



Torsion codimension 2 cycles on supersingular abelian varieties

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Abstract. We prove that torsion codimension 2 algebraic cycles modulo rational equivalence on supersingular abelian varieties are algebraically equivalent to zero. As a consequence, we prove that homological equivalence coincides with algebraic equivalence for algebraic cycles of codimension 2 on supersingular abelian varieties over the algebraic closure of finite fields.

1 Introduction

Let k be a field, and let X be a smooth and projective variety over k . Write $Z^n(X)$ for the group of codimension n algebraic cycles on X , and let $Z_{\text{rat}}^n(X) \subseteq Z_{\text{alg}}^n(X) \subseteq Z_{\text{hom}}^n(X)$ be the subgroups of $Z^n(X)$ consisting of those cycles which are rationally (resp. algebraically, resp. homologically (with respect to a fixed Weil cohomology theory; see Remark 2.1)) equivalent to zero. Let $\text{CH}^n(X) \supseteq \text{CH}_{\text{hom}}^n(X) \supseteq \text{CH}_{\text{alg}}^n(X)$ denote the quotients of $Z^n(X) \supseteq Z_{\text{hom}}^n(X) \supseteq Z_{\text{alg}}^n(X)$ by $Z_{\text{rat}}^n(X)$. Let $\text{Griff}^n(X) := \text{CH}_{\text{hom}}^n(X)/\text{CH}_{\text{alg}}^n(X)$ be the Griffiths group of codimension n cycles on X .

Griffiths [Gri69] was the first to show that smooth projective varieties can have nontrivial Griffiths groups— $\text{Griff}^2(X) \otimes \mathbb{Q}$ is nontrivial for a very general quintic hypersurface $X \subset \mathbb{P}_{\mathbb{C}}^4$. Clemens [Cle83] later showed that such hypersurfaces have $\dim_{\mathbb{Q}}(\text{Griff}^2(X) \otimes \mathbb{Q}) = \infty$, and then Voisin [Voi00] generalized this by proving that $\dim_{\mathbb{Q}}(\text{Griff}^2(X) \otimes \mathbb{Q}) = \infty$ for any very general Calabi–Yau threefold X over \mathbb{C} . Since we are interested in abelian varieties, let us also point out that Ceresa [Cer83] has shown that, for a very general curve C of genus ≥ 3 , the Ceresa cycle is a nontrivial element in $\text{Griff}^2(J(C))$. In fact, it was shown in [Sch22] that the torsion subgroup of Griff^n need not even be finitely generated (at least for $n \geq 3$, with the situation for $n = 2$ currently open). This phenomena is not specific to “large” base fields either: Harris [Har83] has given an explicit abelian threefold defined over \mathbb{Q} with nontrivial Griffiths group—the Ceresa cycle on the Jacobian of the Fermat quartic is not algebraically equivalent to zero. Bloch [Bl84, Theorem 4.1] gave a different proof of Harris’ result on the Jacobian of the Fermat quartic, and showed moreover that the Ceresa cycle is nontorsion. In general, for a smooth projective variety defined over a number field, it is part of the Bloch–Beilinson conjectures that the dimensions of the

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Griffiths groups tensor \mathbb{Q} are finite, and the dimensions are controlled by the orders of vanishing at integers of appropriate L -functions [Bl85, p. 381].

In this note, we are interested in the situation where the base field has positive characteristic. In this setting, again the Griffiths groups of smooth projective varieties can be nontrivial and even infinite. For example, Schoen [Sch95, Theorem 0.1] showed that if k is a finite field of characteristic $p \equiv 1 \pmod{3}$ and E denotes the Fermat cubic, then $\text{Griff}^2(E_{\bar{k}}^3)$ is nontrivial and has a nontrivial divisible part. Here, \bar{k} denotes the algebraic closure of k , and subscript \bar{k} means the base change to \bar{k} .

