COMPLETIONS OF SEMILATTICES OF CANCELLATIVE SEMIGROUPS

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Introduction. A semilattice of cancellative semigroups $S$ is a p.o. semigroup with the order relation $a \leq b$ iff $ab = a^2$. If $S$ is a strong semilattice of cancellative semigroups (i.e., multiplication in $S$ is given by structure maps $\phi_e, f (f \equiv e \in E)$), for each supremum-preserving completion $\tilde{E}$ of the semilattice $E$ there is a strong semilattice of cancellative semigroups $T$ over $\tilde{E}$ which is a supremum-preserving completion of $S$ in $\leq$. Given $\tilde{E}$, $T$ is constructed directly. In this paper it is shown that multiplication by an element of $S$ distributes over suprema in $\leq$ if $E$ has this property (called strong distributivity). Next it is shown that the completion construction also applies to a semilattice of cancellative semigroups which is not strong if $S$ is commutative and $\tilde{E}$ is strongly distributive. Finally, it is shown that for semilattices of cancellative monoids a completion is completely determined, up to isomorphism over $S$, by completions of $E$.

We begin by noting that if $S$ is a semilattice of cancellative semigroups $S(e \in E)$ then there are three particular ways of defining an order relation on $S$, namely

$$a \leq_1 b \iff ab = a^2, \quad a \leq_2 b \iff ba = a^2$$

and

$$a \leq_3 b \iff asb = bsa = asa \quad \text{for all } s \in S$$

(see [5] and [10]). These all coincide in this case. For if $a \in S_e$, $b \in S_f$ and $a \leq_1 b$ then $ab = a^2$ (giving $e \leq f$), so that $aba = a^3$ and $ba = a^2$, since $S_e$ is cancellative. Hence $\leq_1$ and $\leq_2$ coincide. If $a \leq_3 b$ then $a^2b = a^3$ giving $a \leq_1 b$, while if $a \leq_1 b$ and $s \in S$, the equation

$$asb = asa$$

and cancellation give the remaining equivalence. Necessary and sufficient conditions for these relations to be order relations are found in [5] and [10]. This is the case for semilattices of cancellative semigroups.

In the case of inverse semigroups whose idempotents are central, this order coincides with the natural order for an inverse semigroup [6, p. 40]. In particular this applies to semilattices.

The order relation on $S$ makes $S$ into a p.o. semigroup [5, Proposition 3] and the relation is called Abian’s order. A subset $X$ of $S$ can have an upper bound in $S$ only if it is boundable, i.e., for $x, y \in X$, $xy^2 = x^2y$. A semigroup $S$ is complete if every boundable set in $S$ has a supremum. An embedding $S \subseteq T$ of semigroups is a completion if (i) $T$ is a semilattice of cancellative semigroups, (ii) $T$ is complete and (iii) every element of $T$ is the supremum of some boundable set in $S$. We shall be dealing with completions such that the inclusion $S \subseteq T$ preserves suprema which exist in $S$.

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Finally, if $S = \bigcup_{e \in E} S_e$ is a strong semilattice of cancellative semigroups (i.e., for $f \leq e$ in $E$ there are homomorphisms $\phi_{e,f} : S_e \to S_f$ such that for $a \in S_e$, $b \in S_f$, $ab = \phi_{e,ef}(a)\phi_{f,ef}(b)$) then a set $X \subseteq S$ is boundable if and only if for $x, y \in X$, $x \in S_e$, $y \in S_f$ then $\phi_{e,ef}(x) = \phi_{f,ef}(y)$. Note that distinct elements of a boundable set are in distinct cancellative parts of $S$.

1. The completion of a strong semilattice of cancellative semigroups. Throughout this part let $S$ be a semigroup which is a semilattice $E$ of cancellative semigroups $S_e$ ($e \in E$), where multiplication in $S$ is given by structure maps $\phi_{e,f} : S_e \to S_f$ for $e, f \in E, f \leq e$.

The construction of a completion $T$ for $S$ is done by directly constructing $T$ as a lattice of cancellative semigroups. This construction owes something to the construction of semigroups of quotients of semilattices of groups as found in [9] and [11], but here a completion of $E$ is at the base of it all and the cancellative components need not be groups.

