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SOME DISTRIBUTIONS OF ORDERED VALUES FROM BURR AND BETA DISTRIBUTIONS

BY
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Summary. In this paper we use some known transformations available in the Theory of Multiple Integrals to give straightforward, simpler, and elegant proofs of some distributions of ordered values from Burr and beta distributions. The exact distribution (under the null hypothesis) of Wilks' Λ criterion is obtained by considering it as a certain minimum value distribution problem.

1. **Introduction.** The evaluation of multiple integrals which occur in order statistics distribution theory is involved due to the fact that the integration is to be carried on over an ordered range of variables of integration. This difficulty is sometimes completely obviated by transforming the ordered variates to the unordered ones. Several such transformations are available in the Theory of Multiple Integrals. In previous papers, Kabe ([5], [6], [7]), the author used some such transformations and gave alternative simpler proofs of several known results in order statistics distribution theory. This is yet another attempt in the same direction. Here we use some known transformations to derive moments (and distributions if necessary) of ordered values from Burr and beta distributions.

Burr distribution includes as particular cases Pareto, power function, and logistic distributions. Malik ([8], [9]), has derived moments of ordered values from Pareto and power function distributions without transforming the ordered variates to the unordered ones. The process of direct integration used by Malik becomes complicated for dealing with moments of more than two ordered values. Our process of integration is so simple and straightforward that the moments of any number of ordered values may be obtained without any complicated steps in integration. Since several of Wilks' Λ criteria (under null hypothesis) may be looked upon as the product of p beta variates of first kind this product is a minimum amongst the p products $x_1x_2\ldots x_i$, $0<x_i<1$, $i=1,\ldots,p$, and hence is a minimum value distribution problem. Finally, we outline a procedure to obtain moments of ordered values from second kind of beta distributions.

Some useful transformations and results are stated in the next section.

2. **Some useful results.** Let $0<x_1<x_2<\cdots<x_N<\infty$, be an ordered sample, the transformation with Jacobian unity

$$(1)\qquad x_i = \sum_{j=1}^{i} y_j, \qquad i,j = 1,\ldots,N, \quad 0 < y_j < \infty,$$

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transforms the ordered x variates to unordered y variates. The transformation

(2) $$Z_i = \frac{1+y_1+\cdots+y_{i-1}}{1+y_1+\cdots+y_i}, \quad y_0 = 0,$$

with Jacobian

(3) $$J(y\colon Z) = \prod_{i=1}^{N} Z_i^{-(N-i+2)}$$

is useful to deal with ordered values from Burr distribution. The transformation, $0<\theta_i<1$, $i=1,\ldots,N-1$, $0<\theta_N<\infty$,

(4) $$y_1+\cdots+y_j = \theta_j\theta_{j+1}\ldots\theta_N, \quad j = 1,\ldots,N,$$

with Jacobian

(5) $$J(y\colon \theta) = \theta_2\theta_3^2\ldots\theta_N^{N-1},$$

is useful to evaluate certain multiple integrals occurring in Wilks' Λ distribution problems.

The transformation, $0<y_j<\infty$, $j=1,\ldots,N$

(6) $$y_j = t_j(1-t_1-t_2-\cdots-t_N)^{-1}, \quad \sum t_j \leq 1,$$

with Jacobian

(7) $$J(y\colon t) = (1+y_1+\cdots+y_N)^{N+1},$$

transforms first kind of Dirichlet's distribution to second kind of Dirichlet's distribution.

The transformation, $0<y_j<1$, $j=1,\ldots,N$,

(8) $$t_1 = y_1, \qquad t_j = (1-y_1)(1-y_2)\ldots(1-y_{j-1})y_j$$

with Jacobian

(9) $$J(t\colon y) = (1-y_1)^{N-1}(1-y_2)^{N-2}\ldots(1-y_{N-1})$$

transforms dependent first kind of Dirichlet's variates to independent first kind of beta variates.

