ${}_RM$) to indicate that M is being viewed as a right (left) R-module, and, for a subset A of M, write $r(A) \equiv \{r \in R \mid Ar = 0\}$ and $l(A) \equiv \{r \in R \mid rA = 0\}$. The key to the theorem is contained in

LEMMA 3. *Let R be a right Noetherian ring all of whose simple modules are finitely generated over a central subring C. If I is an ideal of R with I_R Artinian then ${}_RI$ and $R/l(I)$ are Artinian.*

Proof. By Noetherian induction, it may be assumed that I contains no smaller non-zero ideals of R. Let $I = \sum_{i=1}^n \alpha_i C$. Then $P = r(I)$ is a prime ideal and, since $r(I) = \bigcap_{i=1}^n r(\alpha_i)$, the ring R/P is Artinian. Therefore, R/P is a finitely generated

$(C/C \cap P)$-module; and so $C/C \cap P$ is Artinian, by [**2**, Theorem 1]. Let $K = C/C \cap P$. Thus $I = \sum_{i=1}^{n} K\alpha_i$, so ${}_R I$ is Artinian. Since I_R is finitely generated, the last part follows easily.

Proof of the Theorem. Nakayama's Lemma ensures that any module with a composition series is annihilated by a power of $J(R)$. Thus the theorem follows from Lemma 1 provided that, if S is simple and M is a finitely generated submodule of $E_R(S)$, then M is Artinian. Let M be such a submodule, with largest Artinian submodule $A(M)$. If $M \neq A(M)$ and N is any other non-Artinian submodule of M then $S \subseteq N$ and $r(M) \subseteq r(N)$. Also, since $A(N) = N \cap A(M)$, $r(M/A(M)) \subseteq r(N/A(N))$. Hence, using Noetherian induction and replacing M by a suitable non-Artinian submodule if necessary, we may assume that $r(M) = r(N)$ and that $r(M/A(M)) = r(N/A(N))$, whenever N is a non-Artinian submodule of M. It follows easily from the second of these two assumptions that $Q = r(M/A(M))$ is a prime ideal. Put $X = r(A(M))$. Then R/X and $(Q/QX)_R$ are Artinian. Therefore, by Lemma 3, if $I = l(Q/QX)$ then R/I is an Artinian ring. Note that $MIQ \subseteq MQX = 0$. If $MI \subseteq A(M)$ then M is Artinian, as required. Otherwise, MI is not Artinian, so $MQ = 0$ by assumption. Since $Q = r(M/A(M))$, clearly $Q = r(M)$. By Lemma 2, $Q = X$, and again M is Artinian, as required.

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