**Examples.** Consider the following lattices of groups.

$$
\begin{array}{ccc}
S: & \{e, a\} & \{f, b\} \\
\{0\} & & \{0\} \\
\{1\} & & \{1, u, v, w\}
\end{array}
$$

where, in both, $e^2 = e$, $f^2 = f$. In $T$, ker $\phi_{1,e} = \{1, u\}$, ker $\phi_{1,f} = \{1, v\}$. The boundable sets of $S$ are: the singletons, $\{e, f\}$, $\{e, b\}$ $\{a, f\}$, $\{a, b\}$ and these four with $0$ added and $\{1, e, f, 0\}$. We have that $S$ is not complete since, for example, $\{a, b\}$ has no upper bound. However $T$ is a completion of $S$ with the obvious embedding. This is a model for the general construction.

The semilattice $E$ can be completed in various ways, $E \subseteq \widehat{E}$, $\widehat{E}$ a complete lattice. In particular we may take $\widehat{E}$ to be the Dedekind–MacNeille completion where the element $f \in \widehat{E}$ corresponds to the subset $A = \{e \in E \mid e \leq f\}$ of $E$ (see [12, p. 44]). The embedding $E \subseteq \widehat{E}$ preserves all suprema which exist in $E$. The completion to be constructed will be a lattice of cancellative semigroups $T = \bigcup_{E} T_f$. (In order to obtain the theorem below, any supremum-preserving completion of $E$ will suffice, the particular one being mentioned only for concreteness.) The construction of $T$ and the verification of its properties will be done in six steps. In order to establish notation for the remainder of the article, to $f \in \widehat{E}$ we make correspond a subset of $E$ as follows: if $f = \sup\{e \in E \mid e < f\}$ then we let $A = \{e \in E \mid e < f\}$ (this occurs if $f \in \widehat{E} \setminus E$ and for some elements of $E$); if $f \neq \sup\{e \in E \mid e < f\}$ we let $A = \{e \in E \mid e \leq f\}$ (this can only occur for some elements of $E$, for
example 0, e, f in the preceding example, but not 1). Whichever case occurs A will be called the subset of $E$ corresponding to $f$.

**Step 1.** For $f \in \overline{E}$, let $A$ be the corresponding subset of $E$. Define $T_f$ to be the inverse limit of the system

$$\{S_e; \phi_{e,e'}, e, e' \in A, e' \leq e\}$$

(see [8, p. 291]); that is, $T_f$ is the subsemigroup of $\prod S_e$ consisting of the elements $(x_e)_A$ such that if $e' \leq e$ ($e, e' \in A$), then $\phi_{e,e'}(x_e) = x_{e'}$. The result is clearly cancellative and if the $S_e$ ($e \in A$) are groups, so is $T_f$. Note that $T_f$ could be empty although not in the case where each $S_e$ ($e \in A$) contains an idempotent. Also if $f \in E$ and $A = \{e \in E \mid e \leq f\}$ then $T_f = S_f$.

The elements of $T_f$ are precisely the boundable sets $X$ of $S$ such that

(i) if $s \leq x$ for some $s \in S$, $x \in X$, then $s \in X$,

and

(ii) $\{e \in E \mid x \in S_e \text{ for some } x \in X\} = A$.

**Step 2.** Put $T = \bigcup_{f \in E} T_f$ and define multiplication via structure maps as follows. If $f, f' \in \overline{E}$, $f' \leq f$ with corresponding subsets of $E, B \subseteq A$, then define $\psi_{f,f'} : T_f \to T_{f'}$, by $\psi_{f,f'}((x_e)_A) = (x_{e'})_B$, the restriction of $(x_e)_A \in T_f \subseteq \prod_{A} S_e$ to $B$. Then in general if $f, f' \in \overline{E}$ have corresponding subsets $A$ and $B$ of $E$, respectively,

$$(x_e)_A (y_e)_B = \psi_{f,f'}((y_e)_B).$$

Abian's order is defined on $T$.

**Step 3.** The embedding of $S \subseteq T$ is as follows. If $e \in E$ then $e \in \overline{E}$ and we assign to $x \in S_e$ the element $(x_e)_A$ where $A$ is the subset of $E$ corresponding to $e$ and for $e' \in A$, $x_{e'} = \phi_{e,e'}(x)$. This embedding is clearly order-preserving.

