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ON THE DE VYLDER AND GOOVAERTS CONJECTURE ABOUT RUIN FOR EQUALIZED CLAIMS

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Abstract

In ruin theory, the conjecture given in De Vylder and Goovaerts (2000) is an open problem about the comparison of the finite time ruin probability in a homogeneous risk model and the corresponding ruin probability evaluated in the associated model with equalized claim amounts. In this paper we consider a weaker version of the conjecture and show that the integrals of the ruin probabilities with respect to the initial risk reserve are uniformly comparable.

Keywords: De Vylder and Goovaerts' conjecture; homogeneous risk model; finite time ruin probability; stochastic dominance

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1. Introduction

In [1] De Vylder and Goovaerts considered a homogeneous risk model on a fixed time interval $[0, t]$. They proposed a conjecture about the uniform comparison of the ruin probability in the homogeneous risk model and the corresponding ruin probability evaluated in the associated model with equalized claim amounts. In this paper we consider a weaker version of the conjecture and show that the integrals of the ruin probabilities with respect to the initial risk reserve are uniformly comparable.

More formally, De Vylder and Goovaerts made the following assumptions.

(i) The point process is homogeneous.

Let N_t denote the number of claims in the interval $[0, t]$, and let $T_1, T_2, \ldots, T_{N_t}$ be the arrival times of the claims. For all $n > 0$ and fixed $N_t = n > 0$, the points $T_1, \ldots, T_n$ are uniformly distributed over $(0, t]$, i.e. the conditional vector $(T_1, \ldots, T_n) \mid N_t = n$ has a constant density equal to $t^n/n!$ on the subset $W_{t,n} = \{(t_1, \ldots, t_n) \in \mathbb{R}^n : 0 < t_1 < \cdots < t_n \le t\}$. We denote the interarrival times of the claims by $E_i = T_i - T_{i-1}$ for $i = 1, \ldots, n$, with $T_0 = 0$, and let $E_{n+1} = t - T_n$. The homogeneous point process is an extension of the classical Poisson process (see [3]).

(ii) The claim amounts $X_1, X_2, \ldots$ are independent and identically distributed, and they are independent from the arrival times of the claims.

The risk reserve process is denoted by $R_\tau = u + c\tau - S_\tau$ $(0 \le \tau \le t)$, where $u \ge 0$ is the initial risk reserve, $c > 0$ is the premium income rate, and S_τ is the total claim amount in

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$[0, \tau]$, i.e. $S_\tau = \sum_{i=1}^{N_\tau} X_i$. We denote by $\psi(t, u)$ the ruin probability before t for the initial risk reserve u , and by $\psi_n(t, u)$ the corresponding conditional probability of ruin for fixed $N_t = n$. Then

$$\psi(t, u) = \mathbb{P}\Big(\inf_{0<\tau\le t} R_\tau < 0\Big) = \sum_{n=1}^{\infty} \mathbb{P}(N_t = n)\psi_n(t, u)$$

with

$$\psi_n(t, u) = \mathbb{P}\Big(\inf_{0<\tau\le t} R_\tau < 0 \,\Big|\, N_t = n\Big) = \mathbb{P}(Z_n > u),$$

where $Z_n = \max_{1\le k\le n}(\sum_{i=1}^{k} Y_i)^+$ with $Y_i = X_i - cE_i$ for $i = 1, \dots, n$.

The associated model is the model where each claim amount is replaced by the average amount $X_k^\sim = \sum_{i=1}^{N_t} X_i/N_t = \bar{X}_{N_t}$. The arrival times of the claims are the same as those in the first model. The ruin probability is then given by

$$\psi^\sim(t, u) = \sum_{n=1}^{\infty} \mathbb{P}(N_t = n)\psi_n^\sim(t, u)$$

with

$$\psi_n^\sim(t, u) = \mathbb{P}(Z_n^\sim > u),$$

where $Z_n^\sim = \max_{1\le k\le n}(\sum_{i=1}^{k} Y_i^\sim)^+$ with $Y_i^\sim = X_i^\sim - cE_i$ for $i = 1, \dots, n$.

The conjecture proposed by De Vylder and Goovaerts in [1] is as follows.

