Glasgow Math. J. **47** (2005) 205–212. © 2005 Glasgow Mathematical Journal Trust.
DOI: 10.1017/S001708950400223X. Printed in the United Kingdom

THREE SOLUTIONS FOR A PERTURBED SUBLINEAR ELLIPTIC PROBLEM IN $\mathbb{R}^N$

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(Received 3 August, 2004; accepted 4 October, 2004)

Abstract. In this paper we study a perturbed sublinear elliptic problem in $\mathbb{R}^N$. In particular, using variational methods, we establish a result that ensures the existence of at least three weak solutions.

2000 *Mathematics Subject Classification.* 35J20.

1. Introduction. In this paper we are concerned with the following perturbation problem

$$\begin{cases} -\Delta u = h(x)|u|^{s-2}u + \lambda f(x,u) \text{ in } \mathbb{R}^N \\ u \in D^{1,2}(\mathbb{R}^N) \end{cases} \qquad (P_\lambda)$$

where $s \in]1,2[$, $N \geq 3$, $\lambda \in \mathbb{R}_+$, $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $h \in L^\infty_{loc}(\mathbb{R}^N) \cap L^{\frac{2N}{2N-s(N-2)}}(\mathbb{R}^N)$ is a function almost everywhere nonnegative in $\mathbb{R}^N$ such that, for some set $A \subseteq \mathbb{R}^N$ of positive measure, $\operatorname{ess\,inf}_A h > 0$ and

$$D^{1,2}(\mathbb{R}^N) = \left\{ u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N) / \nabla u \in L^2(\mathbb{R}^N) \right\}$$

is the completion of

$$C_0(\mathbb{R}^N) = \{u \in C(\mathbb{R}^N) : \text{supp}(u) \text{ (support of } u) \text{ is compact}\}$$

with respect to the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{1}{2}},$$

induced by the scalar product

$$(u, v) = \int_{\mathbb{R}^N} \nabla u(x) \nabla v(x) \, dx.$$

Moreover, we assume that the function f satisfies the following conditions.

(a) There exist $q \in [0, \frac{N+2}{N-2}[$, $\alpha \in L^t(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$, where we set $t = \frac{2N}{N+2-q(N-2)}$, $\beta \in L^{\frac{2N}{N+2}}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ such that

$$|f(x,t)| \leq \alpha(x)|t|^q + \beta(x),$$

for a.e. $x \in \mathbb{R}^N$ and all $t \in \mathbb{R}$.

(b) There exist three functions $\gamma \in L^{\frac{N}{2}}(\mathbb{R}^N, \mathbb{R}_+)$, $\delta \in L^{\frac{2N}{N+2}}(\mathbb{R}^N, \mathbb{R}_+)$ and $\chi \in L^1(\mathbb{R}^N, \mathbb{R}_+)$ such that

$$\int_0^t f(x,s)\,ds \le \gamma(x)|t|^2 + \delta(x)|t| + \chi(x),$$

for a.e. $x \in \mathbb{R}^N$ and all $t \in \mathbb{R}$.

For each $u \in D^{1,2}(\mathbb{R}^N)$,

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx - \frac{1}{s}\int_{\mathbb{R}^N} h(x)|u(x)|^s - \lambda \int_{\mathbb{R}^N}\left(\int_0^{u(x)} f(x,s)\,ds\right) dx$$

is the energy functional relative to problem (P_λ). In proving our result, we will show that the above assumptions on h and f ensure, in particular, that E is continuously differentiable on $D^{1,2}(\mathbb{R}^N)$. Hence the weak solutions of (P_λ) corresponds to the critical points of E.

Our result is as follows.

THEOREM 1.1. *Under the assumptions stated above there exist $\sigma, \bar{\lambda} > 0$ such that, for each $\lambda \in [0, \bar{\lambda}[$, problem (P_λ) admits at least three distinct solutions in $D^{1,2}(\mathbb{R}^N)$ whose norms are not greater than σ.*

In the literature, most of the papers that deal with multiple solutions for elliptic equations on $\mathbb{R}^N$ consider particular types of nonlinearities as power functions, convex or concave functions; see for instance [**4**, **9**]. This is due to the lack of compact embeddings for Sobolev spaces on unbounded domains which, in many cases, represents the main difficulty in using a variational approach. Such a difficulty is often overcome by exploiting the particular properties of the nonlinearity as homogeneities or symmetries.