**Step 4.** Every element of $T$ is the supremum of a subset of $S$. Consider $x = (x_e)_A \in T_f$ then $(x_e)_A = \sup\{x_e \mid e \in A\}$. Indeed, for $e' \in A$,

$$xx_{e'} = \psi_{f,e'}((x_e)_A)x_{e'} = x_{e'}^2.$$

Thus $y$ is an upper bound of $X = \{x_e \mid e \in A\}$. Suppose that $y = (y_e)_B \in T_f$ is an upper bound of $X$. Then $(y_e)_B x_{e'} = x_{e'}^2$, showing that $e' \leq f'$ for all $e' \in A$. Hence, $A \subseteq B$ and $f \leq f'$. By the cancellation property, for $e \in A$, $y_e = x_e$ so that $\psi_{f,f'}(y) = x$ and so $xy = x^2$, giving the result.

**Step 5.** The semigroup $T$ is complete. Let $X = \{(x_e^\alpha)_{A_e} \mid \alpha \in \Lambda\}$ be a boundable set in $T$. We may assume that if $t \in T$, $x \in X$ and $t \leq x$ then $t \in X$. We have that for $(x_e^\alpha)_{A_e}$ and $(x_e^\beta)_{A_e}$ in $X$,

$$(x_e^\alpha)^2 (x_e^\beta)_{A_e} = (x_e^\alpha)_{A_e} (x_e^\beta)_A,$$

Calculating we get

$$(x_e^\alpha)^2 (x_e^\beta)_{A_e} = ((x_e^\alpha)^2 (x_e^\beta)_A, = (x_e^\alpha (x_e^\beta)^2)_{A_e}.$$
where if $A_{\alpha}$ corresponds to $f_\alpha$ and $A_{\beta}$ to $f_\beta$ then $A_{\gamma}$ corresponds to $f_\alpha f_\beta$; thus $A_{\gamma} = A_{\alpha} \cap A_{\beta}$. Hence for all $e \in A_{\gamma}$, $x_\alpha^e = x_\beta^e$. Put

$$U = \{x_\alpha^e \mid e \in A_{\alpha}, \alpha \in \Lambda\} \subseteq S.$$

Then

$$(x_\alpha^e)^2 x_\beta^e = \phi_{e,e'}((x_\alpha^e)^2) \phi_{e',e''}(x_\beta^e)$$

$$= (x_\alpha^e)^2 x_\beta^{e'} = x_\alpha^{e''} (x_\beta^{e'})^2$$

$$= x_\alpha^e (x_\beta^{e'})^2.$$

Hence $U$ is boundable.

Put $A = \{e \in E \mid x_\alpha^e \in S_e \text{ for some } x_\alpha^e \in U\}$. This set has a supremum $f$ in $\bar{E}$ and $f = \sup(\sup A_e)$ (see [1, p. 53]); hence if $e \in E$, $e \leq f$ then $e \leq \sup A_{\alpha}$ for some $e \in A$ and thus $e \in A_{\alpha}$. Now put $x = (x_e)_{A_e}$ where $x_e$ is any $x_\alpha^e$ ($\alpha \in \Lambda$). This is well-defined, since if $e \in A_{\alpha} \cap A_{\beta}$ then $x_\alpha^e = x_\beta^e$. We claim that $x = \sup X$.

For $(x_\alpha^e)_{A_e} \in X$ we have

$$(x_e)_{A_e} (x_\alpha^e)_{A_e} = (x_e x_\alpha^e)_{A_e},$$

since $A_{\alpha} \subseteq A$; hence for $e' \in A_{\alpha}$, $x_{e'} = x_\alpha^{e'}$ and we conclude that

$$x(x_\alpha^e)_{A_e} = (x_\alpha^e)^2_{A_e}.$$

Hence $x$ is an upper bound. But $x = \sup U$ and every $x_\alpha^e \in U$ is below an element of $X$ (indeed if $x_\alpha^e \in \{x_\alpha^e\}_{A_e}$ then $x_\alpha^e \leq (x_\alpha^e)_{A_e}$ by the nature of the embedding $S \subseteq T$). Hence $x = \sup X$.

**Step 6.** The embedding $S \subseteq T$ preserves all suprema which exist in $S$. Let $s = \sup X$ ($X$ a subset of $S$, $s \in S$). Then if $A = \{e' \mid x \in S_{e'}$ for some $x \in X\}$ and $s \in S_{e'}$, it follows that $e = \sup A$. If $g \leq e$ is another upper bound of $A$ in $E$, it is readily seen that $y = \phi_{e,g}(x)$ is an upper bound of $X$ and that $y \leq x$. Hence $y = x$ and $e = g$. But $e = \sup A$ in $\bar{E}$ as we and the boundable set $X$ has a supremum $t$ in $T_e$, and hence $t \leq s$. But $T_e$ is cancellative, so $s = t$.

