

FUNCTIONS WHICH HAVE GENERALIZED RIEMANN DERIVATIVES

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1. Introduction. Let $f(x)$ be a measurable function defined in the interval (a, b) , and let

$$\Delta_n(x, 2h; f) = \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} f(x + 2jh - nh) \quad (h > 0; n = 1, 2, \dots).$$

If the limit of $(2h)^{-n} \Delta_n(x, 2h; f)$ exists and is finite at the point x , as $h \rightarrow 0$, it is called the n th generalized Riemann derivative of $f(x)$ at the point x , $D^n f(x)$. Considering the upper and lower limits of the above expression we can similarly define the upper and lower n th generalized Riemann derivatives, $\bar{D}^n f(x)$ and $\underline{D}^n f(x)$ respectively. If $\bar{D}^n f(x) = \underline{D}^n f(x)$, their common value is the n th generalized Riemann derivative $D^n f(x)$.

If two functions $F(x)$ and $G(x)$ are such that the n th ordinary derivative of $F(x) - G(x)$ is equal to zero then $F(x)$ and $G(x)$ differ by a polynomial of degree at most $n - 1$. The main purpose of this paper is to study the relations between two functions $F(x)$ and $G(x)$ where the n th generalized Riemann derivative of the continuous function $F(x) - G(x)$ is equal to zero, first for derivatives of second order and later for derivatives of higher order.

In the case $n = 2$, if $D^2(F - G) = 0$ then $F(x) - G(x)$ is linear. This follows from Denjoy's work. In order to form a background for a study of the cases in which $n > 2$ we first give a proof for $n = 2$ in conformity with our notations and methods. It turns out that for $n > 2$ additional conditions must be imposed on $F(x) - G(x)$ to ensure that $D^n(F - G) = 0$ makes $F(x) - G(x)$ a polynomial of degree at most $n - 1$. These conditions are considered in §4. Our main result is Theorem 4.2.

2. Definition of the operators H_2 and H_3 . Let $F(x)$ be a single valued function defined over a given domain. Then

$$(2.1) \quad H_2(F; \alpha, \beta, \gamma) = F(\gamma) - \frac{\gamma - \alpha}{\beta - \alpha} F(\beta) - \frac{\gamma - \beta}{\alpha - \beta} F(\alpha),$$

$$(2.2) \quad H_3(F; \alpha, \beta, \gamma, \delta) = F(\delta) - \frac{(\delta - \alpha)(\delta - \beta)}{(\gamma - \alpha)(\gamma - \beta)} F(\gamma) - \frac{(\delta - \alpha)(\delta - \gamma)}{(\beta - \alpha)(\beta - \gamma)} F(\beta) \\ - \frac{(\delta - \beta)(\delta - \gamma)}{(\alpha - \beta)(\alpha - \gamma)} F(\alpha),$$

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where $\alpha, \beta, \gamma, \delta$ are on the domain of $F(x)$, and α, β, γ are distinct points except in the case of H_2 when γ may coincide with α or with β . The operator H_n is defined in §4 (Definition 4.1).

3. The fundamental theorem for $n = 2$. *If $F(x)$ and $G(x)$ are defined on $[a, b]$ and are such that $F(x) - G(x)$ is continuous on $[a, b]$, and $D^2(F - G) = 0$ at all points of (a, b) , then*

$$H_2(F; x_1, x_2, x_3) = H_2(G; x_1, x_2, x_3)$$

for every three points of $[a, b]$ with $x_1 \neq x_2$.

This theorem has been proved for the case where $F(x)$ and $G(x)$ are both continuous by James (3) and by Jeffery (4) where use is made of convex functions. In our proof no use is made of convex functions.

In order to prove the fundamental theorem for $n = 2$ we need the following result due to Denjoy (2, pp. 18–19). We give a proof in conformity with the notations and methods which we shall use for $n > 2$.

