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ON NORMAL NUMBERS

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Abstract

Schmidt has shown that if r and s are positive integers and there is no positive integer power of r which is also a positive integer power of s, then there exists an uncountable set of reals which are normal to base r but not even simply normal to base s. We give a structurally simple proof of this result.

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I. Introduction

For $r, s \in Z^+$, we write $r \sim s$ if there exist $m, n \in Z^+$ with $r^n = s^m$, otherwise $r \nsim s$. (As subsequently, we put $Z^+ = \{1, 2, \ldots\}$, $Z = \{0, \pm 1, \pm 2, \ldots\}$.) We have the following well-known results:

THEOREM A. *Assume $r \sim s$. Then any real normal to base r is normal to base s.*

THEOREM B. *If $r \nsim s$, then the set of reals which are normal to base r but not even simply normal to base s has the cardinality of the reals.*

This theorem has been established by Schmidt (1960). Theorem B is also established independently by Cassels (1959) for the case $s = 3$. Part A is trivial and the treatments of Schmidt and Cassels of the non-trivial Part B utilise chains of number-theoretic lemmas. As noted by Pelling (1980), no simple proof appears to exist. Theorem B admits an equivalent formulation in terms of weak convergence of measures. In this paper, by combining a version of a theorem of Serfling (1970) on almost sure convergence with two elementary number-theoretic lemmas of Schmidt we give a short and structurally simple proof of the proposition. Schmidt's proofs for Theorem A and these two lemmas are short, self-contained and do not involve his other lemmas.

Consider the set $E \subset [0, 1]$ of points x with s-adic expansions

$$x = \sum_{j=1}^{\infty} e_j(s-1)s^{-j}, \qquad e_j \in \{0, 1\}.$$

The set E consists of an uncountable collection of points which are clearly not even simply normal to base s if $s > 2$. Theorem B is established for $s > 2$ if we can show that E has an uncountable subset of points which are normal to base r.

Suppose we define a map V from E onto $[0, 1]$ by $Vx = y$, where

$$y = \sum_{j=1}^{\infty} e_j 2^{-j}.$$

We note that this map is well-defined even thought a point with terminating s-adic expansion has an alternative non-terminating s-adic representation.

Through the map V Lebesgue measure λ and the Borel σ-field on $[0, 1]$ induce a measure μ carried by E and an associated σ-field $\mathfrak{B}$.

Let δ_x denote the measure concentrated at x and T the operator T: $[0, 1) \to [0, 1)$ defined by

$$Tx = rx \pmod 1, \qquad x \in [0, 1).$$

To establish Theorem B it suffices to show for $r \nsim s$ that except for a μ-null subset of E, points x of E have the sequence $(x, Tx, T^2x, \ldots)$ uniformly distributed on $[0, 1)$, that is

$$\frac{1}{n}\sum_{k=0}^{n-1} \delta_{T^k x} \to \lambda \quad \text{weakly almost everywhere } (\mu)$$

by Weyl's criterion (see Cassels (1957), Chapter 4).

A necessary and sufficient condition for this to hold is that for each $l \in Z \setminus \{0\}$ we have

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) \to \int_{[0,1)} f\, d\lambda \quad \text{almost everywhere } (\mu)$$

where

$$f(x) = \exp(2\pi i l x),$$

or equivalently that

$$\frac{1}{n}\sum_{k=0}^{n-1} X_k \to 0 \quad \text{almost everywhere } (\mu), \tag{1}$$

where

$$X_k(x) = \exp(2\pi i l r^k x). \tag{2}$$

We shall derive the stronger

THEOREM 1. *Suppose $r \nsim s$ with $s > 6$. For X_k defined by* (2), *there exists an η, $0 < \eta < 1$, such that*

$$n^{-\eta}\sum_{k=0}^{n-1} X_k \to 0 \quad \textit{almost everywhere } (\mu). \tag{3}$$

By virtue of the foregoing discussion, Theorem 1 has as an immediate corollary that Theorem B holds for $s > 6$. The restiction $s > 6$ may then be removed easily by an appeal to Theorem A, since $s \sim s^k$ and $s^k > 6$ for all sufficiently large k.