This completes the construction, giving the following theorem; its corollary follow from it and the remark in Step 1.

**Theorem 1.** Let $S$ be a semigroup with decomposition $S = \bigcup E$, where $E$ is semilattice and the $S_e$ are cancellative. Suppose further that multiplication in $S$ is given by structure maps $\phi_{e,e'} : S_e \to S_{e'}$ for $e' \leq e$ in $E$. Then $S$ has a completion in Abian’s order, where $T$ is a semigroup of the same type as $S$ and the inclusion $S \subseteq T$ preserves suprema from $S$.

**Corollary 2.** Let $S$ be a semilattice of groups. Then $S$ has a completion $T$ in Abian order, where $T$ is a lattice of groups.
The completion of a semigroup is not unique (unlike the case for rings \([2, \text{Theorem 12}]\)) since even a lattice may be completed in several non-isomorphic ways. Uniqueness will be discussed further in part 4 below. Theorem 1 does yield an internal characterization of complete semigroups (of the type being studied here). The proof is clear from the proof of Theorem 1.

**Proposition 3.** Let \( S \) be a semigroup with decomposition \( S = \bigcup_{e \in E} S_e \), where \( E \) is a semilattice, the \( S_e \) are cancellative and the multiplication in \( S \) is given by structure maps \( \phi_{e,e'}: S_e \to S_{e'} (e' \leq e \text{ in } E) \). Then \( S \) is complete if and only if (i) \( E \) is a complete lattice, (ii) if \( f \in E \) is such that \( f = \sup A \) where \( A = \{ e \in E \mid e < f \} \) then \( S_f = \lim_{\leftarrow A} \{ S_e; \phi_{e,e} \} \) and (iii) if \( e' \leq e \text{ in } E \) then \( \phi_{e,e'} \) is the homomorphism induced by the universal property of inverse limits.

**Example.** Let \( E \) be a semilattice with 0 such that \( ef = 0 \) for all \( e \neq f \). Then with \( S_0 = \{ 0 \} \) and \( S_e \) arbitrary \((e \neq 0)\), a semigroup \( S = \bigcup_{e \in E} S_e \) can be formed. By adjoining an element 1 to \( E \) we get a completion \( \bar{E} \). Clearly \( T_1 = \prod_{e \in E} S_e \) and \( T_e = S_e \) for all \( e \in E \). Here \( \bar{E} \) is the Dedekind–MacNeille completion of \( E \). Using the same \( E \) we can also form the ideal completion \( F \) of \( E \), which in this case is supremum-preserving (it is not always \([7]\); \( F \) is the set of all subsets of \( E \) which contain 0. For \( U \in F \), \( T_U = \prod_{U} S_e \). These two completions are clearly not isomorphic.

2. Distributivity. In the case of semiprime rings, Abian’s order and Conrad’s order satisfy an infinite distributivity: if \( R \) is a semiprime ring and if \( x = \sup X \), \( a \in R \) then \( \sup aX = ax \) and \( \sup Xa = xa \) ([4, Corollary 3]). For semigroups this is false since there are lattices which are not distributive. However, for the type of semigroups we have been studying, distributivity will be seen to be a property of the underlying semilattice. Let us say that a semilattice \( L \) is strongly distributive if for any subset \( X \) of \( L \) and any \( e \in L \), if \( \sup X \) exists then \( \sup eX = e(\sup X) \).

**Proposition 4.** Let \( S \) be a semigroup with a decomposition \( S = \bigcup_{e \in E} S_e \), where \( E \) is a semilattice, the \( S_e \) are cancellative and multiplication in \( S \) is given by structure maps \( \phi_{e,e'}: S_e \to S_{e'} (e' \leq e \text{ in } E) \). Suppose that \( E \) is strongly distributive. Then for any boundable set \( X \) of \( S \) and any \( a \in S \), \( \sup aX = a(\sup X) \) and \( \sup Xa = (\sup X)a \) if \( \sup X \) exists.

