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GENERALISED UMBILICS ON EMBEDDED SPHERES

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ABSTRACT. We study two kinds of generalized umbilics on smoothly embedded n-manifolds in $\mathbb{R}^{n+1}$. A sectional umbilic occurs where two of the principal curvatures are equal, and a split sectional umbilic is a more general notion.

1. **Introduction.** Let N be a smooth oriented n-manifold ($n \geq 2$), smoothly embedded in $\mathbb{R}^{n+1}$, and let ν be the smooth unit normal field to N obtained by requiring $\{v_1, v_2, \ldots, v_n, \nu(w)\}$ to be a positively oriented basis for $\mathbb{R}^{n+1}$ whenever $\{v_1, v_2, \ldots$ $\ldots, v_n\}$ is a positively oriented basis for TN_w. Identify ν with a smooth map from N to the unit sphere S^n in $\mathbb{R}^{n+1}$, and TN_w with $TS^n_{\nu(w)}$ where $w \in N$. Then the derivative $d\nu_w$ is self-adjouint with respect to the usual metric on $TS^n_{\nu(w)}$; the eigenvalues are therefore real, and their negatives are the principal curvatures $c_1(w) \leq c_2(w) \leq \cdots \leq c_n(w)$ at w of the hypersurface N. The eigenvectors are called principal directions. The function c_i is differentiable at points w such that $c_{i-1}(w) \neq c_i(w) \neq c_{i+1}(w)$, by the implicit function theorem, and c_i is continuous on the whole of N, by Rouchés theorem. A point $x \in N$ is an umbilic, in the classical sense, when $c_1(x) = c_n(x)$.

For example, if $N \cong S^2$ then there must be at least one umbilic, because the principal directions would otherwise define continuous nonsingular line element fields on S^2, and this cannot happen because S^2 has nonzero Euler characteristic. On the other hand, there is no proof as yet that a convex immersion of S^2 in $\mathbb{R}^3$ has at least two umbilics; this was conjectured by Carathéodory and proved only in the ananlytic case by Hamburger [3]. Another proof, also in the analytic case, was given by Bol and Klotz [2], [4], and also by Titus [9]. Some examples where $N \cong S^2$ are described by Spivak in [8]; note also that the ellipsoids of revolution which are not spheres have just two umbilics namely the fixed points of the rotation group.

When $n > 2$ there is no particular reason to suppose that a hypersurface N will have umbilics, except in special circumstances, and in the present paper we consider the existence of generalized umbilics when $N \cong S^n$.

Generalized umbilics are related to results on Bott periodicity and symplectic geometry.

In [6, §4] Section 2, for example a real vector bundle $E_{\mathbb{R}}$ over S^n is used to construct a hypersurface $M_{E_{\mathbb{R}}}$ in $\mathbb{R}^{m+1}$ without generalized umbilics of a particular kind; here $2(n+p)+1$. In [6, §4] $E_{\mathbb{R}}$ is an orthogonal real $2p$-plane bundle associated with a unitary complex p-plane bundle E over S^n, and $M_{E_{\mathbb{R}}}$ is used to construct a Lagrangian immersion

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$\hat{\iota}_E\colon W \cong S^{n+1} \times \mathbb{R}^{m-n} \longrightarrow T^*\mathbb{R}^{m+1}$. The Gauss class of $\hat{\iota}_E$ is then related to the classifying map of E by means of a Bott homotopy equivalence. The hypersurface $M_{E_\mathbb{R}}$ is noncompact, and the search for compact examples leads to questions about generalised umbilics when $N \cong S^n$.

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2. **Sectional Umbilics.** We call x a sectional umbilic of N at the i-th position when $c_i(x) = c_{i+1}(x)$ for some i between 1 and $n-1$. The multplicity m_x of a sectional umbilic x is defined to be the number of positions i between 1 and $n-1$ at which x is a sectional umbilic. An umbilic in the classical sense is precisely a sectional umbilic of multiplicity $n-1$.

Unless TN is a direct sum of line bundles, there must be at least one sectional umbilic, because otherwise the eigenspaces of the $d\nu_w$ would define n mutually orthogonal line-sub-bundles of TN. To verify local triviality for these line sub-bundles we prove

LEMMA 1. *Let $h\colon E \longrightarrow F$ be a continuous vector bundle map whose fibrewise kernels $K_w(w \in N)$ have constant dimension d. Then $K \equiv \cup_{w\in N} K_w$ is a continuous vector bundle over N.*

(We have in mind the case where $E = F = TN$, and $h|TN_w$ is $d\nu_w + c_i(w)1$.)