2. Preliminaries to proofs

Suppose $(X_n)_{n=0}^{\infty}$ is a sequence of random variables on some probability space $(X, \mathfrak{B}, \mu)$ and $F_{a,n}$ is the joint distribution function of $X_{a+1}, \ldots, X_{a+n}$. Then for $c > 0$, $0 < \delta < 1$,

$$g(F_{a,n}) \equiv cn^{2-\delta} \tag{4}$$

is a trivial functional in the sense of Serfling (1970) for which an inequality of the form

$$g(F_{a,n}) \leqslant Kn^2(\log n \log_2 n)^{-2} \qquad (n \geqslant 1, a \geqslant 0) \tag{5}$$

is satisfied. A theorem of Serfling (1970) (see also Stout (1974), pp. 204–5) establishes that if

$$E\left[\left(\sum_{i=a+1}^{a+n} X_i\right)^2\right] \leqslant g(F_{a,n}), \tag{6}$$

we have

$$n^{-1}\sum_{k=0}^{n-1} X_k \to 0 \quad \text{almost everywhere } (\mu). \tag{7}$$

It is easily seen that if (X_n) is replaced by a complex-valued sequence defined on $(X, \mathfrak{B}, \mu)$, relation (7) still holds provided (6) is replaced by

$$E\left[\left|\sum_{i=a+1}^{a+n} X_i\right|^2\right] \leqslant g(F_{a,n}). \tag{8}$$

In fact, given the tighter constraint (4) in place of (5), the proof of Serfling's result may be modified to tell us that if

$$q(n) = n^{\delta/2}(\log n)^{-1-\delta/2}(\log_2 n)^{-(1+\phi)/2}$$

for ϕ an arbitrary positive constant, then (8) entails that

$$[q(n)]^{-1}\sum_{k=0}^{n-1} X_k \to 0 \quad \text{almost everywhere } (\mu).$$

It follows at once that there exists an η, $0 < \eta < 1$, such that

$$n^{-\eta}\sum_{k=0}^{n-1} X_k \to 0 \quad \text{almost everywhere } (\mu).$$

Thus to prove Theorem 1, it suffices to show that for (X_k) defined by (2),

$$E_\mu\left[\left|\sum_{i=a+1}^{a+n} X_i\right|^2\right] \leqslant cn^{2-\delta} \quad \text{for all } l \in Z \setminus \{0\} \tag{9}$$

for some δ, $0 < \delta < 1$.

The argument is conveniently carried out in terms of the Fourier-Stieltjes coefficients $\hat{\mu}(n)$ corresponding to the measure μ and given by

$$\hat{\mu}(n) = \int_0^1 \exp(-2\pi inx)\, d\mu.$$

The set E is of Cantor type and the Fourier-Stieltjes coefficients corresponding to its natural measure μ are well known. We have

$$\hat{\mu}(n) = (-1)^n(2\pi)^{-1}\prod_{k=1}^{\infty}\cos[(s-1)\pi n/s^n] \tag{10}$$

(see Zygmund (1959), page 196).

In terms of the Fourier-Stieltjes coefficients,

$$E_\mu\left[\left|\sum_{i=a+1}^{a+n} X_i\right|^2\right] = \sum_{i=a+1}^{a+n}\sum_{j=a+1}^{a+n}\hat{\mu}((r^i - r^j)l),$$

so that by (10) we have

$$E_\mu\left[\left|\sum_{i=a+1}^{a+n} X_i\right|^2\right] \leqslant n + \pi^{-1}\sum_{i=1}^{n-1}\sum_{j=a+1}^{a+n-i}|u_j((r^i - 1)l)|, \tag{11}$$

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where

$$u_j(q) = \prod_{k=1}^{\infty} \cos[(s-1)\pi q r^j / s^k], \qquad q \in Z. \tag{12}$$

From (9) and (11), Theorem 1 follows as a consequence of

THEOREM 2. *If* $s > 6$, $r \nsim s$, *then for each* $l \in Z \setminus \{0\}$ *there exists a* $c > 0$, $0 < \delta < 1$ *such that*

$$\sum_{i=1}^{n-1} \sum_{j=a+1}^{a+n-i} |u_j((r^i - 1)l)| < cn^{2-\delta}. \tag{13}$$

It is clear from (12) that without loss of generality we may take $l \in Z^+$.

The proof of Theorem 2, which is derived in section 4, utilises three simple number-theoretic lemmas given in the next section.

3. Number-theoretic notation and lemmas

For $m, n \in Z^+$, denote by $\text{ord}_n\, m$ the order of m mod n, that is, the smallest positive integer t such that

$$m^t \equiv 1 \pmod n.$$

Following Schmidt, we use the notation $(m)_n$ for the "n part" of m, the largest power of n dividing m, so that for some positive integers k, m'

$$m = n^k m', \qquad (m)_n = n^k, \qquad n \nmid m'.$$

LEMMA 1. *Assume* p *is a prime with* $p \nmid r$. *Then for all positive integers* k

$$\text{ord}_{p^k}\, r \geqslant c_1(r, p) p^k,$$

where, as subsequently the notation $c_1(r, p)$ *is used to denote a constant depending only on* r *and* p, *not on* k.