THEOREM 3.1. *Let $\bar{D}^2f(x)$ and $D^2f(x)$ be the upper and lower second generalized Riemann derivatives of $f(x)$ which is continuous on $[a, b]$. Then, for every three distinct points of $[a, b]$, x_1, x_2, x_3 ,*

$$(3.1) \quad \inf_{a < x < b} \bar{D}^2f(x) \leq \frac{2H_2(f; x_1, x_2, x_3)}{(x_3 - x_1)(x_3 - x_2)} \leq \sup_{a < x < b} D^2f(x).$$

To establish this theorem we consider the function

$$(3.2) \quad g(x) = H_3(f; x_1, x_2, x_3, x) \quad (a < x < b).$$

According to (2.2), $g(x_1) = g(x_2) = g(x_3) = 0$. Let us assume $x_1 < x_3 < x_2$. Then, the continuous function $g(x)$ attains a non-negative maximum at some point q of the interval (x_1, x_2) ; this is obvious if $g(x) > 0$ at one point of (x_1, x_2) . The point q may coincide with x_3 , as it happens when $g(x) \leq 0$ at all points of (x_1, x_2) . Consequently

$$[g(q + 2h) - g(q)] - [g(q) - g(q - 2h)] \leq 0,$$

whence, according to (2.1)

$$(3.3) \quad (2h)^{-2} H_2(g; q - 2h, q, q + 2h) \leq 0$$

for any h , $0 < 2h < \min(q - x_1, x_2 - q)$.

Returning now to (3.2) we can obtain by simple computation

$$(3.4) \quad \frac{H_2(g; p_1, p_2, p_3)}{(p_3 - p_1)(p_3 - p_2)} = \frac{H_2(f; p_1, p_2, p_3)}{(p_3 - p_1)(p_3 - p_2)} - \frac{H_2(f; x_1, x_2, x_3)}{(x_3 - x_1)(x_3 - x_2)}$$

where p_1, p_2, p_3 are three arbitrary distinct points. Thus, setting $p_1 = q - 2h$, $p_2 = q$, $p_3 = q + 2h$:

$$(3.5) \quad \frac{H_2(g: q - 2h, q, q + 2h)}{(2h)^2} = \frac{H_2(f: q - 2h, q, q + 2h)}{(2h)^2} - \frac{2 H_2(f: x_1, x_2, x_3)}{(x_3 - x_1)(x_3 - x_2)}.$$

Relations (3.3) and (3.5) combine to give

$$(3.6) \quad \frac{H_2(f: q - 2h, q, q + 2h)}{(2h)^2} \leq \frac{2 H_2(f: x_1, x_2, x_3)}{(x_3 - x_1)(x_3 - x_2)}.$$

Considering a sequence of h for which the left side of (3.6) tends to the upper second generalized Riemann derivate of $f(x)$ at the point q , $\bar{D}^2 f(q)$, we have

$$\bar{D}^2 f(q) \leq \frac{2 H_2(f: x_1, x_2, x_3)}{(x_3 - x_1)(x_3 - x_2)}$$

and consequently

$$(3.7) \quad \inf_{a < x < b} \bar{D}^2 f(x) \leq \frac{2 H_2(f: x_1, x_2, x_3)}{(x_3 - x_1)(x_3 - x_2)}.$$

By a similar argument dealing with the minimum attained by the continuous function $g(x)$ on the interval (x_1, x_2) , we arrive at the relation

$$(3.8) \quad \sup_{a < x < b} \bar{D}^2 f(x) \geq \frac{2 H_2(f: x_1, x_2, x_3)}{(x_3 - x_1)(x_3 - x_2)}.$$

Relations (3.7) and (3.8) establish Theorem 3.1.

In proving relation (3.1) we assumed $x_1 < x_3 < x_2$. However, (3.1) holds for x_1, x_2, x_3 arbitrary but distinct since the expression

$$\frac{2 H_2(f: x_1, x_2, x_3)}{(x_3 - x_1)(x_3 - x_2)}$$

remains invariant under all permutations of x_1, x_2, x_3 .