Now, let k be a perfect field with $\text{char}(k) = p > 0$. The positive characteristic analog of the Bloch–Beilinson philosophy says that cycles on varieties, at least after tensoring with \mathbb{Q} , are controlled by the slopes of the Frobenius on crystalline cohomology. Recall that a smooth proper variety X over k is called ordinary if $H^m(X, d\Omega_{X/k}^r) = 0$ for all m, r . When the crystalline cohomology groups $H_{\text{cris}}^n(X/W(k))$ of X are torsion-free, X is ordinary if and only if, for each n , the Newton polygon of X coincides with the Hodge polygon [BK86, Proposition 7.3]. If A is an abelian variety, then A is ordinary if and only if $A(\bar{k})[p] = (\mathbb{Z}/p\mathbb{Z})^{\dim A}$. For example, the condition that $p \equiv 1 \pmod{3}$ in Schoen’s theorem forces $E_{\bar{k}}^3$ to be an ordinary abelian threefold. Generalizing Schoen’s result, under the rubric of the Tate conjecture for surfaces over finite fields, Brent Gordon and Joshi [BGJ02, Proposition 6.2] proved that the codimension 2 Griffiths group of ordinary abelian threefolds over the algebraic closure of a finite field are nontrivial, and contain a nontrivial divisible part.

At the opposite extreme to ordinarity is supersingularity, and in this situation, the Bloch–Beilinson philosophy suggests that Griffiths group should be smaller because of the extreme degeneracy in the slopes of Frobenius. Recall that a smooth projective variety X over k is said to be supersingular if the Newton polygons of X are isoclinic. If A is an abelian variety, then A is supersingular if and only if $A_{\bar{k}}$ is isogenous to the self-product of an (any!) supersingular elliptic curve, where an elliptic curve E is supersingular if and only if $E(\bar{k})[p] = 0$ (see [Oor74, Theorem 4.2]). Schoen [Sch95, Theorem 14.4] showed that if k is a finite field of characteristic $p \equiv 2 \pmod{3}$ and E denotes the Fermat cubic, then $\text{Griff}^2(E_{\bar{k}}^3)$ is at most a p -primary torsion group. The condition that $p \equiv 2 \pmod{3}$ implies that E^3 is a supersingular abelian threefold. Using the work of Fakhruddin [Fak02], Brent Gordon and Joshi [BGJ02, Theorem 5.1] generalized Schoen’s result to all supersingular abelian varieties—the codimension 2 Griffiths groups of supersingular abelian varieties defined over the algebraic closure of a finite field are at most p -primary torsion. The question of whether these groups possess nontrivial p -torsion was left open.

In this note, we prove the following (see Theorem 4.1(1)).

Theorem 1.1 *Let k be an algebraically closed field of characteristic $p > 0$, and let A be a supersingular abelian variety over k . Then the inclusions*

$$\text{CH}_{\text{alg}}^2(A)_{\text{tors}} \subseteq \text{CH}_{\text{hom}}^2(A)_{\text{tors}} \subseteq \text{CH}^2(A)_{\text{tors}}$$

are equalities. (Here, G_{tors} denotes the torsion subgroup of the group G .)

It was already shown in the proof of [BGJ02, Theorem 5.1] that $\text{CH}_{\text{alg}}^2(A)[\ell^\infty] = \text{CH}_{\text{hom}}^2(A)[\ell^\infty] = \text{CH}^2(A)[\ell^\infty]$ for each prime $\ell \neq p$ (where $G[\ell^\infty]$ denotes the ℓ -primary torsion subgroup of the group G). Our only new result is that this is also true for p -primary torsion. To handle the $\ell = p$ case, we initially follow the proof of Brent Gordon and Joshi, but then conclude using an inductive argument based on the Bloch–Srinivas method [BS83].

As a consequence of Theorem 1.1, we settle the p -primary torsion case of [BGJ02, Theorem 5.1]. Indeed, we have the following corollary (see Theorem 4.1(2)).

Theorem 1.2 *Let k be a finite field of characteristic $p > 0$, and let A be a supersingular abelian variety over \bar{k} . Then $\text{Griff}^2(A)[p^\infty]$ is trivial.*

Together with [BGJ02, Theorem 5.1], this shows that $\text{Griff}^2(A)$ is trivial. That is, homological equivalence coincides with algebraic equivalence for codimension 2 cycles on supersingular abelian varieties over the algebraic closure of finite fields.

