BULL. AUSTRAL. MATH. SOC. 20D05
VOL. 17 (1977), 157-160.

A classification of groups with a centralizer condition II: Corrigendum and addendum

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The aim of this note is to prove a theorem which extends the results of the authors' earlier paper, *Bull. Austral. Math. Soc.* 16 (1977), 55-60. As one of the corollaries we prove Theorem 2 of that paper, the proof of which was incomplete.

We prove the following theorem.

THEOREM 1. *Let* G *be a finite group and let* M *be a CC-subgroup of* G. *Denote the set of primes dividing* $|M|$ *by* π. *Then, either* $O_\pi(G) \neq 1$ *and*

(1) G *is a Frobenius group with* M *as the Frobenius kernel,*

or $O_\pi(G) = 1$ *and one of the following holds:*

(2) G *is a Frobenius group with* M *as a Frobenius complement;*

(3) $G = O_{\pi'}(G)N_G(M)$, *a solvable group;*

(4) *there exists* $H \triangleleft G$ *satisfying*

(a) $H \cap M \neq 1$, *and*

(b) $H/O_{\pi'}(H)$ *is simple.*

Proof. If G is simple, then clearly (4) holds. Thus assume that $N \neq 1$ is a minimal normal subgroup of G and Theorem 1 is true for groups

Received 1 June 1977.

of order less than $|G|$.

Case 1. N is a π-group. Then $N \subseteq M$ and consequently $V = \cap\{M^g \mid g \in G\} \supset 1$. If $V = M$ then $M \triangleleft G$ and (1) holds. So assume that $1 \subset V \subset M$. As V is a normal CC-subgroup of M and of G , both M and G are Frobenius groups with the kernel V . Let C be a complement of V in M . Then C is a CC-subgroup of G and by [2, Lemma 1], CV/V is a CC-subgroup of G/V . However, by [4, Theorem V, 8.18], G/V has a nontrivial center, a contradiction.

Thus we may assume that $O_\pi(G) = 1$.

Case 2. N is a π'-group. In view of Lemma 1 in [2], G/N satisfies the assumptions of Theorem 1 with respect to MN/N . By the inductive hypotehsis G/N satisfies one of (1)-(4). Since N is a non-trivial normal π'-subgroup of G , $N_G(M)$ is solvable (see Theorem 2.3.h in [3]). Thus if G/N satisfies (1), then by [2, Lemma 2] G satisfies (3). If G/N satisfies (2), then also G satisfies (2) and if G/N satisfies (3) then by [2, Lemma 2] so does G . Finally, if G/N satisfies (4), then it is easy to see that so does G . Thus, in Case 2, the theorem holds.

Consequently we may assume that $O_\pi(G) = O_{\pi'}(G) = 1$. As M is a Hall subgroup of G , it follows that $M \cap N \neq 1$ and since $M \cap N$ is a CC-subgroup of N , N is simple. Thus (4) holds with $H = N$, and the proof of Theorem 1 is complete.

The theorem immediately yields

COROLLARY 1. *If G is solvable, then $G = O_{\pi'}(G)N_G(M)$.*

We also have

COROLLARY 2. *If $N_G(M) = M$, then $O_\pi(G) = 1$ and either (2) holds or $O_{\pi'}(G) = 1$ and (4) holds with a simple H .*

Proof. Suppose that (2) doesn't hold. It suffices to show that $L \equiv O_{\pi'}(G) = 1$. Otherwise, ML is a Frobenius group with a complement M . Thus $Z(M) \neq 1$ [4, Theorem V, 8.18] and M is a TI-group [3, Theorem 2.1]. As $N_G(M) = M$, it follows that G satisfies (2), a contradiction.

https://doi.org/10.1017/S0004972700025612 Published online by Cambridge University Press

As a final corollary we prove Theorem 2 of [2]:

THEOREM 2. *If in Theorem* 1, $N_G(M) = M$ *and* $3 \mid |M|$, *then either* (2) *holds or* $G \cong PSL(2, q)$ *for some* $q \geq 4$.

Proof. If $2 \mid |M|$, then by [5], either (2) holds or $G \cong PSL(2, 2^{n})$. Thus assume that $2 \nmid |M|$ and by Corollary 2 we may assume that (4) holds, with a simple H . In addition, we shall assume that Theorem 2 holds for groups of order less than $|G|$.

Case 1. $H \subset G$. If $3 \nmid |H|$, then $H \cong Sz(q)$ by Thompson's classification of simple $3'$-groups, and if $3 \mid |H|$, then $H \cong PSL(2, q)$ for some q or $H \cong PSL(3, 4)$ by [1, Theorem B]. In all cases H has one class of involutions and all involutions of MH belong to H . As M is a CC-subgroup of MH and $2 \nmid |M|$, counting of involutions forces $M \subset H$. By induction $H \cong PSL(2, q)$ and as $N_H(M) = M$, M is the normalizer of a Sylow group in H . Thus G-conjugates of M are already conjugate in H and counting conjugates of M yields $H = G$, a contradiction.

Case 2. $H = G$. Thus G is simple and M satisfies $3 \mid |M|$, $2 \nmid |M|$, and $N_G(M) = M$. By [1, Theorem B], $G \cong PSL(2, q)$, q odd. The proof of Theorem 2 is complete.

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https://doi.org/10.1017/S0004972700025612 Published online by Cambridge University Press