The transformation $0<y_j<\infty$, $0<t_j<\infty$, $j=1,\ldots,N$

(10) $$y_1 = t_1, \qquad y_j = (1+t_1)(1+t_2)\ldots(1+t_{j-1})t_j$$

with Jacobian

(11) $$J(y\colon t) = (1+t_1)^{N-1}(1+t_2)^{N-2}\ldots(1+t_{N-1})$$

transforms the dependent second kind of Dirichlet's variates to independent second kind of beta variates.

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The following known definite integrals, Erdelyi *et al.* [3, pp. 114–115], are written down for ready reference

$$(12)\quad \int_0^1 x^{a-1}(1-x)^{b-1}(1+\alpha x)^{-q}\,dx = B(a,b)(1+\alpha)^{-q}F[a,q,a+b,\alpha(1+\alpha)^{-1}],$$

$$(13)\quad \int_0^\infty t^{p-1}(1+t)^{-q}(1+tZ)^{-a}\,dt = B(p,a+q-p)F[a,p,a+q,(1-Z)],$$

$$(14)\quad \begin{aligned}&\int_0^1 Z^{m-1}(1-Z)^{N-1}(1+\alpha Z)^{-(M+N+p)}\,dZ\\ &\qquad = (1+\alpha)^{-(M+p)}\int_0^1 t^{M-1}(1-t)^{N-1}(1+\alpha(1-t))^p\,dt.\end{aligned}$$

The formula (14) when p is not an integer reduces to formula (12). However, our purpose in noting (14) is to draw the attention of the reader to the fact that some of the series occurring in Wilks' Λ distribution may be finite.

Finally, we note down the product formula for gamma functions, Gibson [4, p. 499, Example 25]

$$(15)\quad \prod_{r=0}^{N-1}\Gamma\left(px+\frac{rp}{N}\right) = (2\pi)^{(N-p)/2}(p/N)^{Npx+(Np-N-p)/2}\prod_{r=0}^{p-1}\Gamma\left(Nx+\frac{rN}{p}\right).$$

We assume that all integrals occurring in this paper are evaluated over appropriate ranges of the variables of integration.

3. **Order statistics from Burr distribution.** Burr's [2] distribution in standard form has the density function

$$(16)\qquad f(x) = \frac{K\alpha x^{\alpha-1}}{(1+x^\alpha)^{K+1}},\qquad K>0,\quad \alpha>1,\quad 0<x<\infty.$$

Let $0<x_1<x_2<\cdots<x_N<\infty$ be an ordered sample and let $x_i^\alpha=t_i=\sum_{j=1}^{i} y_j$, i, $j=1,\ldots,N$, then the density function of y's is

$$(17)\quad f(y_1,\ldots,y_N) = N!\,K^N\prod_{i=1}^{N}(1+y_1+y_2+\cdots+y_i)^{-(K+1)},\quad 0<y_j<\infty.$$

Now transform the y variates to Z variates by (2), and find that

$$(18)\qquad f(Z_1,\ldots,Z_N) = N!\,K^N\prod_{i=1}^{N}Z_i^{K(N-i+1)-1},\quad 0<Z_i<1.$$

Notice that Z's have independent power function distributions, and that the hth moment or Mellin transform of

$$(1+t_i) = (1+y_1+\cdots+y_i) = (Z_1Z_2\ldots Z_i)^{-1}$$

is

$$(19)\quad \begin{aligned}\mu_h' &= N!\,K^N\int\prod_{\alpha=1}^{i}Z_\alpha^{K(N-\alpha+1)-h-1}\prod_{j=i+1}^{N}Z_j^{K(N-j+1)-1}\pi\,dZ\\ &= \frac{\Gamma(N+1)\,\Gamma(N-i-(h/K)+1)}{\Gamma(N-i+1)\,\Gamma(N-(h/K)+1)}.\end{aligned}$$