2 Chow groups of supersingular abelian varieties

We repeat the discussion from [BGJ02, Sections 2 and 3]. Let A be an abelian variety of dimension g over a field k , and let n be a nonnegative integer. Then, by the work of Mukai [Muk81], Beauville [Bea86], and Deninger–Murre [DM91], the rational Chow groups of A admit a direct sum decomposition

$$\text{CH}^n(A) \otimes \mathbb{Q} = \bigoplus_i \text{CH}_i^n(A),$$

where $\text{CH}_i^n(A) := \{Z \in \text{CH}^n(A) \otimes \mathbb{Q} : m_A^*(Z) = m^{2n-i}Z \text{ for all } m \in \mathbb{Z}\}$ and m_A^* denotes the flat pullback by multiplication-by- m on A .

Now, suppose that k is an algebraically closed field of characteristic $p > 0$. Then Fakhruddin [Fak02] has proved that if A is a supersingular abelian variety over k , $\text{CH}_i^n(A) = 0$, for $i \neq 0, 1$. Moreover, the ℓ -adic cycle class map induces an isomorphism

$$\text{CH}_0^n(A) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} H_{\text{ét}}^{2n}(A, \mathbb{Q}_\ell(n))$$

for all primes $\ell \neq p$. The same proof shows that the crystalline cycle class map induces an isomorphism

$$\text{CH}_0^n(A) \otimes K \xrightarrow{\sim} H_{\text{cris}}^{2n}(A/W(k)) \otimes_{W(k)} K,$$

where $K = W(k)[1/p]$ is the fraction field of the Witt vectors $W(k)$ of k . In particular, $\text{CH}_1^n(A) = \text{CH}_{\text{hom}}^n(A) \otimes \mathbb{Q}$, where $\text{CH}_{\text{hom}}^n(A)$ is the kernel of the cycle class map.

Remark 2.1 A priori, the definition of the group $\text{CH}_{\text{hom}}^n(X)$ of codimension n cycles homologically equivalent to zero on a smooth projective variety X depends on the choice of Weil cohomology theory for X . Of course, it is a consequence of the standard conjectures (specifically that homological equivalence coincides with numerical equivalence) that $\text{CH}_{\text{hom}}^n(X)$ is independent of the choice of Weil cohomology theory. Notice, though, that Fakhruddin’s result shows that $\text{CH}_{\text{hom}}^n(A)$ is independent of

any choice when A is a supersingular variety over an algebraically closed field of characteristic $p > 0$. Since this is the setting that we are interested in, there is no ambiguity in the definition.

As pointed out in [BGJ02, Section 2], if k is moreover the algebraic closure of a finite field, then it is known by the results of Soulé [Sou84] and Künnemann [Kün93] that $\text{CH}_1^n(A) = 0$. In particular, $\text{CH}_{\text{hom}}^n(A)$ is torsion.

Remark 2.2 Beilinson [Bei87, 1.0] has conjectured that $\text{CH}_{\text{hom}}^n(X)$ is torsion for any smooth projective variety X over the algebraic closure of a finite field.

3 Abel–Jacobi maps

In this section, we fix notation involving ℓ -adic Abel–Jacobi maps, for primes ℓ (including $\ell = p$).

Let X be a smooth projective variety over an algebraically closed field k of characteristic $p \geq 0$. Let ℓ be a prime. Define

$$H^i(X, \mathbb{Z}_\ell(j)) := \begin{cases} H_{\text{ét}}^i(X, \mathbb{Z}_\ell(j)), & \text{if } \ell \neq p \\ H^{i-j}(X_{\text{ét}}, W\Omega_{X, \log}^j), & \text{if } \ell = p \end{cases}$$

and

$$H^i(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j)) := \begin{cases} H_{\text{ét}}^i(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j)), & \text{if } \ell \neq p, \\ \varinjlim_r H^{i-j}(X_{\text{ét}}, W_r\Omega_{X, \log}^j), & \text{if } \ell = p, \end{cases}$$

where $W_r\Omega_{X, \log}^j$ denotes the logarithmic Hodge–Witt sheaf of X (see [Ill79, Chapter I, 5.7]) and the limit is taken over the maps $\underline{p} : W_r\Omega_X^j \rightarrow W_{r+1}\Omega_X^j$ [Ill79, Chapter I, Proposition 3.4]. Let

$$\lambda_{n, X} : \text{CH}^n(X)[\ell^\infty] \rightarrow H^{2n-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$$

be Bloch’s ℓ -adic Abel–Jacobi map [Blo79] if $\ell \neq p$, and the Gros–Suwa p -adic Abel–Jacobi map [GS88] if $\ell = p$. Here, $\text{CH}^n(X)[\ell^\infty]$ denotes the ℓ -primary torsion subgroup of $\text{CH}^n(X)$.