PROOF. Without loss E, F are trivial bundles: there is no need for K to be trivial in such a case, even when h is surjective; for example h might be orthogonal projection from $T\mathbb{R}^{n+1}|S^n$ onto the normal bundle of S^n in $\mathbb{R}^{n+1}$. To prove local triviality of K, it is sufficient to observe that any matrix of nullity d can be reduced to the matrix with d 1's down the diagonal and 0's elsewhere, by means of elementary row and column operations, and that a continuous perturbation of the original matrix requires only continuous perturbation of the operations. The row and column operations correspond to coordinate changes in the fibres of E, F. Note that for the purposes of Lemma 1 there is no need for N to be a manifold.

It follows that if $N \cong S^n$ where $n \neq 3, 7$, then N has at least one sectional umbilic, because line bundles over simply-connected spaces are trivial, and the only parallelisable spheres are those of dimensions $1, 3, 7$. We take this observation further as follows.

Let $i_1 < i_2 < i_3 < \cdots < i_k$ be the positions at which sectional umbilics do not occur for any value of x in N. We call the ordered k-tuple $(i_1, i_2, i_3, \cdots, i_k)$ the profile of N. When $k = 0$, which happens for example when $N \cong S^2$, the profile is empty.

LEMMA 2. *TN is a direct sum of $k+1$ mutually orthogonal sub-bundles $E_1, E_2, \ldots,$ E_{k+1} of dimensions $d_1 = i_1$, $d_2 = i_2 - i_1$, $d_3 = i_3 - i_2$, …, $d_k = i_k - i_{k-1}$, $d_{k+1} = n - i_k$.*

The fibre E_{jw} of E_j over $w \in N$ is spanned by the principal directions corresponding to $c_{i_{j-1}+1}(w), c_{i_{j-1}+2}(w), \ldots, c_{i_j}(w)$, and the definition of the profile gives $\dim_\mathbb{R} E_{jw} = d_j$

for all $w \in N$. To prove that the union of these fibres is a vector bundle we refer again to Lemma 1. (We do not say that the E_j are smooth sub-bundles of TN.)

We call the sub-bundles E_j the summands of TN, and the d_j the summand dimensions. It would be interesting to argue in the opposite direction; for example, is there an embedding of S^3 in $\mathbb{R}^4$ without Section 1 umbilics? We take $N \cong S^n$ for the remainder of this section. This would be the case, for example if N was connected, compact, and strictly convex (namely $c_1(w) > 0$ for all $w \in N$), by [5] Theorem 5.6.

LEMMA 3. *Let $E_{(1)}$ and $E_{(2)}$ be vector bundles over S^n with positive fibre dimensions $p_1 \leq p_2$, and let E be the direct sum of these bundles. Then E has at least p_1 everywhere linearly independent cross-sections.*

PROOF. Let H^+ and H^- be the closed hemispheres in S^n consisting of those points whose $n+1$-st coordinates are non-negative and non-positive respectively. Because hemispheres are contractible we can choose everywhere linearly independent cross-sections $s_{1,1}, s_{1,2}, s_{1,3}, \ldots, s_{1,p_1}$ of the restriction of $E_{(1)}$ to H^+. Similarly we can choose everywhere linearly independent cross-sections $s_{2,1}, s_{2,2}, s_{2,3}, \ldots, s_{2,p_2}$ of the restrictions of $E_{(2)}$ to H^-. We extend these cross-sections to cross-sections $e_{1,1}, e_{1,2}e_{1,3}, \ldots e_{1,p_1}$ of $E_{(1)}$ and cross-sections $e_{2,1}, e_{2,2}e_{2,3}, \ldots e_{2,p_2}$ of $E_{(2)}$; we do not say that the extended cross-sections are linearly independent everywhere.

For $1 \leq i \leq p_1$ define s_i to be the cross-section of E given by the sum $e_{1,i} + e_{2,i}$. Since the $e_{1,i}$ are linearly independent over one hemisphere and the $e_{2,i}$ are linearly independent over the other, it follows that the s_i are everywhere linearly independent.

The following combinatorial procedure is needed. Suppose we are given a set S of $\underset{\sim}{k} + 1$ positive integers $d_1, d_2, \ldots, d_{\underset{\sim}{k}+1}$ whose sum is a given integer $\underset{\sim}{n}$. Partition S into two disjoint subsets U, V and let σ be the smaller of $\sum_U d_j$ and $\sum_V d_j$. Define $\mu(S)$ to be the maximum value of σ as the partitions vary; a partition which achieves this maximum value is called optimal.