COROLLARY 1. *Let* n *run through a residue system modulo* p^k. *Then at most* $c_2(r, p)$ *of the numbers* r^n *will fall into the same residue class modulo* p^k.

COROLLARY 2. *For* p, r *as above and any positive integer* n

$$(r^n - 1)_p \leqslant c_3(r, p) n.$$

PROOFS. Lemma 1 and Corollary 1 are Lemma 4 of Schmidt and its corollary, proved by him (page 666) by elementary number theory.

For Corollary 2, suppose $(r^n - 1)_p = p^k$. Then $r^n \equiv 1 \pmod{p^k}$ and hence

$$\operatorname{ord}_{p^k} r | n.$$

Thus

$$\operatorname{ord}_{p^k} r < n$$

from which the result follows from the lemma.

In (12) we may, without loss of generality, replace r^j by the number ρ_j defined as $r^j/(r^j)_s$, that is,

$$(14) \qquad r^j = (r^j)_s \rho_j, \qquad s \nmid \rho_j.$$

This gives

$$(15) \qquad u_j(q) = \prod_{k=1}^{\infty} \cos\left[(s-1)\pi q \rho_j / s^k\right], \qquad q \in Z.$$

Suppose r, s factorise as

$$r = p_1^{d_1} p_2^{d_2} \cdots p_h^{d_h},$$
$$s = p_1^{e_1} p_2^{e_2} \cdots p_h^{e_h},$$

where we may assume that never both $d_i = 0$, $e_i = 0$. The primes p_i are so ordered that $e_1/d_1 \geqslant e_2/d_2 \geqslant \cdots \geqslant e_h/d_h$, and we put $e_i/d_i = +\infty$ if $d_i = 0$.

LEMMA 2. *Suppose $r \not\sim s$ and $q \in Z^+$. If j runs through a complete residue system modulo s^m, then at most $c_4(r,p)(s/p)^m q_p$ of the numbers $q\rho_j$ are in the same residue class modulo s^m. Here ρ_j is defined by* (14) *and p is the prime p_1 defined above.*

PROOF. This is Theorem 5A of Schmidt (1960) and is deduced by him (page 667) from Corollary 1 above.

LEMMA 3. *If e, $f \in \{0, 1, \ldots, s-1\}$ and $e \neq f$, then* $|\cos[(s-1)\pi \times 0.\, ef \cdots]| \leqslant \theta = \cos(\pi/s^2)$.

The proof is elementary.

Let Y be the set of all ordered m-tuples $y = (y_{m-1}, \ldots, y_1, y_0)$ with $y_i \in \{0, 1, \ldots, s-1\}$ and let τ: $Z^+ \cup \{0\} \to Y$ be the natural projection operator defined as follows:

If $n \in Z^+ \cup \{0\}$ has the representation

$$n = e_0 + e_1 s + e_2 s^2 + \cdots, \quad e_i \in \{0, 1, \ldots, s-1\},$$

in the scale of s, then $\tau n = (e_{m-1}, \ldots, e_1, e_0)$. Further, define

$$\sigma(n) = \operatorname{card}\{i: e_i \neq e_{i+1}, i \geqslant 0\},$$
$$\sigma_0(y) = \operatorname{card}\{i: y_i \neq y_{i+1}, 0 \leqslant i < m-1\}.$$

With this notation we are in a position to establish Theorem 2.

4. Proof of Theorem 2

By definition $\sigma_0(\tau v) > \sigma$ entails $\sigma(v) > \sigma$ for any $v \in Z^+ \cup \{0\}$. From (15) and Lemma 3, we thus have that $\sigma_0(\tau(q\rho_j)) > \sigma$ implies $|u_j(q)| < \theta^\sigma$. Equation (15) also gives that $|u_j(q)| \leqslant 1$ for all $j, q \in Z^+$ so that

$$|u_j(q)| \leqslant \theta^\sigma\{1 - H[\sigma - \sigma_0(\tau(q\rho_j))]\} + H[\sigma - \sigma_0(\tau(q\rho_j))]$$
$$\leqslant \theta^\sigma + H[\sigma - \sigma_0(\tau(q\rho_j))] \quad \text{for all } j \in Z^+, \sigma \in Z^+ \cup \{0\},$$

where H denotes the Heaviside function $H(x) = 1 (x \geqslant 0)$, 0 otherwise. Hence, for all $\sigma \geqslant 0$

$$n^{-1} \sum_{j=a+1}^{a+n} |u_j(q)| \leqslant \theta^\sigma + n^{-1} \sum_{j=a+1}^{a+n} H[\sigma - \sigma_0(\tau(q\rho_j))].$$