4 The result

Theorem 4.1 *Let k be an algebraically closed field of characteristic $p > 0$, and let A be a supersingular abelian variety over k . Then:*

(1) *We have*

$$\text{CH}_{\text{alg}}^2(A)_{\text{tors}} = \text{CH}_{\text{hom}}^2(A)_{\text{tors}} = \text{CH}^2(A)_{\text{tors}}.$$

(2) *If k is the algebraic closure of a finite field, then $\text{Griff}^2(A)$ is trivial, i.e.,*

$$\text{CH}_{\text{alg}}^2(A) = \text{CH}_{\text{hom}}^2(A).$$

Proof (1) Let ℓ be a prime. Consider the following commutative diagram:

$$(4.1) \quad \begin{array}{ccc} \mathrm{CH}^2(A)[\ell^\infty] & \xrightarrow{\lambda_{2,A}} & H^3(A, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \\ \uparrow & \nearrow \lambda'_{2,A} & \\ \mathrm{CH}^2_{\mathrm{alg}}(A)[\ell^\infty] & & \end{array}$$

where $\lambda'_{2,A}$ denotes the restriction of $\lambda_{2,A}$. It is known that $\lambda_{2,A}$ is injective (as a consequence of the Merkurjev–Suslin theorem [CTSS83, Corollary 4] for $\ell \neq p$, [GS88, Section III, Proposition 3.4] for $\ell = p$), and hence $\lambda'_{2,A}$ is injective. It therefore suffices to show that $\lambda'_{2,A}$ is surjective, since then all maps in (4.1) are isomorphisms and in particular

$$\mathrm{CH}^2_{\mathrm{alg}}(A)[\ell^\infty] = \mathrm{CH}^2_{\mathrm{hom}}(A)[\ell^\infty] = \mathrm{CH}^2(A)[\ell^\infty]$$

for each prime ℓ .

For any $n \geq 0$, consider the following diagram:

$$\begin{array}{ccccccc} & & & & \mathrm{CH}^n(A)[\ell^\infty] & & \\ & & & & \downarrow \lambda_{n,A} & & \\ 0 & \longrightarrow & H^{2n-1}(A, \mathbb{Z}_\ell(n)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell & \longrightarrow & H^{2n-1}(A, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) & \longrightarrow & H^{2n}(A, \mathbb{Z}_\ell(n)). \end{array}$$

Up to a sign, the induced map $\mathrm{CH}^n(A)[\ell^\infty] \rightarrow H^{2n}(A, \mathbb{Z}_\ell(n))$ is the restriction of the cycle class map ([CTSS83, Corollary 4] for $\ell \neq p$, [GS88, Section III, Propositions 1.16 and 1.21] for $\ell = p$). The bottom row of the diagram is exact (see [GS88, (3.33)] for exactness when $\ell = p$). Therefore, the restriction of $\lambda_{n,A}$ to $\mathrm{CH}^n_{\mathrm{hom}}(A)[\ell^\infty]$ has image in $H^{2n-1}(A, \mathbb{Z}_\ell(n)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell$. In particular, $\lambda'_{2,A}$ has image in $H^3(A, \mathbb{Z}_\ell(2)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell$. Therefore, the cokernel of $\lambda'_{2,A}$ is divisible. Note that $H^3(A, \mathbb{Z}_\ell(2))$ is torsion-free (the nontrivial case when $\ell = p$ follows from $H^3_{\mathrm{cris}}(A/W(k))$ being torsion-free [GS88, Lemme 3.12], which is because H^1_{cris} is always torsion-free and $H^3_{\mathrm{cris}} = \wedge^3 H^1_{\mathrm{cris}}$ for abelian varieties), and hence $H^3(A, \mathbb{Z}_\ell(2)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell$ is a direct sum of a finite number of copies of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$.