In this paper, we consider a nonlinearity which is a pertubation of a particular symmetric sublinear nonlinearity (i.e. $h(x)|t|^{s-2}t$) by a function f that must satisfy only the growth conditions (a) and (b). This kind of nonlinearity allows us to study our problem by means of variational methods and, in particular, to apply a consequence (Theorem 2.1 in [**1**]) of a variational principle stated in [**6**].

The paper is organized into three sections. In Section 2 some preliminary results are stated and proved. Section 3 contains the proof of Theorem 1.1.

2. Preliminary lemmas. For the reader's convenience, we begin this section with the following known result.

LEMMA 2.1. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence weakly converging to u in $D^{1,2}(\mathbb{R}^N)$. Then, for $p \in [1, \frac{2N}{N-2}[$ and for any bounded measurable set $\Omega \subseteq \mathbb{R}^N$, $\{u_{n|\Omega}\}_{n\in\mathbb{N}}$ strongly converges to $u_{|\Omega}$ in $L^p(\Omega)$.*

Proof. Consider $\{u_n\}_{n\in\mathbb{N}}$ weakly converging to u in $D^{1,2}(\mathbb{R}^N)$. We note that $u_n \to u$ weakly in $L^{\frac{2N}{N-2}}(\mathbf{R}^N)$. Hence, in particular, for every $v \in C_0^\infty(\mathbb{R}^N)$, one has

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} u_n(x)v(x)\,dx = \int_{\mathbf{R}^N} u(x)v(x)\,dx. \tag{2.1}$$

https://doi.org/10.1017/S001708950400223X Published online by Cambridge University Press

Fix $R > 0$, $p \in [1, \frac{2N}{N-2}[$ and set $B_R = \{x \in \mathbf{R}^N : |x| \le R\}$. We prove that $u_n \to u$ strongly in $L^p(B_R)$.

Indeed, denote by $u_{|B_R}$ the restriction of u to B_R and suppose that $\{u_n\}$ does not converge to $u_{|B_R}$ weakly in $W^{1,2}(B_R)$. Since $\{u_n\}$ is norm bounded in $W^{1,2}(B_R)$, there exist a subsequence $\{u_{n_k}\}$ and $\bar{u} \in W^{1,2}(B_R)$, with $\bar{u} \neq u_{|B_R}$, such that $u_{n_k} \to \bar{u}$ weakly in $W^{1,2}(B_R)$. By the Rellich-Kondrachov's Theorem, $u_{n_k} \to \bar{u}$ strongly in $L^p(B_R)$.

Taking into account (2.1), we obtain

$$\int_{B_R} u(x)v(x)\,dx = \lim_{k\to\infty} \int_{B_R} u_{n_k}(x)v(x)\,dx = \int_{B_R} \bar{u}(x)v(x)\,dx,$$

for every $v \in C_0^\infty(B_R)$. Then $u(x) = \bar{u}(x)$, for almost all $x \in B_R$, against the fact that $\bar{u} \neq u_{|B_R}$. Hence $\{u_n\}$ weakly converges to $u_{|B_R}$ in $W^{1,2}(B_R)$. Applying the Rellich-Kondrachov's Theorem again, we conclude that $\{u_n\}$ strongly converges to u in $L^p(B_R)$. □

LEMMA 2.2. *Condition* (*a*) *implies that the functional*

$$\Phi : u \to \int_{\mathbb{R}^N} \left(\int_0^{u(x)} f(x,t)\,dt \right) dx,$$

is well-defined, strongly continuous and Gateâux differentiable in $D^{1,2}(\mathbb{R}^N)$ *with compact Gateâux derivative.*

Proof. By standard arguments, Φ is proved to be well-defined, strongly continuous and Gateâux differentiable in $D^{1,2}(\mathbb{R}^N)$ and so we limit ourselves to prove that the Gateâux derivative Φ' is compact.

Let $\{u_n\}$ be a bounded sequence in $D^{1,2}(\mathbb{R}^N)$. There exist $\bar{u} \in D^{1,2}(\mathbb{R}^N)$ and a subsequence, which we denote by $\{u_n\}$ again, weakly converging to $\bar{u}$ in $D^{1,2}(\mathbb{R}^N)$. Now, fix $\epsilon > 0$ arbitrarily and choose $M > 0$ such that

$$\|\alpha\|_{L^t(|x|>M)} < \epsilon, \quad \|\beta\|_{L^{\frac{2N}{N+2}}(|x|>M)} < \epsilon.$$