In order to prove the fundamental theorem for $n = 2$ we consider the functions $F(x)$ and $G(x)$, where $F(x) - G(x)$ is continuous on $[a, b]$ and $D^2(F - G) = 0$ at all points of (a, b) . Then, by (3.1)

$$\frac{2 H_2(F - G: x_1, x_2, x_3)}{(x_3 - x_1)(x_3 - x_2)} = 0$$

for every three distinct points of $[a, b]$, x_1, x_2, x_3 . It follows that

$$H_2(F - G: x_1, x_2, x_3) = 0$$

whence, according to (2.1)

$$H_2(F: x_1, x_2, x_3) = H_2(G: x_1, x_2, x_3).$$

4. The fundamental theorem for $n > 2$. The fundamental theorem fails for $n = 3$ as we can easily show by considering two functions $F(x)$ and $G(x)$ that are defined on the interval $[-2, +3]$ and are such that $F(x)$

$-G(x) = |x|$. The function $|x|$ is continuous on $[-2, +3]$ and is such that $D^3|x| = 0$ at all points of $(-2, +3)$. Yet, $H_3(|x|; -1, 0, 1, 2)$ is not zero as can be seen by applying (2.2). Similarly, relation (3.1) fails, because $H_3(|x|; -1, 0, 1, 2) \neq 0$.

As we have mentioned in §1, in order to generalize the fundamental theorem we must impose additional conditions on the difference $F(x) - G(x)$. One procedure is to impose conditions on the generalized Riemann derivatives of $F(x) - G(x)$; in this way, we can show the following:

THEOREM 4.1. *If $F(x)$ and $G(x)$ are defined on $[a, b]$ and are such that at every point of (a, b) , $D^3(F - G) = 0$, $D^2(F - G)$ exists, and $D^1(F - G)$ exists, then*

$$H_3(F; x_1, x_2, x_3, x_4) = H_3(G; x_1, x_2, x_3, x_4)$$

for every four points x_1, x_2, x_3, x_4 of $[a, b]$, the first three being distinct.

Remark 4.1. We can make Theorem 4.1 stronger by replacing the existence of $D^2(F - G)$ with the weaker condition that

$$\lim_{h \rightarrow 0} (2h)^{-1} H_2(F - G; x - 2h, x, x + 2h) = 0$$

at every point x of (a, b) .

LEMMA 4.1. *If $(2h)^{-1} H_2(f; x - 2h, x, x + 2h)$ tends to zero with h for every $x \in (a, b)$, and if $D^1f(x)$ exists at every point x of (a, b) , then the derivative exists at every point x of (a, b) : in fact*

$$\frac{d}{dx}f(x) = D^1f(x).$$

It follows that $f(x)$ is continuous on $[a, b]$.

The truth of this lemma follows from the identities

$$\begin{aligned} & + (4h)^{-1}[f(x + 2h) - 2f(x) + f(x - 2h)] \\ & = (4h)^{-1}[f(x + 2h) - f(x - 2h)] - (-2h)^{-1}[f(x - 2h) - f(x)], \\ & - (4h)^{-1}[f(x + 2h) - 2f(x) + f(x - 2h)] \\ & = (4h)^{-1}[f(x + 2h) - f(x - 2h)] - (2h)^{-1}[f(x + 2h) - f(x)]. \end{aligned}$$

Taking limits as $h \rightarrow 0$ ($h > 0$), we get

$$0 = D^1f(x) - \frac{d}{dx}f(x).$$

In order to prove Theorem 4.1 we observe that the function $F(x) - G(x)$ satisfies the conditions of Lemma 4.1, and consequently its derivative exists everywhere on (a, b) . Then, according to a theorem of Verblunsky (6, p. 393), together with the condition that $D^3(F - G) = 0$ everywhere on (a, b) , we conclude that $F(x) - G(x)$ is a polynomial of degree at most 2 on $[a, b]$. We have by direct application of (2.2)

$$H_3(F: x_1, x_2, x_3, x_4) = H_3(G: x_1, x_2, x_3, x_4).$$

We now determine a set of conditions, different from those of Theorem 4.1, under which $H_3(F: x_1, x_2, x_3, x_4) = H_3(G: x_1, x_2, x_3, x_4)$.