We are therefore reduced to showing that $\mathrm{coker}(\lambda'_{2,A})$ is annihilated by a positive integer. (Indeed, $\mathrm{coker}(\lambda'_{2,A})$ is a quotient of the divisible group $H^3(A, \mathbb{Z}_\ell(2)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell \cong (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^r$ for some r . If $\mathrm{coker}(\lambda'_{2,A})$ is finite, then it must be trivial since any finite divisible group is trivial. So we must rule out the case that $\mathrm{coker}(\lambda'_{2,A})$ is infinite, in which case it is a finite number of copies of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$. However, this group is not annihilated by a positive integer.) We shall prove this by induction on the dimension g of A . Of course, the entire theorem is trivial if $g = 1$, so suppose that $g > 1$ and suppose that $\lambda'_{2,B}$ is surjective for supersingular abelian varieties of dimension $\leq g - 1$. By [Fak02, Lemma 3] and its proof, there exist g -dimensional abelian subvarieties $Y_1, \dots, Y_n \subset A \times A$ such that the class of the diagonal Δ_A decomposes as

$$\Delta_A = \sum_i c_i [Y_i] \in \mathrm{CH}^g(A \times A) \otimes \mathbb{Q}$$

for some $c_i \in \mathbb{Q}$, and such that, for each i , the image of Y_i under at least one of the projections $\text{pr}_1, \text{pr}_2 : A \times A \rightarrow A$ has dimension $\leq g - 1$. By clearing denominators, we see that there exists an integer $N > 0$ such that

$$N\Delta_A = \sum_i d_i [Y_i] \in \text{CH}^g(A \times A)$$

for some $d_i \in \mathbb{Z}$. Label the Y_1, \dots, Y_n so that $A_i := \text{pr}_1(Y_i)$ has dimension $\leq g - 1$ for $i = 1, \dots, m$, and $A_i := \text{pr}_2(Y_i)$ has dimension $\leq g - 1$ for $i = m + 1, \dots, n$. Let $V_1 := A_1 \cup \dots \cup A_m$ and $V_2 := A_{m+1} \cup \dots \cup A_n$, and let $j_1, j_2 : V_1, V_2 \rightarrow A$ be the natural inclusions. Then

$$N\Delta_A = Z_1 + Z_2 \in \text{CH}^g(A \times A),$$

where $\text{Supp}(Z_1) \subset V_1 \times A$ and $\text{Supp}(Z_2) \subset A \times V_2$. Let $\tilde{V}_1 := A_1 \sqcup \dots \sqcup A_m$ and $\tilde{V}_2 := A_{m+1} \sqcup \dots \sqcup A_n$ be the disjoint unions, and let $\tau_1 : \tilde{V}_1 \rightarrow V_1, \tau_2 : \tilde{V}_2 \rightarrow V_2$ be the natural morphisms. We claim that there is a correspondence $\tilde{Z}_1 \in \text{CH}^{g-1}(\tilde{V}_1 \times A)$ such that

$$Z_1 = \tilde{Z}_1 \circ \gamma_1,$$

where $\gamma_1 \in \text{CH}^g(A \times \tilde{V}_1)$ is the correspondence given by the transpose of the graph of $\tilde{j}_1 := j_1 \circ \tau_1$. Indeed, let V_1^{sm} denote the smooth locus of V_1 , and consider the pullback square

$$\begin{array}{ccc} \tilde{V}_1 \times A & \xrightarrow{\tau_1 \times \text{id}_A} & V_1 \times A \\ \uparrow & & \uparrow \\ \tau_1^{-1}(V_1^{\text{sm}}) \times A & \xrightarrow{\tau_1 \times \text{id}_A} & V_1^{\text{sm}} \times A. \end{array}$$

Since $V_1^{\text{sm}}, \tau_1^{-1}(V_1^{\text{sm}})$, and \tilde{V}_1 are smooth, the morphisms in the diagram admit refined Gysin pullbacks (see [Ful84, Section 6.6]). Set \tilde{Z}_1 to be the closure in $\tilde{V}_1 \times A$ of the pullback of Z_1 along $\tau_1^{-1}(V_1^{\text{sm}}) \times A \rightarrow V_1^{\text{sm}} \times A \rightarrow V_1 \times A$, where consider Z_1 as a cycle on $V_1 \times A$. Then $Z_1 = (j_1 \circ \tau_1 \times \text{id}_A)_* \tilde{Z}_1 = \tilde{Z}_1 \circ \gamma_1$ by [Ful84, Proposition 16.1.1], as desired. The same argument applied to the transpose of Z_2 shows that there exists a correspondence $\tilde{Z}_2 \in \text{CH}^{g-1}(A \times \tilde{V}_2)$ such that