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The result (19) is also given by Malik [8, p. 147, equation (3.6)]. Note that $u=1+t$ has a Pareto distribution

$$f(u) = Ku^{-(K+1)}, \qquad 1 < u < \infty, \quad K > 0, \tag{20}$$

and (19) is the hth moment of ith ordered u. Further, the transformation $u=v^{-1}$ shows that v has a power function distribution and hence (19) is the hth moment of v_{N-i+1} if $v_1 > v_2 > \cdots > v_N$ is an ordered sample from this power function distribution. Again, if we set $t=\exp\{-W\}$ and take $K=1$, then W has a logistic distribution

$$f(W) = \exp\{W\}(1+\exp\{W\})^{-2}, \quad 0 < W < \infty, \tag{21}$$

On inverting the Mellin transform (19), we find the density function of $(1+t_i)$, i.e. of t_i to be

$$f(t_i) = \binom{N}{N-i+1} \sum_{i=N-i+1}^{N} (-1)^{N+i+j} \binom{i-1}{i+j-N-1} K(N-i+1)(1+t_i)^{-(Kj+1)}, \quad 0 < t_i < \infty. \tag{22}$$

Now it follows from (22) that

$$E(t_i^h) = \binom{N}{N-i+1} \sum_{i=N-i+1}^{N} (-1)^{N+i+j} \binom{i-1}{i+j-N-1} \times \frac{K(N-i+1)\Gamma(h+1)\Gamma(Kj-h)}{\Gamma(Kj+1)}. \tag{23}$$

Note that (23) is the moment generating function of $(-W_{N-i+1})$ if $W_1 > W_2 > \cdots > W_N$ is an ordered sample from (21).

Again to find the joint moments of the ith and jth ordered values of t we set $(1+t_i) = (Z_1Z_2 \ldots Z_i)^{-1}$ and $(1+t_j) = (Z_1Z_2 \ldots Z_j)^{-1}$, $j > i$, and evaluate the integral

$$\begin{aligned} \mu'_{p,q} &= N!K^N \int \prod_{\alpha=1}^{i} Z_\alpha^{K(N-\alpha+1)-(p+q)-1} \\ &\quad \times \prod_{\beta=i+1}^{j} Z_\beta^{K(N-\beta+1)-q-1} \prod_{\delta=j+1}^{N} Z_\delta^{K(N-\delta+1)-1} \pi \, dZ \\ &= \frac{\Gamma(N+1)\Gamma(N-i-((p+q)/v)+1)\Gamma(N-j-(q/v)+1)}{\Gamma(N-((p+q)/v)+1)\Gamma(N-i-q/v)\Gamma(N-j+1)}. \end{aligned} \tag{24}$$

On setting $p=1$, $q=1$ in (24) we find that the result (24) agrees with the one given by Malik [8, p. 148, equation (4.5)]. The joint density of $(1+t_i)$ and $(1+t_j)$, i.e. of t_i and t_j may be obtained by inverting the joint Mellin transform (24). The inversion integral

$$(25)\quad f((1+t_i),(1+t_j)) = \frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \mu'_{p,q}\,(1+t_i)^{-(p+1)}(1+t_j)^{-(q+1)}\,dp\,dq$$

is a known Mellin–Barnes type contour integral, Erdelyi *et al.* [3, p. 232, section (5.8.3)], and we find that

$$(26)\quad \begin{aligned} f(t_i,t_j) &= \frac{N!}{(i-1)!(N-j)!(J-i-1)!}\sum_{r=0}^{i-1}\sum_{s=0}^{j-i-1}\binom{i-1}{r}\binom{j-i-1}{s} \\ &\quad\times(-1)^{r+s}(1+t_i)^{-K(s+i-r-1)}(1+t_j)^{-K(N-i-s)},\quad 0<t_i<t_j<\infty. \end{aligned}$$

Now in (26) set $t_i=\theta_1\theta_2$, $t_j=\theta_2$, and then evaluate

$$(27)\quad \int_0^1 d\theta_1\int_0^\infty \frac{\theta_1^p\theta_2^{p+q+1}\,d\theta_2}{(1+\theta_1\theta_2)^{K(s+i-r-1)}(1+\theta_2)^{K(N-i-s)}}$$

by using (13) and (12) and find that

$$(28)\quad \begin{aligned} E(t_i^pt_j^q) &= \frac{N!}{(i-1)!(N-j)!(j-i-1)!}\sum_{r=0}^{i-1}\sum_{s=0}^{j-i-1}\binom{i-1}{r}\binom{j-i-1}{s} \\ &\quad\times B(p+q+2,\,K(N-r-1)-(p+q+2)) \\ &\quad\times\sum_{t=0}^{\infty}\frac{(-1)^{r+s}\Gamma(K(s+i-r-1)+t)\Gamma(p+q+2+t)\Gamma[K(N-r-1)]\Gamma(p+1)}{\Gamma[K(s+i-r-1)]\Gamma(p+q+2)\Gamma[K(N-r-1)+t]\Gamma(p+t+2)}. \end{aligned}$$