DEFINITION 4.1. Let $F(x)$ be any single valued function defined over a given domain. Then we define

$$H_n(F: x_1, x_2, \dots, x_n, x_{n+1}) = w_{n+1}(x_{n+1}) \cdot \sum_{j=1}^{n+1} \frac{F(x_j)}{w_{n+2}(x_j)} \quad (n = 0, 1, 2, \dots),$$

where

$$w_{n+1}(y) = \prod_{i=1}^n (y - x_i),$$

and the "primes" denote ordinary differentiations. For $n = 1$, the above relation reads

$$H_1(F: x_1, x_2) = F(x_2) - F(x_1);$$

for $n = 0$, we have $H_0(F: x_1) = F(x_1)$.

Now, let $f(x)$ be defined and continuous on $[a, b]$, and suppose that

$$\inf_{a < x < b} D^n f(x) \text{ and } \sup_{a < x < b} \bar{D}^n f(x)$$

are finite.

Set

$$(4.1) \quad g(x) = H_{n+1}(f: x_1, x_2, \dots, x_{n+1}, x) \quad (a \leq x \leq b),$$

$$(4.2) \quad y(x, h) = H_{n-2}(g: x - nh + 2h, x - nh + 4h, \dots, x + nh - 2h) \quad (h > 0)$$

where x_1, x_2, \dots, x_{n+1} are $n + 1$ arbitrary points of $[a, b]$ such that $x_1 < x_2 < \dots < x_n < x_{n+1}$.

It follows directly from (4.1) that $g(x_i) = 0$, where $i = 1, 2, \dots, n + 1$. Consequently, the continuous function $g(x)$ attains n extrema, each of which is an absolute extremum over one of the intervals (x_j, x_{j+1}) , ($j = 1, 2, \dots, n$).

Let q be the point of (x_1, x_2) at which $g(x)$ attains its absolute extremum over the interval (x_1, x_2) . Then, for h fixed and small, we can find two points x' and x'' of the intervals (x_1, q) and (q, x_2) respectively, such that

$$[g(x' + h) - g(x' - h)] [g(x'' + h) - g(x'' - h)] \leq 0$$

where $x' + h = q = x'' - h$. The function $u_1(x, h) = g(x + h) - g(x - h)$ is continuous in x for h fixed. It follows that $u_1(x, h)$ vanishes at some point of the interval $[x', x'']$ ($x_1 < x' < x'' < x_2$), because $u_1(x, h)$ changes sign between x' and x'' .

Dealing in a similar way with the absolute extrema of $g(x)$ over the remaining $n - 1$ intervals (x_s, x_{s+1}) ($s = 2, 3, \dots, n$), we conclude that the function $u_1(x, h)$, for h fixed and small, vanishes at n points of the interval (x_1, x_{n+1}) .

Applying successively the same argument to the functions

$$u_t(x, h) = u_{t-1}(x + h, h) - u_{t-1}(x - h, h) \quad (t = 2, 3, \dots, n - 2),$$

we conclude that the function $y(x, h) \equiv u_{n-2}(x, h)$, for h fixed and small, vanishes at three distinct points of the interval (x_1, x_{n+1}) , L_h , M_h , and N_h with $L_h < M_h < N_h$. It follows then that the function $y(x, h)$, which is continuous in x , attains an absolute non-negative maximum at some point Q_h of the interval (L_h, N_h) if $y(x, h) > 0$ at one point of (L_h, N_h) . The point Q_h may coincide with M_h if $y(x, h) \leq 0$ at all points of (L_h, N_h) . Similarly, $y(x, h)$ attains an absolute non-positive minimum at some point R_h of the interval (L_h, N_h) and $Q_h \neq R_h$. Consequently

$$y(Q_h + 2h, h) - y(Q_h, h) \leq 0, \quad y(Q_h - 2h, h) - y(Q_h, h) \leq 0$$

whence

$$(4.3) \quad y(Q_h + 2h, h) - 2y(Q_h, h) + y(Q_h - 2h, h) \leq 0$$

for any h , $0 < 2h < \min(Q_h - L_h, N_h - Q_h)$.