$$Z_2 = \Gamma_{\tilde{j}_2} \circ \tilde{Z}_2,$$

where $\Gamma_{\tilde{j}_2} \in \text{CH}^g(\tilde{V}_2 \times A)$ is the correspondence given by the graph of $\tilde{j}_1 := j_2 \circ \tau_2$. Hence,

$$N\Delta_A = Z_1 + Z_2 = \tilde{Z}_1 \circ \gamma_1 + \Gamma_{\tilde{j}_2} \circ \tilde{Z}_2$$

and the self-correspondence $N\Delta_A^* : \text{CH}^2(A) \rightarrow \text{CH}^2(A)$ factors as

$$\text{CH}^2(A) \xrightarrow{\tilde{Z}_1^* \oplus \tilde{j}_2^*} \text{CH}^1(\tilde{V}_1) \oplus \text{CH}^1(\tilde{V}_2) \xrightarrow{\tilde{j}_1^* + \tilde{Z}_2^*} \text{CH}^2(A).$$

Since the ℓ -adic Abel–Jacobi maps are compatible with correspondences ([Blo79, Proposition 3.5] for $\ell \neq p$, [GS88, Proposition 2.9] for $\ell = p$), we get a commutative diagram

$$(4.2) \quad \begin{array}{ccc} \mathrm{CH}_{\mathrm{alg}}^2(A)[\ell^\infty] & \xrightarrow{\lambda'_{2,A}} & H^3(A, \mathbb{Z}_\ell(2)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell \\ \tilde{z}_1^* \oplus \tilde{j}_2^* \downarrow & & \tilde{z}_1^* \oplus \tilde{j}_2^* \downarrow \\ \mathrm{CH}_{\mathrm{alg}}^1(\tilde{V}_1)[\ell^\infty] \oplus \mathrm{CH}_{\mathrm{alg}}^2(\tilde{V}_2)[\ell^\infty] & \xrightarrow{\lambda'_{1,\tilde{V}_1} \oplus \lambda'_{2,\tilde{V}_2}} & H^1(\tilde{V}_1, \mathbb{Z}_\ell(1)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell \oplus H^3(\tilde{V}_2, \mathbb{Z}_\ell(2)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell \\ \tilde{j}_{1*} + \tilde{z}_2^* \downarrow & & \tilde{j}_{1*} + \tilde{z}_2^* \downarrow \\ \mathrm{CH}_{\mathrm{alg}}^2(A)[\ell^\infty] & \xrightarrow{\lambda'_{2,A}} & H^3(A, \mathbb{Z}_\ell(2)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell \end{array}$$

where the composition of the vertical arrows is $N\Delta_A^*$. However, Δ_A^* is the identity, so $N\Delta_A^*$ is multiplication-by- N .

The map $\lambda'_{1,\tilde{V}_1} : \mathrm{CH}_{\mathrm{alg}}^1(\tilde{V}_1)[\ell^\infty] \rightarrow H^1(\tilde{V}_1, \mathbb{Z}_\ell(1)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell$ is a bijection [ACMV21, Proposition A.28]. We claim that the map $\lambda'_{2,\tilde{V}_2} : \mathrm{CH}_{\mathrm{alg}}^2(\tilde{V}_2)[\ell^\infty] \rightarrow H^3(\tilde{V}_2, \mathbb{Z}_\ell(2)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell$ is also a bijection. Indeed, it is injective by the same reasoning that showed that $\lambda'_{2,A}$ is injective. To see that λ'_{2,\tilde{V}_2} is surjective, recall that $\tilde{V}_2 := A_{m+1} \sqcup \dots \sqcup A_n$ and λ'_{2,\tilde{V}_2} is the direct sum

$$\bigoplus_{i=m+1}^n \mathrm{CH}_{\mathrm{alg}}^2(A_i)[\ell^\infty] \xrightarrow{\oplus \lambda'_{2,A_i}} \bigoplus_{i=m+1}^n H^3(A_i, \mathbb{Z}_\ell(2)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell.$$

The A_i are supersingular abelian varieties of dimension $\leq g - 1$ (they are supersingular since they are subvarieties of A), so the induction hypothesis implies that $\lambda'_{2,\tilde{V}_2} = \oplus \lambda'_{2,A_i}$ is surjective as claimed.