**Proof.** Let \( y = \sup X \). If \( A = \{ e \in E \mid x \in X \cap S_e \text{ for some } x \} \), then clearly if \( y \in S_f \) we have \( f = \sup A \). Let \( a \in S_g \) and consider

\[
ayx = \phi_{g,ge}(a)\phi_{f,ge}(y)\phi_{g,ge}(a)\phi_{e,ge}(x) = \phi_{g,ge}(a)\phi_{e,ge}(x)\phi_{g,ge}(a)\phi_{e,ge}(x)
= axax \quad \text{for } x \in X \cap S_e.
\]

Hence \( ay \) is an upper bound for \( aX \). Let \( u \in S_h \) be another upper bound for \( aX \). Since \( h \) is
an upper bound for \( gA \),
\[
h \geq \sup gA = g(\sup A) = gf.
\]
It follows that \( \phi_{n, sf}(u) \) is an upper bound of \( aX \) in \( S_{sf} \). By cancellation, \( \phi_{n, sf}(u) = ay \) and \( ay \leq u \).

An analogous statement for inverse semigroups is [13, Lemma 1.13].

Note that strong distributivity for semilattices and, more generally, for semigroups with Abian’s order implies the following distributive property: if \( S \) is a strong semilattice of cancellative semigroups such that for \( s \in S \) and a boundable set \( X \), \( \sup sX = s(\sup X) \) if either exists, then for boundable sets \( X \) and \( Y \) we get that \( XY = \{xy | x \in X, y \in Y\} \) is boundable and \( \sup XY = (\sup X)(\sup Y) \) if either side exists.

### 3. A generalization.

In this section we attempt to construct a completion of a semilattice of cancellative semigroups where there are no structure maps available. It will be necessary to impose supplementary conditions on the cancellative semigroups and on the semilattice.

**Theorem 5.** Let \( S \) be a commutative semigroup which is a semilattice \( \bigcup_{e \in E} S_e \) of cancellative semigroups. Assume further that \( E \) has a supremum-preserving completion \( \bar{E} \) which is strongly distributive. Then \( S \) has a supremum-preserving completion.

**Proof.** We first construct for each \( e \in E \) the group \( G_e \) of fractions of \( S_e \). For \( ab^{-1} \in G_e \) and \( cd^{-1} \in G_{e'} \) define \( ab^{-1} \cdot cd^{-1} = ac(bd)^{-1} \in G_{ee'} \). Let \( G = \bigcup_{e \in E} G_e \) with the indicated multiplication; it is a semigroup of the type studied in Part 1. Let \( T = \bigcup_{e \in E} T_e \) be the completion of \( G \) as constructed in Theorem 1.

For \( f \in \bar{E} \) let \( A \) be the corresponding subset of \( E \) (see Part 1 for notation) and recall that an element of \( T_f \) has the form \( (x_e)_A \) where if \( e' \leq e \) in \( A \) then \( \phi_{e',e}(x_e) = x_{e'} \). Put

\[
U_f = \{(x_e)_A \in T_f | \text{ for some } B \subseteq A, \sup B = \sup A = f, \; x_e \in S_e \; \text{ for all } \; e \in B\}
\]

These are elements of \( T_f \) which are, in a sense, “almost everywhere” in \( S \). We put \( U = \bigcup_{e \in E} U_f \) and we shall show that \( U \) is the desired completion. Note that, as remarked in Part 2, if \( \bar{E} \) is strongly distributive and \( A, B \subseteq \bar{E} \) then \( \sup AB = (\sup A)(\sup B) \); indeed

\[
\sup AB = \sup_A (\sup aB) = \sup_A (a \sup B) = \sup (A \sup B) = (\sup A)(\sup B).
\]

Firstly, \( U \) is a subsemigroup of \( T \). Let \( (x_e)_A \in U_f \) and \( (y_e)_{A'} \in U_{f'} \) where \( A \) and \( A' \) are the subsets of \( E \) corresponding to \( f \) and \( f' \) respectively and for some \( B \subseteq A \), \( B = f \), \( x_e \in S_e \) for all \( e \in B \) and for some \( B' \subseteq A' \), \( B' = f' \), \( y_e \in S_e \) for all \( e \in B' \). Then

\[
(x_e)_A \cdot (y_e)_{A'} = (x_e y_e)_{AA'}.
\]
(of course \( AA' = \{e \in E | e \equiv ff'\} \)). But \( \sup BB' = (\sup B)(\sup B') = ff' \) (by hypothesis) and \( x_e y_e \in S_e \) for all \( e \in BB' \).
LEMMA. If \((x_e)_A \in T_f\), \(A\) is the subset of \(E\) corresponding to \(f\) and \(B \subseteq A\) is such that \(\sup B = f\) then \(\sup\{x_e | e \in A\} = \sup\{x_e | e \in B\}\).