Conjecture 1. ([1].) *In any homogeneous risk model with time interval* $[0, t]$ *and its associated model with equalized claim amounts,*

$$\psi^\sim(t, u) \le \psi(t, u), \qquad u \ge 0.$$

As explained by De Vylder and Goovaerts, the conjecture is equivalent to the proposition that $\psi_n^\sim(t, u) \le \psi_n(t, u)$ for all $n = 1, 2, \dots$. They proved that $\psi_1^\sim(t, u) = \psi_1(t, u)$ and that $\psi_2^\sim(t, u) \le \psi_2(t, u)$, but the general case has still to be established. Moreover, note that, by Theorem 2 of [1], $\psi^\sim(t, 0) = \psi(t, 0)$.

2. Main result

The conjecture is equivalent to the proposition that, for all $n = 1, 2, \dots$, $Z_n^\sim$ is smaller than Z_n in stochastic dominance (see Definition 3.3.1 and Equation (3.7) of [2]), or to the proposition that, for all $n = 1, 2, \dots$, $\mathbb{E}[h(Z_n^\sim)] \le \mathbb{E}[h(Z_n)]$ for all nondecreasing functions h such that the expectations exist (see Theorem 3.3.14 of [2]). In the next theorem we prove that $\mathbb{E}[h(Z_n^\sim)] \le \mathbb{E}[h(Z_n)]$ for the subset of nondecreasing convex functions h, which is equivalent to saying that $Z_n^\sim$ is smaller than Z_n in the stop-loss order (see Theorem 3.4.6 of [2]).

Theorem 1. *For all* $n = 1, 2, \dots$, $Z_n^\sim$ *is smaller than* Z_n *in the stop-loss order. Therefore, for all* $u \ge 0$,

$$\int_u^\infty \psi^\sim(t, v)\,\mathrm{d}v \le \int_u^\infty \psi(t, v)\,\mathrm{d}v. \tag{1}$$

Proof. First, note that the statement $Z_n^\sim$ is smaller than Z_n in the stop-loss order is equivalent to

$$\pi_{Z_n^\sim}(u) \le \pi_{Z_n}(u), \qquad u \ge 0,$$

where $\pi_{Z_n^\sim}(u) = \mathbb{E}[(Z_n^\sim - u)^+] = \int_u^\infty \mathbb{P}(Z_n^\sim > v)\,\mathrm{d}v = \int_u^\infty \psi_n^\sim(t, v)\,\mathrm{d}v$ (see Definition 3.4.2 of [2]). Hence, to prove (1), it is sufficient to show that $Z_n^\sim$ precedes Z_n in the stop-loss order for all $n = 1, 2, \ldots$.

Second, $(E_i)_{i=1,\ldots,n+1}$ are exchangeable random variables, that is, the distribution of $(E_i)_{i=1,\ldots,n+1}$ is the same as that of $(E_{\sigma(i)})_{i=1,\ldots,n+1}$ for every permutation σ of $\{1, \ldots, n+1\}$, since there exist independent standard exponential random variables Γ_i for $i = 1, \ldots, n+1$ such that

$$(E_1, \ldots, E_{n+1}) \overset{\mathrm{D}}{=} t\biggl(\frac{\Gamma_1}{\sum_{i=1}^{n+1}\Gamma_i}, \ldots, \frac{\Gamma_{n+1}}{\sum_{i=1}^{n+1}\Gamma_i}\biggr);$$

see, e.g. Theorem 1.6.7 of [5]. Moreover, the vectors of the random variables $(X_i, E_i)_{i=1,\ldots,n}$ are also exchangeable random vectors, as are those of $(X_i^\sim, E_i)_{i=1,\ldots,n}$. Then we derive that

$$\mathbb{E}(h(Z_n)) = \frac{1}{n!}\sum_\pi \mathbb{E}\biggl[h\biggl(\max_{1\le k\le n}\biggl(\sum_{i=1}^k Y_{\pi(i)}\biggr)^+\biggr)\biggr],$$

$$\mathbb{E}(h(Z_n^\sim)) = \frac{1}{n!}\sum_\pi \mathbb{E}\biggl[h\biggl(\max_{1\le k\le n}\biggl(\sum_{i=1}^k Y_{\pi(i)}^\sim\biggr)^+\biggr)\biggr],$$

where π denotes a permutation of $\{1, \ldots, n\}$.