For every $v \in H$ with $\|v\| = 1$ and $n \in \mathbb{N}$, one has

$$\begin{aligned} |(\Phi(u_n) - \Phi(\bar{u}))(v)| \le & \left| \int_{|x|\le M} (f(x,u_n(x)) - f(x,\bar{u}(x)))v(x)\,dx \right| \\ & + \left| \int_{|x|>M} f(x,u_n(x)) - f(x,\bar{u}(x)))v(x)\,dx \right| \\ & \le C_1 \left(\int_{|x|\le M} |f(x,u_n(x)) - f(x,\bar{u}(x))|^{\frac{2N}{N+2}} \right)^{\frac{N+2}{2N}} \\ & + C_2 \|\alpha\|_{L^t(|x|>M)}^{\frac{N+2}{2N}} + C_3 \|\beta\|_{L^{\frac{2N}{N+2}}(|x|>M)}, \end{aligned}$$

where C_1, C_2, C_3 are suitable positive constants. By Lemma 2.1, $\{u_n\}$ converges strongly in $L^{q+1}(|x| \le M)$, so that $f(\cdot, u_n(\cdot)) \to f(\cdot, \bar{u}(\cdot))$ strongly in $L^{\frac{2N}{N+2}}(|x| \le M)$. Hence, there exists $\nu \in \mathbb{N}$ such that, for $n > \nu$, one has

$$\|\Phi(u_n) - \Phi(\bar{u})\|_{H^*} = \sup_{v\in H, \|v\|=1} |(\Phi(u_n) - \Phi(\bar{u}))(v)| \le C_1\epsilon + C_2\epsilon^{\frac{N+2}{2N}} + C_3\epsilon.$$

This proves that Φ' is a compact operator. □

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LEMMA 2.3. *Let* $\Psi : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ *be defined by*

$$\Psi(u) = \frac{1}{2}\|u\|^2 - \frac{1}{s}\int_{\mathbb{R}^N} h(x)|u(x)|^s\,dx.$$

There exists a unique $u_0 \in D^{1,2}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$*, with* $u_0 > 0$ *in* $\mathbb{R}^N$*, such that*

$$\Psi(u_0) = \inf_{u \in D^{1,2}(\mathbb{R}^N)} \Psi(u). \tag{2.2}$$

Proof. By Lemma 2.2, the functional

$$u \to \int_{\mathbb{R}^N} h(x)|u(x)|^s\,dx$$

has compact Gateâux derivative in $D^{1,2}(\mathbb{R}^N)$. Hence, in particular, it follows that Ψ is weakly sequentially lower semicontinuous. Since Ψ is also coercive, $\Psi(u) = \Psi(|u|)$ and $\inf_{D^{1,2}(\mathbb{R}^N)} \Psi < 0$, there exists $u_0 \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}$ such that

$$\Psi(u_0) = \inf_{D^{1,2}(\mathbb{R}^N)} \Psi, \tag{2.3}$$

with $u_0 \geq 0$ almost everywhere in $\mathbb{R}^N$. Consequently u_0 is a weak solution of the problem

$$\begin{cases} -\Delta u = h(x)|u|^{s-2}u \ \text{ in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N). \end{cases} \qquad (P_0)$$

From Lemma B.3 in [**8**] and Theorem 8.22 in [**3**], it follows that $u_0 \in C(\mathbb{R}^N)$. Moreover the strong maximum principle ensures that $u_0(x) > 0$, for all $x \in \mathbb{R}^N$.

Now, for each $R > 0$, set $B_R = \{x \in \mathbb{R}^N : |x| < R\}$ and let $u_R \in W_0^{1,2}(B_R)$ such that

$$\Psi(u_R) = \inf_{u \in W_0^{1,2}(B_R)} \Psi(u),$$

and $u_R \geq 0$ a.e. in B_R. The function u_R exists and is unique. Also the net $\{u_R\}_{R>0}$ is increasing with respect to R; see [**2**]. Moreover, by definition, the net $\{\Psi(u_R)\}$ is not increasing with respect to R and is bounded below by $\inf_{D^{1,2}(\mathbb{R}^N)} \Psi$. We deduce that

$$\lim_{R\to+\infty} \Psi(u_R) \geq \inf_{u \in D^{1,2}(\mathbb{R}^N)} \Psi(u).$$