To state Theorem 1 we make a particular choice of S as follows. Let m_1, m_2 be the number of summand dimensions that are $1, 2$ respectively. The remaining $\underset{\sim}{k}+1 = k+1-m_1-m_2$ summand dimensions are our d_j and these make up S. Note that $\underset{\sim}{n} = n-m_1-2m_2$.

THEOREM 1. $\mu(S) + m_1 + 2m_2 < \rho(n+1)$.

Here $\rho(n+1)$ is the Radon-Hurwitz number, defined as follows. Write $n+1$ in the form $(2a+1)2^b$, and then write $b = c + 4d$ where a, b, c, d are integers and $0 \leq c \leq 3$. Then $\rho(n+1) = 2^c + 8d$. For example $\rho(n+1) = 1$ when n is even, $\rho(4) = 4$, $\rho(6) = 2$, $\rho(8) = 8$, $\rho(10) = 2$, $\rho(12) = 4$, $\rho(14) = 2$, $\rho(16) = 9$, $\rho(18) = 2$, $\rho(20) = 4$.

When $n = 2$ Theorem 1 holds trivially, and so we take $n > 2$ for the purposes of the proof. Let $E_{(1)}$, $E_{(2)}$ be obtained by taking the direct sums of the summands of TS^n corresponding to elements of an optimal partition of S. Then by Lemma 3 E has at least $\mu(S)$ everywhere linearly independent cross-sections. Note that TS^n is the direct sum of E with the summands of dimensions $1, 2$. Since $n > 2$, all vector bundles over S^n of fibre dimensions $1, 2$ are trivial and so TS^n has at least $\mu(S) + m_1 + 2m_2$ everywhere linearly independent cross-sections.

But a famous result of J. F. Adams [1] asserts that TS^n has at most $\rho(n+1) - 1$ everywhere linearly independent cross-sections. This proves Theorem 1.

REMARK 1. In the situation of Theorem 1, S contains only integers ≥ 3, and it follows that $\mu(S) \geq 3\left[\frac{k+1}{2}\right]$.

REMARK 2. $\sum m_x \geq n - k - 1$, where the multiplicities m_x are summed over all sectional umbilics x.

To justify Remark 2 consider the profile $(i_1,\ i_2,\ i_3,\ \ldots\ i_k)$ of N. By definition of the profile, for every integer $q \in (i_{j-1}, i_j)$ there is at least one x which is a sectional umbilic at the qth position; the sum of the multiplicities of these sectional umbilics is therefore at least as large as $d_j - 1$. Allowing j to vary, and summing, we obtain Remark 2.

Theorem 1 is easy to apply.

Examples:

(1) When n is even $\rho(n+1) = 1$, and so $\mu(S) = m_1 = m_2 = 0$ and the profile is therefore empty. So $\sum m_x \geq n - 1$.

(2) Theorem 1 says nothing about the cases where $n = 3, 7$.

(3) When $n = 5$, $\rho(n+1) = 2$ and so $\mu(S) + m_1 + 2m_2 \leq 1$. Therefore $m_2 = 0$ and $(m_1, \mu(S))$ is either $(0,0)$, $(1,0)$ or $(0,1)$. In the first case the profile is empty and therefore $\sum m_x \geq 4$. If the profile is nonempty then, because $\mu(S) \geq 3$ unless $k = 0$, we have $m_1 = 1$, $m_2 = 0$, $k = 0$. So $\sum m_x \geq 5 - 2 = 3$, and the only possible nonempty profiles are (1), (4).

(4) When $n = 9$ we have $\rho(n+1) = 2$ again and so $m_2 = 0$ and $(m_1, \mu(S))$ is either $(0,0)$, $(1,0)$ or $(0,1)$. In the first case the profile is empty and therefore $\sum m_x \geq 8$. If the profile is nonempty then the only possibilities are (1), (8), and then we have $\sum m_x \geq 7$.

(5) When $n = 11$ we have $\rho(n+1) = 4$ and so $\mu(S) + m_1 + 2m_2 \leq 3$. If the profile is empty then $\sum m_x \geq 10$. The other possibilities are $(\mu(S), m_1, m_2) = (0,1,0)$, $(0,2,0)$, $(0,3,0)$, $(0,0,1)$, $(0,1,1)$, $(3,0,0)$ and we would have $\sum m_x \geq 9, 8, 7, 9, 8, 9$ respectively.