Using Theorem 2.2 and Equation (1.1) of [6], we have

$$\frac{1}{n!}\sum_\pi \mathbb{E}\biggl[h\biggl(\max_{1\le k\le n}\biggl(\sum_{i=1}^k Y_{\pi(i)}\biggr)^+\biggr)\biggr] = \sum^*\biggl(\prod_{\nu=1}^n \frac{1}{\nu^{k_\nu}(k_\nu!)}\biggr)\mathbb{E}\biggl[h\biggl(\sum_{\nu=1}^n\sum_{i_\nu=1}^{k_\nu}\biggl(\sum_{j_\nu=1}^{\nu} Y_{l_{\nu-1}+(i_\nu-1)\nu+j_\nu}\biggr)^+\biggr)\biggr]$$

and

$$\frac{1}{n!}\sum_\pi \mathbb{E}\biggl[h\biggl(\max_{1\le k\le n}\biggl(\sum_{i=1}^k Y_{\pi(i)}^\sim\biggr)^+\biggr)\biggr] = \sum^*\biggl(\prod_{\nu=1}^n \frac{1}{\nu^{k_\nu}(k_\nu!)}\biggr)\mathbb{E}\biggl[h\biggl(\sum_{\nu=1}^n\sum_{i_\nu=1}^{k_\nu}\biggl(\sum_{j_\nu=1}^{\nu} Y_{l_{\nu-1}+(i_\nu-1)\nu+j_\nu}^\sim\biggr)^+\biggr)\biggr],$$

where the summation $\sum^*$ extends over all n -tuples $\boldsymbol{k} = (k_1, k_2, \ldots, k_n)$ of non-negative integers with the property that $k_1 + 2k_2 + \cdots + nk_n = n$, $l_j = k_1 + 2k_2 + \cdots + jk_j$ for $j = 1, \ldots, n$, and $l_0 = 0$.

Let $\boldsymbol{X} = (X_1, \ldots, X_n)$, $\boldsymbol{E} = (E_1, \ldots, E_n)$, $\boldsymbol{X}^\sim = (X_1^\sim, \ldots, X_n^\sim) = (\bar{X}_n, \ldots, \bar{X}_n)$,

$$\rho(\boldsymbol{x}, \boldsymbol{e}; \boldsymbol{k}, h) = h\biggl(\sum_{\nu=1}^n\sum_{i_\nu=1}^{k_\nu}\biggl(\sum_{j_\nu=1}^{\nu}(x_{l_{\nu-1}+(i_\nu-1)\nu+j_\nu} - ce_{l_{\nu-1}+(i_\nu-1)\nu+j_\nu})\biggr)^+\biggr),$$

and

$$\Theta_{\boldsymbol{X}}(\boldsymbol{e}; \boldsymbol{k}, h) = \mathbb{E}(\rho(\boldsymbol{X}, \boldsymbol{e}; \boldsymbol{k}, h)).$$

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Since $\boldsymbol{X}$ and $\boldsymbol{E}$ are independent, we deduce that

$$\mathbb{E}\Big[h\Big(\sum_{\nu=1}^{n}\sum_{i_\nu=1}^{k_\nu}\Big(\sum_{j_\nu=1}^{\nu} Y_{l_{\nu-1}+(i_\nu-1)\nu+j_\nu}\Big)^{+}\Big)\Big] = \mathbb{E}[\Theta_{\boldsymbol{X}}(\boldsymbol{E};\boldsymbol{k},h)],$$

and since $\boldsymbol{X}^{\sim}$ and $\boldsymbol{E}$ are independent,

$$\mathbb{E}\Big[h\Big(\sum_{\nu=1}^{n}\sum_{i_\nu=1}^{k_\nu}\Big(\sum_{j_\nu=1}^{\nu} Y_{l_{\nu-1}+(i_\nu-1)\nu+j_\nu}\Big)^{+}\Big)\Big] = \mathbb{E}[\Theta_{\boldsymbol{X}^{\sim}}(\boldsymbol{E};\boldsymbol{k},h)].$$

Let us now show that, for all $\boldsymbol{e}$ and $\boldsymbol{k}$ with the property that $\sum_{i=1}^{n} ik_i = n$, and all nondecreasing convex functions h,

$$\Theta_{\boldsymbol{X}^{\sim}}(\boldsymbol{e};\boldsymbol{k},h) \le \Theta_{\boldsymbol{X}}(\boldsymbol{e};\boldsymbol{k},h).$$

If this is true, we obtain $\mathbb{E}[h(Z_n^{\sim})] \le \mathbb{E}[h(Z_n)]$ for all nondecreasing convex functions h.