Since Ψ is coercive, there exist a subsequence $\{R_n\}$, with $\lim_{n\to\infty} R_n = +\infty$, and $\tilde{u} \in D^{1,2}(\mathbb{R}^N)$ such that $u_{R_n} \to \tilde{u}$ weakly in $D^{1,2}(\mathbb{R}^N)$ as $n \to \infty$. Hence, we exploit the sequentially weak lower semicontinuity of Ψ and obtain

$$\Psi(\tilde{u}) \leq \lim_{n\to\infty} \Psi\left(u_{R_n}\right) = \lim_{R\to+\infty} \Psi(u_R).$$

Furthermore, by Lemma 2.1 and the monotony of $\{u_R\}_{R>0}$, we also have that

$$\lim_{R\to+\infty} u_R(x) = \lim_{n\to\infty} u_{R_n}(x) = \tilde{u}(x),$$

for a.e. $x \in \mathbb{R}^N$.

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We claim that $\Psi(\tilde{u}) = \inf_{D^{1,2}(\mathbb{R}^N)} \Psi$. Suppose that

$$\Psi(\tilde{u}) > \inf_{D^{1,2}(\mathbb{R}^N)} \Psi.$$

Consequently, there would exist $v \in C_0^\infty(\mathbb{R}^N)$ such that

$$\Psi(v) < \Psi(\tilde{u}). \tag{2.4}$$

On the other hand, if we choose $\bar{R} > 0$ such that $\mathrm{supp}(v) \subseteq B_{\bar{R}}$, we also have

$$\Psi(v) \geq \inf_{u \in W_0^{1,2}(B_{\bar{R}})} \Psi(u) = \Psi(u_{\bar{R}}) \geq \Psi(\tilde{u}),$$

contradicting (2.4).

Let $\bar{u} \in D^{1,2}(\mathbb{R}^N)$ be such that

$$\Psi(\bar{u}) = \inf_{D^{1,2}(\mathbb{R}^N)} \Psi,$$

with $\bar{u} \geq 0$ a.e. in $\mathbb{R}^N$. It turns out that $\bar{u} = \tilde{u}$.

Indeed, for each $R > 0$, $\bar{u}$ is a supersolution for the problem

$$\begin{cases} -\Delta u = h(x)u^{s-1} & \text{in } B_R \\ u = 0 & \text{on } \partial B_R, \end{cases}$$

so that $u_R \leq \bar{u}$ a.e. in $\mathbb{R}^N$. Passing to the limit as $R \to +\infty$, we find that

$$\tilde{u}(x) \leq \bar{u}(x), \tag{2.5}$$

for a.e. $x \in \mathbb{R}^N$. From the fact that $\tilde{u}$ and $\bar{u}$ are, in particular, critical points of Ψ such that $\Psi(\tilde{u}) = \Psi(\bar{u})$, it follows that

$$\int_{\mathbb{R}^N} h(x)[(\tilde{u}(x))^s - (\bar{u}(x))^s]\, dx = 0. \tag{2.6}$$

Consequently, from (2.5) and (2.6), $\tilde{u}(x) = \bar{u}(x)$ for a.e. $x \in \mathbb{R}^N$. □

LEMMA 2.4. *Let Ψ and u_0 be as in Lemma 2.3. For every $u \in D^{1,2}(\mathbb{R}^N)$, the real function*

$$r \to \inf_{\|w\|=r} \Psi(u + w)$$

is continuous in $\mathbb{R}_+$. Moreover, there exists $r_0 > 0$ such that, for every $0 < r < r_0$, we have

$$\Psi(\pm u_0) < \inf_{\|v\|=r} \Psi(v \pm u_0).$$

Proof. Fix $u \in D^{1,2}(\mathbb{R}^N)$. For each $r > 0$, one has

$$\inf_{\|v\|=r} \Psi(u+v) = \frac{1}{2}r^2 + \frac{1}{2}\|u\|^2 - \sup_{\|v\|=r} \left((u, v) - \frac{1}{s} \int_{\mathbb{R}^N} h(x)|u(x) + v(x)|^s\, dx \right).$$

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Hence the continuity follows from the weak sequential continuity of the function

$$v \to (u, v) - \frac{1}{s}\int_{\mathbb{R}^N} h(x)|u(x) + v(x)|^s\, dx.$$

See Lemma 6 of [7].