By Lemma 2, we have for $n = s^m$ that

$$\sum_{j=a+1}^{a+n} H[\sigma - \sigma_0(\tau(q\rho_j))] \leqslant c_4(r, s)(s/p)^m q_p \operatorname{card}\{y\ Y: \sigma_0(y) \leqslant \sigma\}$$
$$= c_4(r, s)(s/p)^m q_p \sum_{j=0}^{\sigma} \binom{m-1}{j} s(s-1)^j.$$

It follows that for $n \geqslant s^m$ and $\sigma \geqslant 0$

$$n^{-1} \sum_{j=a+1}^{a+n} |u_j(q)| \leqslant \theta^\sigma + 2c_4(r, s)(s/p)^m q_p \times \sum_{j=0}^{\sigma} \binom{m-1}{j} ((s-1)/s)^j (1/s)^{m-1-j}. \tag{16}$$

If we choose $m = [\log_s n]$, the constraint $n \geqslant s^m$ is automatically satisfied and we have (16) holding for all $n \in Z^+$. We shall further choose

$$\sigma = \left[-\frac{\alpha \log_s n}{\log_s \theta}\right]$$

with $\alpha > 0$ small and certainly $\alpha < -\log_s \theta$ so that $\sigma < m$. Since $s\theta = s\cos(\pi/s^2) > 1$ for $s \geqslant 2$ we have in any case that $\alpha < -\log_s \theta$ implies $\alpha < 1$. The normal approximation to the binomial distribution supplies the asymptotic estimate

$$2c_4(r, s)(s/p)^m q_p \Phi[-\{(1-\beta)(s-1)(m-1)\}^{1/2}] \tag{17}$$

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for the second term on the right hand side of (16), where

$$\Phi(-x) = (2\pi)^{-1/2}\int_x^\infty \exp(-t^2/2)\,dt \simeq x^{-1}l\exp(-x^2/2) \quad \text{for } x \text{ large},$$

and β can be made as small as we please by taking α sufficiently small. Hence the estimate (17) is bounded above by

$$(18) \qquad q_p c_5(r, s)\exp\left[(m-1)\{\log_e(s/p) - (1-\beta(s-1)/2)\}\right],$$

or, as $p \geqslant 2$, by

$$(19) \qquad q_p c_6(r, s)\exp(R\log_e n)$$

for $m(n)$ large and suitable constants c_5, c_6, where

$$R = \left[\log_e(s/2) - (1-\beta)(s-1)/2\right]/\log_e s.$$

The expression R is strictly monotone decreasing in s for $s \geqslant 2$ and for $\gamma > 0$ we have $R < -1 - \gamma$ for all sufficiently large s. In fact, if we take $\beta < 1 - (2/7)\log_e 32 \simeq 0.0098$ R is bounded above away from -1 for $s > 7$.

Thus for $s > 7$ we can, if β is sufficiently small, replace the upper bound (19) by

$$(20) \qquad q_p c_6 n^{-1-\gamma}$$

for all n sufficiently large and some $\gamma > 0$. In fact, for $s = 7$ we have $p = 7$ and arguing directly from the tighter bound (18) we see that (18) may be replaced by a bound of form (20) in this case also.

Thus for $s > 6$ and suitable choice of m, σ the second term on the right hand side of (16) can be made less than an expression of the form $q_p c_7 n^{-1-\gamma}$ for all $n \geqslant 1$. Our choice of σ also implies $\theta^\sigma \simeq n^{-\alpha}$. Taking these estimates together we have that if $s > 6$, then for $\alpha > 0$ sufficiently small and some $\gamma > 0$

$$\sum_{j=a+1}^{a+n} |u_j(q)| \leqslant c_8 n^{1-\alpha} + c_7 q_p n^{-\gamma}$$

for all $n, q \geqslant 1$. Hence

$$\begin{aligned}\sum_{i=1}^{n-1}\sum_{j=a+1}^{a+n-i} |u_j((r^i-1)l)| &\leqslant \sum_{i=1}^{n-1}\left[c_8(n-i)^{1-\alpha} + c_7((r^q-1)l)_p/(n-i)^\gamma\right]\\ &\leqslant c_8 n^{2-\alpha} + c_9\sum_{i=1}^{n-1} i/(n-i)^\gamma \quad \text{(by Corollary 2)}\\ &\leqslant c_8 n^{2-\alpha} + c_9 n\sum_{i=1}^{n-1}(n-i)^{-\gamma}\\ &\leqslant cn^{2-\delta} \quad \text{for all } n \geqslant 1\end{aligned}$$

for $\delta = \min(\alpha, \gamma)$ and some $c = c(l, r, s)$. This establishes Theorem 2.

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