4. **Distribution of Wilks' Λ.** Let Z_i, $i=1,\ldots,p$, have the independent density functions

$$(29)\quad f(Z_i) = [B(p_i,q_i)]^{-1}Z_i^{p_i-1}(1-Z_i)^{q_i-1},\quad 0<Z_i<1.$$

Then we note that $Z_1Z_2\ldots Z_p=\Lambda$ is minimum among p products $Z_1Z_2\ldots Z_i$, $i=1,\ldots,p$. This suggests that we transform our Z variates to y variates by using (2) and try to obtain the distribution of $\Lambda=Z_1Z_2\ldots Z_p=(1+y_1+\cdots+y_p)^{-1}$ by using the distribution of y's

$$(30)\quad f(y_1,\ldots,y_p) = K\prod_{i=1}^{p}\frac{y_i^{q_i-1}}{(1+y_1+\cdots+y_i)^{p_i+q_i-p_{i+1}}},$$

where $0<y_i<\infty$, $i=1,\ldots,p$, and $p_{p+1}=0$, and $K=\prod_{i=1}^{p}[B(p_i,q_i)]^{-1}$. Further, by using (4) we find that

$$(31)\quad f(\theta_1,\ldots,\theta_N) = K\frac{\prod_{i=1}^{p}\theta_i^{q_1+q_2+\cdots+q_i-1}(1-\theta_i)^{q_{i+1}-1}}{\prod_{i=1}^{p}(1+\theta_i\theta_{i+1}\ldots\theta_p)^{t_i}},$$

where $0<\theta_i<1$, $i=1,\ldots,p-1$, $0<\theta_p<\infty$, $q_{p+1}=1$, $t_i=q_i+p_i-p_{i+1}$, $i=1,\ldots,p$. Now note that $\Lambda=(1+\theta_p)^{-1}$, and hence (31) must be integrated out with respect to $\theta_1, \ldots, \theta_{p-1}$. The integration may be carried on successively by using (12), and we find that

$$f(\theta_p) = K\Gamma(q_1)\Gamma(q_2)\ldots\Gamma(q_p)\sum_{r_1=0}^{\infty}\cdots\sum_{r_{p-1}=0}^{\infty} \frac{\begin{aligned}&[\Gamma(q_1+r_1)\Gamma(t_1+r_1)\Gamma(q_1+q_2+r_1+r_2)\\ &\times\Gamma(t_1+t_2+r_1+r_2)\Gamma(q_1+q_2+q_3+r_1+r_2+r_3)\\ &\times\Gamma(t_1+t_2+t_3+r_1+r_2+r_3)\ldots\\ &\times\Gamma(q_1+\cdots+q_{p-1}+r_1+\cdots+r_{p-2})\\ &\times(r_1!r_2!\ldots r_{p-1}!)^{-1}\theta_p^{r_1+\cdots+r_{p-1}+q_1+\cdots+q_{p-1}}]\end{aligned}}{\begin{aligned}&[\Gamma(q_1)\Gamma(t_1)\Gamma(q_1+q_2+r_1)\Gamma(t_1+t_2+r_1)\\ &\times\Gamma(q_1+q_2+q_3+r_1+r_2)\\ &\times\Gamma(t_1+t_2+t_3+r_1+r_2)\ldots\\ &\times\Gamma(q_1+\cdots+q_{p-1}+r_1+\cdots+r_{p-3})\\ &\times\Gamma(r_1+r_2+\cdots+r_{p-1}+q_1+\cdots+q_p)\\ &\times(1+\theta_p)^{t_1+\cdots+t_p+r_1+\cdots+r_{p-1}}]\end{aligned}}. \tag{32}$$