3. **Split Sectional Umbilics.** As noted, Theorem 1 says nothing about sectional umbilics when $n = 3, 7$, and this leads us to introduce a weaker notion. A split sectional umbilic at the ith position of a smoothly embedded oriented hypersurface N is an ordered pair (y, z) of points in N such that $c_i(y) = c_{i+1}(z)$. The multiplicity $m_{(y,z)}$ of a split sectional umbilic is the number of positions i between 1 and $n-1$ at which (y, z) is a split sectional umbilic. Then for $x \in N$, (x, x) is a split sectional umbilic precisely when x is a sectional umbilic, and the multiplicities agree.

Suppose again that N is connected, compact, and strictly convex; then $N \cong S^n$.

THEOREM 2. *$\sum m_{(y,z)} \geq n-1$, where the multiplicities are summed over all split sectional umbilics (y,z) of N.*

As the referee points out, Theorem 2 follows from the principal curvature theorem of Smyth and Xavier [7 § 1]. Because the principal curvature theorem applies to hypersurfaces which are not necessarily compact or convex it is to be expected that their proof should be less elementary than my proof of Theorem 2. Their immersion f_0 appears as my f_r, but we seem to argue differently from that point on, and it may helpful to give my relatively simple proof for the special situation in Theorem 2.

So for $r \in \mathbb{R}$ define a smooth map $f_r: N \longrightarrow \mathbb{R}^{n+1}$ by $f_r(w) = w + r\nu(w)$. Here $w \in N$. Given w, let $\{v_j : j = 1, 2, \cdots, n\}$ be an orthonormal basis of $TS^n{}_{\nu(w)} \cong TN_w$ such that v_j is a nonzero eigenvector of the eigenvalue $-c_j(w)$ of $d\nu_w$. We have

$$df_{rw}(v_j) = v_j + r d\nu_w(v_j) = \big(1 - rc_j(w)\big)v_j \tag{1}$$

and this equation determines the linear transformation df_{rw}. We now prove Theorem 2 by contradiction.

Suppose that for some i we can choose $r \in \big(1/\min\{c_{i+1}(z) : z \in N\},\ 1/\max\{c_i(y) : y \in N\}\big)$. Then by (1), f_r is an immersion. Let $\omega_j : \mathbb{R} \longrightarrow N$ $(j = 1, \cdots, n)$ be smooth curves with $\omega_j(0) = w$, and whose velocity vectors at time 0 are v_j. We argue in a familiar way [8, Proposition 8 p. 90]. Choose $p \in \mathbb{R}^{n+1}$ and define $L_p : f_r(N) \longrightarrow \mathbb{R}$ by $L_p(q) = \|q - p\|^2$. Choose $w \in N$ so that $f_r(w)$ is a point of relative maximum of L_p. Then 0 is a critical point of each $L_p\omega_j$, namely

$$2\Big\langle d(f_r\omega_j)/\,dt, f_r\omega_j - p\Big\rangle_{t=0} = 0 \tag{2}$$

and also

$$\begin{aligned} &d^2(L_p f_r\omega_j)/\,dt^2{}_{t=0} \\ &\quad = 2\Big(\Big\langle d^2(f_r\omega_j)/\,dt^2{}_{t=0}, f_r(w) - p\Big\rangle + \Big\langle d(f_r\omega_j)/\,dt_{t=0}, d(f_r\omega_j)/\,dt_{t=0}\Big\rangle\Big) \\ &\quad \leq 0 \end{aligned} \tag{3}$$

Now $\nu(w)$ is orthogonal to the image of df_{rw}, by (1), and so by (2) we have

$$f_r(w) - p = a\nu(w) \tag{4}$$

for some nonzero $a \in \mathbb{R}$. Since $\big\langle \nu\big(\omega_j(t)\big), df_{r\omega_j(t)} d\omega_{jt}(1)\big\rangle = 0$ for all t, we have $\big\langle \nu(w), d^2(f_r\omega_j)/\,dt^2{}_{t=0}\big\rangle = -\big\langle d\nu_w(v_j),\ (1 - rc_j(w))v_j\big\rangle = c_j(w)\big(1 - rc_j(w)\big)$ which is positive or negative according as $j \leq i, j > i$, because of our choice of r. So by (4) we have $\big\langle a(f_r(w) - p),\ d^2(f_r\omega_j)/\,dt^2{}_{t=0}\big\rangle$ positive or negative according to our choice of j. This contradicts (3), and Theorem 2 is proved.

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