Let $r_0 = 2\|u_0\|$ and suppose that, for some $0 < \bar{r} < r_0$, we have

$$\Psi(u_0) = \inf_{\|v\|=\bar{r}} \Psi(u_0 + v). \tag{2.7}$$

Then, we can find a sequence $\{u_k\}$ in $D^{1,2}(\mathbb{R}^N)$ with $\|u_k\| = \bar{r}$, for all $k \in \mathbb{N}$, such that

$$\lim_{k\to+\infty} \Psi(u_0 + u_k) = \Psi(u_0).$$

Up to a subsequence, we can suppose that $u_k \to \bar{u}$ weakly in H. In particular, we have $\|\bar{u}\| \leq \bar{r}$,

$$(u_k, u_0) = \int_\Omega \nabla u_k(x)\nabla u_0(x)\, dx \to \int_{\mathbb{R}^N} \nabla\bar{u}(x)\nabla u_0(x)\, dx = (\bar{u}, u_0) \tag{2.8}$$

and

$$\int_{\mathbb{R}^N} h(x)|u_k(x) + u_0(x)|^s\, dx \to \int_{\mathbb{R}^N} h(x)|\bar{u}(x) + u_0(x)|^s\, dx, \tag{2.9}$$

as $k \to \infty$.

Consequently, from (2.2), (2.7), (2.8) and (2.9), one has

$$\begin{aligned}
\inf_{D^{1,2}(\mathbb{R}^N)} \Psi = \Psi(u_0) &= \lim_{k\to+\infty} \Psi(u_0 + u_k) \\
&= \lim_{k\to+\infty}\left(\frac{1}{2}\|u_0\|^2 + \frac{1}{2}\bar{r}^2 + (u_k, u_0) - \frac{1}{s}\int_{\mathbb{R}^N} h(x)|u_k(x) + u_0(x)|^s\, dx\right) \\
&= \frac{1}{2}\|u_0\|^2 + \frac{1}{2}\bar{r}^2 + (\bar{u}, u_0) - \frac{1}{s}\int_{\mathbb{R}^N} h(x)|\bar{u}(x) + u_0(x)|^s\, dx \\
&\geq \frac{1}{2}\|u_0\|^2 + \frac{1}{2}\|\bar{u}\|^2 + (\bar{u}, u_0) - \frac{1}{s}\int_{\mathbb{R}^N} h(x)|\bar{u}(x) + u_0(x)|^s\, dx = \Psi(\bar{u} + u_0).
\end{aligned}$$

From this we deduce that $\bar{u} + u_0$ as well as $|\bar{u} + u_0|$ are global minima for Ψ and that $\|\bar{u}\| = \bar{r}$. Using the same arguments to prove regularity and positivity of u_0, we have $|\bar{u} + u_0|, \bar{u} + u_0 \in C(\mathbb{R}^N)$ and $|\bar{u}(x) + u_0(x)| > 0$, for all $x \in \mathbb{R}^N$. Note that $\bar{u} \in C(\mathbb{R}^N)$, being $\bar{u} = (\bar{u} + u_0) - u_0$. By applying Lemma 2.3, we obtain

$$|\bar{u}(x) + u_0(x)| = u_0(x), \tag{2.10}$$

for all $x \in \mathbb{R}^N$. At this point, put

$$A = \{x \in \mathbb{R}^N : \bar{u}(x) = 0\}$$

and

$$B = \{x \in \mathbb{R}^N : \bar{u}(x) = -2u_0(x)\}.$$

By (2.10), $A \cup B = \mathbb{R}^N$ and, because $u_0 > 0$ in $\mathbb{R}^N$, $A \cap B = \emptyset$. Moreover, the continuity of $\bar{u}$, u_0, implies that A and B are closed sets in $\mathbb{R}^N$. Since $\mathbb{R}^N$ is connected, it results that $A = \mathbb{R}^N$ or $B = \mathbb{R}^N$.

In the former case, it should be $\bar{u} = 0$ against the fact that $\|\bar{u}\| = \bar{r} > 0$. In the latter case, it should be $\bar{u} = -2u_0$ and hence $\|\bar{u}\| = 2\|u_0\|$ contrary to $\bar{r} < 2\|u_0\|$.

Taking into account that $\Psi(-u_0) = \Psi(u_0)$, the same conclusion holds with $-u_0$ in place of u_0, as can be easily checked. □

3. Proof of Theorem 1.1. Consider the functionals $\Psi, \Phi : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ defined as follows:

$$\Psi(u) = \frac{1}{2}\|u\|^2 - \frac{1}{s}\int_{\mathbb{R}^N} h(x)|u(x)|^s\, dx,$$

$$\Phi(u) = -\int_{\mathbb{R}^N}\left(\int_0^{u(x)} f(x,t)\, dt\right) dx.$$

By Lemma 2.2, it follows that Φ is weakly sequentially continuous and Ψ is weakly sequentially lower semicontinuous in $D^{1,2}(\mathbb{R}^N)$. By condition (b), it also follows that there exists $\lambda_1 > 0$ such that, for every $\lambda \in [0, \lambda_1[$, the functional $\Psi + \lambda\Phi$ is coercive.