Proof. Let \(x = \sup\{x_e | e \in A\}\), \(y = \sup\{x_e | e \in B\}\). Clearly both \(x\) and \(y\) are in \(T_f\) and \(y \leq x\). This gives the equality.

COROLLARY. If \((x_e)_A, (y_e)_A \in T_f\) and for some \(B \subseteq A\), with \(\sup B = f\), \(x_e = y_e\) for all \(e \in B\) then \((x_e)_A = (y_e)_A\).

Proof. By the lemma, \(\sup\{x_e | e \in B\} = (x_e)_A = (y_e)_A\).

Returning to the theorem, we must show that \(U\) is complete; it will follow that \(U\) is a completion of \(S\), since if \((x_e)_A \in U_f\) and \(B \subseteq A\) with \(\sup B = f\) and \(x_e \in S_e\) for all \(e \in B\) then the lemma shows that \((x_e)_A = \sup\{x_e | e \in B\}\), the supremum of a subset of \(S\).

Let \(X = \{(x_e^\alpha)_A | \alpha \in \Lambda\} \subseteq S\) be a boundable set from \(U\), where \(A_\alpha \subseteq E\) corresponds to \(f_\alpha\), \(B_\alpha \subseteq A_\alpha\), \(\sup B_\alpha = f_\alpha\) and \(x_e^\alpha \in S_e\) for all \(e \in B_\alpha\). Put \(x = \sup X\), an element of \(T\). It will be shown that \(x \in U\). Since \(X\) is boundable, for \(e \in A_\alpha \cap A_\beta = A_\alpha A_\beta\) we have \(x_e^\alpha = x_e^\beta\). Let

\[Y = \{x_e | x_e = x_e^\alpha\ \text{for some } e \in \bigcup_{A_\alpha} A_\alpha \text{ and some } \alpha \in \Lambda\} \subseteq U\]

As was shown in Theorem 1, Step 5, \(Y\) is boundable with the same supremum as \(X\). Now consider \(\bigcup B_\alpha \subseteq \bigcup A_\alpha\). We have

\[
\sup \bigcup B_\alpha = \sup\{\sup B_\alpha | \alpha \in \Lambda\} = \sup\{f_\alpha | \alpha \in \Lambda\} = \sup\{\sup A_\alpha | \alpha \in \Lambda\} = \sup \bigcup A_\alpha = f.
\]

Hence \(x \in U_f\).

It would be desirable to weaken the conditions on Theorem 6 to those of Theorem 1.

4. Uniqueness of completions. It has already been mentioned that completions are not unique since semilattices may have non-isomorphic completions. However, in the case of a semilattice of monoids, it will be shown that there is, up to isomorphism over \(S\), one supremum-preserving completion of \(S\), which is a semilattice of cancellative semigroups, for each isomorphism class of supremum-preserving completions of the underlying semilattice \(E\).

THEOREM 6. Let \(S = \bigcup S_e\) be a semilattice of cancellative monoids and let \(U = \bigcup U_f\) be a semilattice of cancellative semigroups which is a supremum-preserving completion of \(S\). Then (i) each \(U_f\) is a monoid, (ii) \(F\) is a supremum-preserving completion of \(E\), (iii) \(U\) is isomorphic over \(S\) to the completion constructed over \(F\) in Theorem 1.