First, note that $\boldsymbol{X}^{\sim}$ is smaller than $\boldsymbol{X}$ in the multivariate convex order because $\mathbb{E}[\boldsymbol{X} \mid \boldsymbol{X}^{\sim}] = \boldsymbol{X}^{\sim}$ (see Definition 3.4.58 and Proposition 3.4.66 of [2]).

Second, let us show that $g\colon (\mathbb{R}^+)^n \to \mathbb{R}$, defined by $g(\boldsymbol{x}) = \rho(\boldsymbol{x},\boldsymbol{e};\boldsymbol{k},h)$, is convex on $(\mathbb{R}^+)^n$. For $\lambda \in [0,1]$, $\boldsymbol{x} \in (\mathbb{R}^+)^n$, $\boldsymbol{y} \in (\mathbb{R}^+)^n$, since $x \mapsto (x-ce)^+$ is a convex function for all $e \ge 0$,

$$\begin{aligned}
&\sum_{\nu=1}^{n}\sum_{i_\nu=1}^{k_\nu}\Big(\sum_{j_\nu=1}^{\nu}(\lambda x_{l_{\nu-1}+(i_\nu-1)\nu+j_\nu} + (1-\lambda)y_{l_{\nu-1}+(i_\nu-1)\nu+j_\nu} - ce_{l_{\nu-1}+(i_\nu-1)\nu+j_\nu})\Big)^{+} \\
&\quad \le \lambda \sum_{\nu=1}^{n}\sum_{i_\nu=1}^{k_\nu}\Big(\sum_{j_\nu=1}^{\nu}(x_{l_{\nu-1}+(i_\nu-1)\nu+j_\nu} - ce_{l_{\nu-1}+(i_\nu-1)\nu+j_\nu})\Big)^{+} \\
&\qquad + (1-\lambda)\sum_{\nu=1}^{n}\sum_{i_\nu=1}^{k_\nu}\Big(\sum_{j_\nu=1}^{\nu}(y_{l_{\nu-1}+(i_\nu-1)\nu+j_\nu} - ce_{l_{\nu-1}+(i_\nu-1)\nu+j_\nu})\Big)^{+}.
\end{aligned}$$

Since h is a nondecreasing function,

$$\begin{aligned}
&g(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \\
&\quad = \rho(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}, \boldsymbol{e};\boldsymbol{k},h) \\
&\quad = h\Big(\sum_{\nu=1}^{n}\sum_{i_\nu=1}^{k_\nu}\Big(\sum_{j_\nu=1}^{\nu}(\lambda x_{l_{\nu-1}+(i_\nu-1)\nu+j_\nu} \\
&\qquad\qquad + (1-\lambda)y_{l_{\nu-1}+(i_\nu-1)\nu+j_\nu} - ce_{l_{\nu-1}+(i_\nu-1)\nu+j_\nu})\Big)^{+}\Big) \\
&\quad \le h\Big(\lambda \sum_{\nu=1}^{n}\sum_{i_\nu=1}^{k_\nu}\Big(\sum_{j_\nu=1}^{\nu}(x_{l_{\nu-1}+(i_\nu-1)\nu+j_\nu} - ce_{l_{\nu-1}+(i_\nu-1)\nu+j_\nu})\Big)^{+} \\
&\qquad + (1-\lambda)\sum_{\nu=1}^{n}\sum_{i_\nu=1}^{k_\nu}\Big(\sum_{j_\nu=1}^{\nu}(y_{l_{\nu-1}+(i_\nu-1)\nu+j_\nu} - ce_{l_{\nu-1}+(i_\nu-1)\nu+j_\nu})\Big)^{+}\Big),
\end{aligned}$$

and since h is a convex function,

$$\begin{aligned} g(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) &\le \lambda \rho(\boldsymbol{x}, \boldsymbol{e}; \boldsymbol{k}, h) + (1-\lambda)\rho(\boldsymbol{y}, \boldsymbol{e}; \boldsymbol{k}, h) \\ &= \lambda g(\boldsymbol{x}) + (1-\lambda) g(\boldsymbol{y}). \end{aligned}$$