Put $x_1 = u_0$, $x_2 = -u_0$ and fix $0 < \bar{r} < \min\{\|u_0\|, r_0\}$. By Lemma 2.4 it follows that the hypotheses of Theorem 2.1 in **[1]** are satisfied and so there exists $0 < \bar{\lambda} < \lambda_1$ such that, for every $\lambda \in]0, \bar{\lambda}[$, $\Psi + \lambda\Phi$ admits two distinct local minima $u_1^{(\lambda)}, u_2^{(\lambda)} \in H$, with $\|x_i - u_i^{(\lambda)}\| < \bar{r}$, for $i = 1, 2$. Fix $\lambda \in [0, \bar{\lambda}[$. Since $\Psi + \lambda\Phi$ satisfies the Palais-Smale condition (see Example 38.25 in **[10]**), Theorem 1 of **[5]** implies the existence of a third critical point $u_3^{(\lambda)}$, distinct from $u_1^{(\lambda)}$ and $u_2^{(\lambda)}$, that satisfies

$$\Psi\big(u_3^{(\lambda)}\big) + \lambda\Phi\big(u_3^{(\lambda)}\big) = \phi(\lambda),$$

where

$$\phi(\lambda) = \inf_{\psi\in\Gamma_\lambda}\sup_{t\in[0,1]} (\Psi(\psi(t)) + \lambda\Phi(\psi(t))),$$

and

$$\Gamma_\lambda = \left\{\psi \in C([0,1], H) : \psi(0) = u_1^{(\lambda)} \text{ and } \psi(1) = u_2^{(\lambda)}\right\}.$$

We note that, for every $\lambda \in [0, \bar{\lambda}[$, the function $\psi_\lambda : t \in [0,1] \to u_1^{(\lambda)} + (1-t)u_2^{(\lambda)}$ belongs to Γ_λ. Moreover, it follows that

$$\sup_{\lambda\in[0,\bar{\lambda}[}\sup_{t\in[0,1]} \|\psi_\lambda(t)\| \leq 2(\|u_0\| + \bar{r}).$$

Exploiting the fact that Ψ is the sum of the norm squared and a sequentially weakly continuous functional and that Φ is sequentially weakly continuous, one has

$$\begin{aligned}\sup_{\lambda\in[0,\bar{\lambda}[}\phi(\lambda) &\leq \sup_{\lambda\in[0,\bar{\lambda}[}\sup_{t\in[0,1]} (\Psi(\psi_\lambda(t)) + \lambda\Phi(\psi_\lambda(t)))\\ &\leq \sup_{\|v\|\leq 2(\|u_0\|+\bar{r})}\Psi(v) + \bar{\lambda}\sup_{\|v\|\leq 2(\|u_0\|+\bar{r})}\Phi(v) < +\infty. \qquad (3.11)\end{aligned}$$

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At this point, we can prove that $\sup_{\lambda\in[0,\bar{\lambda}[} \|u_3^{(\lambda)}\| < \infty$. In fact, if

$$\sup_{\lambda\in[0,\bar{\lambda}[} \left\|u_3^{(\lambda)}\right\| = +\infty,$$

there would exist a bounded sequence $\{\lambda_n\} \subset [0, \bar{\lambda}[$ such that $\|u_3^{(\lambda_n)}\| \to +\infty$, as $n \to \infty$. Consequently, we should have

$$\begin{aligned}\lim_{n\to\infty} \phi(\lambda_n) &= \lim_{n\to\infty} \left(\Psi\left(u_3^{(\lambda_n)}\right) + \lambda_n \Phi\left(u_3^{(\lambda_n)}\right)\right)\\ &\geq \lim_{n\to\infty} \left(\Psi\left(u_3^{(\lambda_n)}\right) + \bar{\lambda} \min\left\{0, \Phi\left(u_3^{(\lambda_n)}\right)\right\}\right) = +\infty,\end{aligned}$$

contradicting (3.11).

Hence, set

$$\sigma = \max\left\{2(\|u_0\| + \bar{r}),\ \sup_{\lambda\in[0,\bar{\lambda}[} \left\|u_3^{(\lambda)}\right\|\right\}.$$

The conclusion follows.

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