Relations (4.2) and (4.3) combine to give

$$(4.4) \quad (2h)^{-n} H_n(g: Q_h - nh, Q_h - nh + 2h, \dots, Q_h + nh) \leq 0 \quad (h > 0).$$

Dealing in a similar way with the point R_h , we obtain

$$(4.5) \quad (2h)^{-n} H_n(g: R_h - nh, R_h - nh + 2h, \dots, R_h + nh) \geq 0 \quad (h > 0)$$

for any h , $0 < 2h < \min(R_h - L_h, N_h - R_h)$.

DEFINITION 4.2. The continuous function $f(x)$ belongs to the class K_n of continuous functions if for arbitrary ϵ there exist h' and h'' , satisfying (4.4) and (4.5) respectively, such that the expressions

$$(4.6) \quad \begin{aligned} & (2h')^{-n} H_n(f: Q_{h'} - nh', Q_{h'} - nh' + 2h', \dots, Q_{h'} + nh') \\ & (2h'')^{-n} H_n(f: R_{h''} - nh'', R_{h''} - nh'' + 2h'', \dots, R_{h''} + nh'') \end{aligned}$$

lie in the interval $[\inf \bar{D}^n f(x) - \epsilon, \sup \bar{D}^n f(x) + \epsilon]$ ($a < x < b$).

THEOREM 4.2. If the continuous function $f(x)$ belongs to the class K_n of continuous functions on $[a, b]$, then

$$(4.7) \quad \inf_{a < x < b} \bar{D}^n f(x) \leq \frac{n! H_n(f: x_1, x_2, \dots, x_{n+1})}{(x_{n+1} - x_1)(x_{n+1} - x_2) \dots (x_{n+1} - x_n)} \leq \sup_{a < x < b} \bar{D}^n f(x)$$

for every $n + 1$ distinct points of $[a, b]$, x_1, x_2, \dots, x_{n+1} .

Consider the identity

$$(4.8) \quad \frac{H_n(g: p_1, \dots, p_{n+1})}{(p_{n+1} - p_1) \dots (p_{n+1} - p_n)} = \frac{H_n(f: p_1, \dots, p_{n+1})}{(p_{n+1} - p_1) \dots (p_{n+1} - p_n)} - \frac{H_n(f: x_1, \dots, x_{n+1})}{(x_{n+1} - x_1) \dots (x_{n+1} - x_n)}$$

where p_1, p_2, \dots, p_{n+1} are $n + 1$ arbitrary distinct points. We substitute $p_1 = Q_{h'} - nh'$, $p_2 = Q_{h'} - (n - 2)h'$, \dots , $p_{n+1} = Q_{h'} + nh'$ and thus we obtain

$$(4.9) \quad \frac{H_n(g: Q_{h'} - nh', \dots, Q_{h'} + nh')}{(2h')^n} = \frac{H_n(f: Q_{h'} - nh', \dots, Q_{h'} + nh')}{(2h')^n} - \frac{n! H_n(f: x_1, \dots, x_{n+1})}{(x_{n+1} - x_1) \dots (x_{n+1} - x_n)}.$$

Similarly, we substitute in (4.8) $p_1 = R_{h''} - nh''$, $p_2 = R_{h''} - (n-2)h''$, ..., $p_{n+1} = R_{h''} + nh''$, and thus we obtain

$$(4.10) \quad \frac{H_n(g: R_{h''} - nh'', \dots, R_{h''} + nh'')}{(2h'')^n} = \frac{H_n(f: R_{h''} - nh'', \dots, R_{h''} + nh'')}{(2h'')^n} - \frac{n! H_n(f: x_1, \dots, x_{n+1})}{(x_{n+1} - x_1) \dots (x_{n+1} - x_n)}.$$

Relations (4.4), (4.5), (4.9), (4.10) combine to give

$$\begin{aligned} \frac{H_n(f: Q_{h'} - nh', \dots, Q_{h'} + nh')}{(2h')^n} &\leq \frac{n! H_n(f: x_1, \dots, x_{n+1})}{(x_{n+1} - x_1) \dots (x_{n+1} - x_n)} \\ &\leq \frac{H_n(f: R_{h''} - nh'', \dots, R_{h''} + nh'')}{(2h'')^n}, \end{aligned}$$

whence Theorem 4.2 follows because the expressions (4.6) lie in the interval $[\inf \underline{D}^n f(x) - \epsilon, \sup \bar{D}^n f(x) + \epsilon]$ ($a < x < b$).