The hth moment of $x_1x_2\ldots x_N$ for Dirichlet's distribution of first kind

$$f(x_1, \ldots, x_N) = Kx_1^{\alpha_1-1}\ldots x_N^{\alpha_N-1}(1-x_1-\cdots-x_N)^{\beta-1}, \tag{33}$$

where

$$x_i \geq 0, \qquad \sum x_i \leq 1, \qquad K = \frac{\Gamma(\alpha_1+\cdots+\alpha_N+\beta)}{\Gamma(\alpha_1+\cdots+\alpha_N)}$$

is

$$\mu_h' = \frac{K\Gamma(\beta)\Gamma(\alpha_1+h)\ldots\Gamma(\alpha_N+h)}{\Gamma(\alpha_1+\cdots+\alpha_N+Nh+\beta)}, \tag{34}$$

which by using (15) may be written as

$$\mu_h' = \frac{K_1\Gamma(\alpha_1+h)\ldots\Gamma(\alpha_N+h)N^{-Nh}}{\prod_{r=0}^{N-1}\Gamma[h+((\alpha_1+\cdots+\alpha_N+\beta+r)/N)]}, \tag{35}$$

and which is the hth moment of a product of N independent beta variates of first kind, here

$$K_1 = K2\,\pi\,\frac{N-1}{2}\,N^{1/2-(\alpha_1+\cdots+\alpha_N+\beta)}\Gamma\beta.$$

Now there are several test criteria in statistics whose hth moment is of type (34), see, e.g. Anderson [1, p. 262, equation (16)]. By using (8) one may transform (34) to the hth moment of a product $\prod_{i=1}^{N} Z_i^a(1-Z_i)^b$, where Z_i are first kind of beta variates and $a \geq 0$, $b \geq 0$. Again, by using (6) we may transform (33) to second kind of Dirichlet's distribution

$$f(y_1, \ldots, y_N) = \frac{Ky_1^{\alpha_1-1}\ldots y_N^{\alpha_N-1}}{(1+y_1+\cdots+y_N)^{\alpha_1+\cdots+\alpha_N+\beta}}, \tag{36}$$

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where $0<y_i<\infty$, which by using (10) may be expressed as a product of second kind of beta densities. Thus (34) may be expressed as the hth moment of $\prod_{i=1}^{N} t_i/(1+t_i)^i$ where each t is second kind beta distributed.

Incidentally, it must be mentioned that the expression given by Mathai and Saxena [10, p. 1442, equation (13)] is only an approximate equation. If that expression were exact probably most of the exact distributions of Wilks' Λ criteria would be given by this exact expression, although noncentral case of Wilks' Λ would still remain unexplored, i.e. exact distribution theory of Wilks' Λ under nonnull hypothesis is as yet practically unexplored.

5. **Ordered values from beta distributions.** Let $0<x_1<x_2\cdots<x_N<\infty$ have the joint density

$$f(x_1,\ldots,x_N) = K\prod_{i=1}^{N}\frac{x_i^{p-1}}{(1+x_i)^{p+q}}, \tag{37}$$

where $K=[B(p,q)]^{-N}N!$ Then the hth moment of $x_i=\theta_i\theta_{i+1}\ldots\theta_N$, $i=1,\ldots,N$ is

$$E(x_i^h) = K\int\prod_{\alpha=1}^{i-1}\frac{\theta_\alpha^{\alpha p-1}}{(1+\theta_\alpha\theta_{\alpha+1}\ldots\theta_N)^{p+q}}\prod_{j=i}^{N}\frac{\theta_j^{jp+h-1}}{(1+\theta_j\theta_{j+1}\ldots\theta_N)^{p+q}}\,\pi\,d\theta \tag{38}$$

where $0<\theta_i<1$, $i=1,\ldots,N-1$, $0<\theta_N<\infty$. The integral (38) may be evaluated by using (12) successively and yields a series of type (32). By setting $(1+x_i)^{-1}=y_{N-i+1}$, we notice that y's are first kind of beta variates.

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