We deduce by the multivariate convex order property that

$$\mathbb{E}[g(\boldsymbol{X}^{\sim})] \le \mathbb{E}[g(\boldsymbol{X})],$$

which is equivalent to

$$\Theta_{\boldsymbol{X}^{\sim}}(\boldsymbol{e}; \boldsymbol{k}, h) \le \Theta_{\boldsymbol{X}}(\boldsymbol{e}; \boldsymbol{k}, h)$$

and the result follows.

Remark 1. Another way to prove Theorem 1 is to use the property of Schur convexity. A function ϕ is said to be Schur convex on $\mathcal{A} \subset \mathbb{R}^n$ if $\phi(\boldsymbol{x}) \le \phi(\boldsymbol{y})$ for every $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathcal{A}$ and $\boldsymbol{y} = (y_1, \dots, y_n) \in \mathcal{A}$ such that $\boldsymbol{x} \prec \boldsymbol{y}$, that is, such that $\sum_{i=1}^k x_{[i]} \le \sum_{i=1}^k y_{[i]}$ for $1 \le k \le n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[i]}$ denotes the ith largest component in $\boldsymbol{x}$ (see, e.g. Sections 1 and 3 of [4]).

By Theorem 2.2 of [6],

$$\frac{1}{n!}\sum_{\pi} \mathbb{E}\bigg[h\bigg(\max_{1\le k\le n}\bigg(\sum_{i=1}^{k} Y_{\pi(i)}\bigg)^{+}\bigg)\bigg] = \frac{1}{n!}\sum_{\tau} \mathbb{E}\bigg[h\bigg(\sum_{i=1}^{n(\tau)}\bigg(\sum_{k\in\alpha_i(\tau)} Y_k\bigg)^{+}\bigg)\bigg],$$

where $\tau = (\alpha_1(\tau))(\alpha_2(\tau))\cdots(\alpha_{n(\tau)}(\tau))$ is a permutation of $\{1, \dots, n\}$ decomposed into disjoint cycles.

Let $\theta\colon (\mathbb{R}^+)^n \to \mathbb{R}$ be defined by

$$\theta(\boldsymbol{x}) = \frac{1}{n!}\sum_{\tau} \mathbb{E}\bigg[h\bigg(\sum_{i=1}^{n(\tau)}\bigg(\sum_{k\in\alpha_i(\tau)} (x_k - cE_k)\bigg)^{+}\bigg)\bigg].$$

It is a symmetric function (with respect to any permutation). If h is also a nondecreasing convex function then θ is convex and, therefore, Schur convex (see, e.g. Section 3.G of [4]). Since $(\bar{x}, \dots, \bar{x}) \prec (x_1, \dots, x_n)$, where $\bar{x} = n^{-1}\sum_{i=1}^n x_i$, it follows that

$$\theta((\bar{x}, \dots, \bar{x})) \le \theta((x_1, \dots, x_n)),$$

and then

$$\begin{aligned} \mathbb{E}(h(Z_n^{\sim})) &= \frac{1}{n!}\sum_{\tau} \mathbb{E}\bigg[h\bigg(\sum_{i=1}^{n(\tau)}\bigg(\sum_{k\in\alpha_i(\tau)} Y_k^{\sim}\bigg)^{+}\bigg)\bigg] \\ &= \mathbb{E}[\theta((\bar{X}_n, \dots, \bar{X}_n))] \\ &\le \mathbb{E}[\theta((X_1, \dots, X_n))] \\ &= \frac{1}{n!}\sum_{\tau} \mathbb{E}\bigg[h\bigg(\sum_{i=1}^{n(\tau)}\bigg(\sum_{k\in\alpha_i(\tau)} Y_k\bigg)^{+}\bigg)\bigg] \\ &= \mathbb{E}[h(Z_n)]. \end{aligned}$$

Remark 2. The assumption that h is convex is crucial here and it seems difficult to weaken this assumption with this method.

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