Proof. \(E\) is contained in \(F\) as semilattices, for if \(e \in E \subseteq S\) and \(e \in U_f\) then for \(s \in S_e\), \(s = se\). It follows that \(s \in U_f\). Further if \(e, e' \in E\) with \(e \in U_f\), \(e' \in U_f\) then \(ee' \in U_{ff}\). Hence if \(e \in U_f\), \(e\) may be identified with \(f\). Further, \(F\) is a supremum-preserving completion of \(E\).
Let $u \in U_f$, $u = \sup X$ for some $X \subseteq S$. Put
$$A = \{ e \in E \mid x \in S_e \text{ for some } x \in X \}.$$  
Now for $x \in X$, $x \in S_e$, $ux = x^2 \in S_e \subseteq U_e$, so that $e \leq f$. Since $U$ is complete, the boundable set $A$ has a supremum $g$ in $U$. Now $g$ is an idempotent, since $g^2 e = ge = e$ for all $e \in A$, which shows that $g^2$ is also an upper bound of $A$; hence $g \leq g^2$ giving $g^2 = g^2$ in a cancellative semigroup. Thus $g = g^2$. We also have
$$gux = gx^2 = gex^2 = ex^2 = x^2$$
for $x \in X \cap S_e$, and so $gu \leq u$. From this $gu^2 = u^2$, showing that $g \in U_f$. We may identify $g$ with $f \in F$.

It follows that each $U_f$ is a monoid. By Proposition 3, $F$ is complete, giving (i) and (ii).

Now let $f \in F$ with corresponding set $A \subseteq E$. Each $u \in U_f$ is the supremum of some $X \subseteq S$. Let
$$B = \{ e \in E \mid x \in X \cap S_e \text{ for some } x \}.$$  
Clearly $B \subseteq A$ and $\sup B = \sup A = f$. Further, if $e \in A$ there is $e' \in B$ with $e' \geq e$, from which it follows that $ue = ue'e$. But if $x \in X \cap S_e$, then $u \geq x$ implies that $ue' = x$. Hence $ue = xe \in S$. Thus multiplication by $e \in A$ gives a homomorphism $\tau_e : U_f \rightarrow S_e$. Let $T_f = \lim_A \{ S_e ; \phi_{e,c} \}$ (as in Part 1). The homomorphisms $\tau_e$ induce a homomorphism $\tau : U_f \rightarrow T_f$ by the universal property of inverse limits. This is readily seen to be an isomorphism. Further, for $f, f' \in F$, $f' \leq f$, multiplication by $f'$ gives $U_f \rightarrow U_{f'}$, which is precisely the induced homomorphism $\psi_{f,f'}$ of Theorem 1. Hence $U$ is isomorphic to $T$ constructed as in Theorem 1 over $F$ and the isomorphism leaves elements of $S$ fixed.

It would be desirable to be able to get this uniqueness result for any strong semilattice of cancellative semigroups.

If the semilattice $E$ is a Boolean algebra then there is only one completion (the Dedekind–MacNeille) and it is strongly distributive. Hence if $R$ is a strongly regular ring then the completion of its multiplicative semigroup is unique; it is based on the completion of $B(R)$, the Boolean algebra of idempotents. This completion is the multiplicative semigroup of the completion of $R$ as a ring which is, in this case, the complete ring of quotients, $Q(R)$ (see [2, Theorem 14] and [3, Theorem 5]). More generally, if $R$ is a reduced p.p. ring (a ring with no non-zero idempotents in which the annihilator of each element is generated by an idempotent; in a reduced ring all idempotents are central and left and right annihilators coincide) then the multiplicative semigroup is a Boolean algebra of cancellative semigroups. Indeed for $e = e^2 \in R$, put
$$R_e = \{ r \in R \mid re = r \text{ and if for some } f = f^2, rf = r \text{ then } e \leq f \}.$$  
Now if $r, s, t \in R_e$, and $rs = rt$ we get $s - t \in \Ann r = gR$ for some $g = g^2$. Thus $r(1 - g) = r$ and $e \leq 1 - g$ giving $eg = 0$ and
$$s - t = g(s - t) = g(es - et) = 0,$$
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showing that $s = t$. Further, $R = \bigcup_{R_e} R_e$. Let $r \in R$; then $\text{Ann } r = eR$ for some $e \in B(R)$ and $r(1-e) = r$. If $rf = r$ for $f \in B(R)$ then $r(1-f) = 0$ and $1-f \in eR$, giving $1-e \leq f$. Hence $r \in R_{1-e}$.

Now if $R$ is commutative p.p. ring, it has a completion in Abian's order, call it $C(R)$, and $B(C(R))$ is the Dedekind–MacNeille completion of $B(R)$ ([3, Theorem 11]). We have shown the following:

**Proposition 7.** Let $R$ be a commutative p.p. ring. Then there is a unique supremum-preserving completion of the multiplicative semigroup of $R$. It is the multiplicative semigroup of the completion of the ring $R$.

**REFERENCES**


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