THEOREM 4.3. *If $F(x)$ and $G(x)$ are defined on $[a, b]$ and are such that $F(x) - G(x)$ belongs to the class K_n , and $D^n(F - G) = 0$ at all points of (a, b) , then*

$$H_n(F: x_1, \dots, x_{n+1}) = H_n(G: x_1, \dots, x_{n+1})$$

for every $n+1$ points of $[a, b]$, x_1, \dots, x_{n+1} , where x_1, \dots, x_n are distinct points.

To prove this theorem we consider the functions $F(x)$ and $G(x)$ where $F(x) - G(x)$ belongs to the class K_n and is such that $D^n(F - G) = 0$ for $a < x < b$. Then according to (4.7) $H_n(F - G: x_1, \dots, x_{n+1}) = 0$. It then follows from the definition 4.1 that

$$H_n(F: x_1, \dots, x_{n+1}) = H_n(G: x_1, \dots, x_{n+1}).$$

5. Additional remarks. Theorem 4.2 reduces to Denjoy's theorem 3.1 for $n = 2$. Indeed, let $f(x)$ be defined and continuous on $[a, b]$ and suppose that

$$\inf_{a < x < b} \underline{D}^2 f(x) \text{ and } \sup_{a < x < b} \bar{D}^2 f(x)$$

are finite. Putting $n = 2$ in relations (4.1) and (4.2) we obtain

$$g(x) = H_3(f: x_1, x_2, x_3, x)$$

$$(a \leq x \leq b)$$

$$y(x, h) = H_0(g: x)$$

whence, by the definition 4.1, we have $y(x, h) = g(x)$. Consequently, the roots and the extrema of $y(x, h)$ are independent of h and are identical with those of the function $g(x)$, respectively. The expressions that correspond to (4.6) are obtained by setting $n = 2$, and thus we get

$$(5.1) \quad (2h)^{-2} H_2(f: Q - 2h, Q, Q + 2h)$$

$$(2h)^{-2} H_2(f: R - 2h, R, R + 2h).$$

Due to the fact that $\bar{D}^2 f(Q)$, $\underline{D}^2 f(Q)$, $\bar{D}^2 f(R)$, $\underline{D}^2 f(R)$, lie in the interval $[\inf \underline{D}^2 f(x), \sup \bar{D}^2 f(x)]$ ($a < x < b$), it follows that for arbitrary ϵ there exist values of h for which the expressions (5.1) lie in the interval $[\inf \underline{D}^2 f(x) - \epsilon, \sup \bar{D}^2 f(x) + \epsilon]$ ($a < x < b$). Consequently, the arbitrary continuous function $f(x)$ belongs to the class K_2 and we conclude that the class K_2 is identical with the class of all continuous functions of x . Thus, Denjoy's theorem 3.1 as well as the fundamental theorem for $n = 2$ are particular cases of the general theorems 4.2 and 4.3, respectively.

Further, it is easy to show that if the function $f(x)$ possesses an n th ordinary derivative or a de La Vallée Poussin derivative of order n , $f_{(n)}(x)$ (5, p. 1), at every point x of (a, b) , then $f(x)$ belongs to the class K_n of continuous functions. Indeed, in both these cases the function $f(x)$ possesses an n th generalized Riemann derivative $D^n f(x)$ equal to the n th ordinary derivative, or to $f_{(n)}(x)$, respectively, at every point x of (a, b) . Moreover, it is known (1, p. 207) that in either of these cases the expression $(2h)^{-n} H_n(f: x - nh, x - nh + 2h, \dots, x + nh)$, where $a < x - nh < x + nh < b$, lies in the interval $[\inf D^n f(x), \sup D^n f(x)]$ ($a < x < b$). It then follows that the function $f(x)$ belongs to the class K_n .

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