In particular, we see that the middle horizontal arrow in (4.2) is a bijection. A diagram chase shows that $\mathrm{coker}(\lambda'_{2,A})$ is annihilated by N .

(2) We have seen in Section 2 that $\mathrm{CH}_{\mathrm{hom}}^2(A)$ is a torsion group when k is the algebraic closure of a finite field. Therefore, the subgroup $\mathrm{CH}_{\mathrm{alg}}^2(A)$ is also torsion. We may then conclude by part (1). ■

Remark 4.2 It was already shown in [BGJ02, Theorem 5.1] that $\mathrm{Griff}^2(A)[\ell^\infty] = 0$ for all primes $\ell \neq p$, so the only new result is that $\mathrm{Griff}^2(A)[p^\infty] = 0$ as well. The proof strategy in Theorem 4.1 of reducing to showing surjectivity of $\lambda'_{2,A}$ is the same as [BGJ02, Theorem 5.1]. When $\ell \neq p$, surjectivity of $\lambda'_{2,A}$ is due to Suwa [Suw88, Théorème 4.7.1]. Suwa’s proof proceeds by considering the following commutative diagram:

$$(4.3) \quad \begin{array}{ccc} \mathrm{CH}^1(A) \otimes \mathbb{Z}_\ell \times \mathrm{CH}_{\mathrm{alg}}^1(A)[\ell^\infty] & \xrightarrow{\quad \cdot \quad} & \mathrm{CH}_{\mathrm{alg}}^2(A)[\ell^\infty] \\ \mathrm{cl} \times \lambda'_{1,A} \downarrow & & \downarrow \lambda'_{2,A} \\ H^2(A, \mathbb{Z}_\ell(1)) \times H^1(A, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) & \xrightarrow{\cup} & H^3(A, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)). \end{array}$$

The cycle class map cl is surjective for supersingular abelian varieties by [Shi75, Appendix], and we have already seen that $\lambda'_{1,A}$ is a bijection. The cup-product map along the bottom of the square is a surjection as a consequence of $H^i(A, \mathbb{Z}/\ell\mathbb{Z}) = \wedge^i H^1(A, \mathbb{Z}/\ell\mathbb{Z})$ for abelian varieties. This forces $\lambda'_{2,A}$ to be surjective.

In the case $\ell = p$, we have the commutative square analogous to (4.3). Unravelling notation, the square is as follows:

$$\begin{array}{ccc}
 \mathrm{CH}^1(A) \otimes \mathbb{Z}_p \times \mathrm{CH}_{\mathrm{alg}}^1(A)[p^\infty] & \xrightarrow{\quad \cdot \quad} & \mathrm{CH}_{\mathrm{alg}}^2(A)[p^\infty] \\
 \mathrm{cl} \times \lambda'_{1,A} \downarrow & & \downarrow \lambda'_{2,A} \\
 H^1(A_{\acute{e}t}, W\Omega_{A,\log}^1) \times \varinjlim_r H^0(A_{\acute{e}t}, W_r\Omega_{A,\log}^1) & \xrightarrow{\quad \cup \quad} & \varinjlim_r H^1(A_{\acute{e}t}, W_r\Omega_{A,\log}^2).
 \end{array}$$

The cup-product map along the bottom horizontal is rarely surjective when A is not an ordinary abelian variety. Indeed, we have $\mathrm{CH}_{\mathrm{alg}}^1(A) = \mathrm{Pic}_{A/k}^0(k)$, so if A is an abelian variety with p -rank 0 (if A is supersingular, for example), then $\mathrm{CH}_{\mathrm{alg}}^1(A)[p^\infty] \cong \varinjlim_r H^0(A_{\acute{e}t}, W_r\Omega_{A,\log}^1)$ is the trivial group. This is why we must use a different argument for the $\ell = p$ case of Theorem 4.1 than the argument for $\ell \neq p$ used in [BGJ02, Theorem 5.1]. Notice that the proof of Theorem 4.1 treats all primes ℓ (including $\ell = p$), and in particular gives a new proof of [BGJ02, Theorem 5.1].

Remark 4.3 The proof of Theorem 4.1 shows that the inclusion

$$H^3(A, \mathbb{Z}_\ell(2)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell \hookrightarrow H^3(A, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$$

is an equality for all primes ℓ . This was known for $\ell \neq p$ as a consequence of the proof of [BGJ02, Theorem 5.1].

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