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A Geometric Approach to Voiculescu-Brown Entropy

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Abstract. A basic problem in dynamics is to identify systems with positive entropy, *i.e.,* systems which are "chaotic." While there is a vast collection of results addressing this issue in topological dynamics, the phenomenon of positive entropy remains by and large a mystery within the broader noncommutative domain of C^*-algebraic dynamics. To shed some light on the noncommutative situation we propose a geometric perspective inspired by work of Glasner and Weiss on topological entropy. This is a written version of the author's talk at the Winter 2002 Meeting of the Canadian Mathematical Society in Ottawa, Ontario.

1 Introduction

Entropy has been very successful in topological dynamics (as in measurable dynamics) as a numerical invariant measuring the complexity of a dynamical system (see, *e.g.,* [9, 15]). Topological entropy was introduced by Adler, Konheim, and McAndrew in [1] with a definition based on open covers. Equivalent definitions in terms of separated and spanning sets with respect to a metric were given by Bowen [4] and Dinaburg [10]. Much more recently Voiculescu introduced a notion of entropy for automorphisms of unital nuclear C^*-algebras based on local approximation [36], and Brown subsequently extended this to automorphisms of exact C^*-algebras using nuclear embeddability [5]. By [36, Prop. 4.8] the topological entropy of a homeomorphism of a compact metric space coincides with the Voiculescu-Brown entropy of the induced automorphism of the C^*-algebra of functions on the space, and so Voiculescu-Brown entropy is an extension of topological entropy to the noncommutative domain (indeed Voiculescu and Brown refer to their invariants as "topological entropy," but we have refrained from this terminology to avoid confusion). However, because Voiculescu's idea of using local approximation constitutes a fundamentally reconceptualized approach to defining entropy (as necessitated by its analytic context, which does not allow for any kind of direct analogue of an open cover with which a genuine dynamical invariant may be obtained), Voiculescu-Brown entropy ultimately requires tools of a completely different formal and technical nature for its study.

To date this study has focused on three aspects: computations for canonical examples (see, *e.g.,* [8, 3, 11, 12]), behaviour under taking crossed products [5, 31] and reduced free products [7], and the variational principle in the presence of a strategic amount of commutativity [26, 19, 20]. See [33] for a survey. The methods for obtaining non-zero lower bounds in computations have been invariably rooted in classical

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considerations. Indeed they have involved either (i) relating the system to a topological dynamical system which is *a priori* known to have positive topological entropy, or (ii) appealing to measure-theoretic invariants like Connes-Narnhofer-Thirring or Sauvageot-Thouvenot entropy, which themselves use classical measurable partitions in their definitions. Thus the fundamental problem of identifying systems with positive entropy (*i.e.,* systems which exhibit "chaotic" behaviour), in however specific or general a setting, has not been addressed from a broad noncommutative viewpoint, and so the phenomenon of positive entropy has remained a mystery beyond the scope of commutativity (and even there a considerable degree of mystery persists, as we do not have a "topological" proof of the equality of Voiculescu-Brown entropy and the topological entropy of the induced homeomorphism on the pure state space in the separable unital commutative case, which was established in [36, Prop. 4.8] using the classical variational principle and continuity properties of Connes-Narnhofer-Thirring entropy). As a consequence Voiculescu-Brown entropy has played a rather isolated role in C^*-dynamics, in contrast to the pervasive presence of topological entropy in topological dynamics.

In fact the theory of C^*-dynamical systems has been much more concerned with questions of C^*-algebraic structure and classification than with the strict investigation of long-term behaviour that is at the heart of topological dynamics and for which entropy is a key tool. Symptomatic of the difficulties in analyzing the long-term behaviour of noncommutative systems is the typical lack of discrete data that can be workably assembled to yield a meaningful statement about some aspect of the dynamics. This is starkly illustrated by the collection of automorphisms of the rotation C^*-algebras A_θ associated to a given matrix in $SL(2, \mathbb{Z})$ with eigenvalues off the unit circle: for a subset of $\theta \in [0, 1)$ of full Lebesgue measure the canonical tracial state is the unique invariant state [25], while in the "degenerate" case $\theta = 0$, in which we recover the corresponding hyperbolic toral automorphism on the pure state space, there is a rich supply of periodic points in terms of which much about the system can be expressed, including the entropic growth (see, *e.g.,* [15]). We can also make a comparison here with the viewpoint of semiclassical analysis, which, as opposed to directly extending topological dynamics to incorporate the noncommutative case, extracts discrete spectral data from within the matrix framework associated to the values $\theta = 1/n$ ("quantization") and performs an asymptotic analysis thereupon, with the possibility of establishing a correspondence with classical information in the limit $n \to \infty$. See the introduction to [40] for a discussion and [22, 23, 24] for some recent results.

The purpose of this article is to introduce a geometric perspective that yields some insight into the mechanisms behind positive Voiculescu-Brown entropy. Our inspiration lies in the link between topological dynamics and the geometric theory of Banach spaces that was established by Glasner and Weiss in one of the two proofs they gave in [13] for the following striking result.

Theorem 1.1 ([13, Theorem A]) *If a homeomorphism from a compact metric space X to itself has zero topological entropy, then so does the induced homeomorphism on the space of probability measures on X.*

Since the Voiculescu-Brown entropy agrees with the topological entropy on the pure state space for automorphisms of separable unital commutative C^*-algebras, Glasner and Weiss's result is equivalent to the assertion that zero Voiculescu-Brown entropy for such an automorphism implies zero topological entropy for the induced homeomorphism on the state space. We have shown that this assertion in fact holds for automorphisms of any separable unital exact C^*-algebra, and we can additionally drop the assumption of a unit by replacing the state space with the quasi-state space in the general exact setting (Theorem 4.2). Full details of this result can be found in [18]. The key geometric tool is the asymptotic exponential dependence of k on n given an approximately isometric embedding of ℓ_1^n into the C^*-algebra M_k of $k \times k$ matrices. This geometric fact also has the consequence that the presence of a certain supply of dynamically generated subspaces approximately isometric to ℓ_1^n is sufficient to obtain positive Voiculescu-Brown entropy (Proposition 2.2). In particular we can show, without relying in any way on classical dynamical concepts, that certain C^*-dynamical systems constructed in an operator-theoretic manner have positive entropy. We also obtain some information about the behaviour of Voiculescu-Brown entropy under taking extensions.

The main body of the paper consists of three sections. Sections 2 and 4 revolve around the results described above involving positive entropy and approximately isometric embeddings of ℓ_1^n into M_k at the C^*-algebra and state space levels, respectively, while in Section 3 we apply our geometric perspective in a complementary way with a look at the free shift on $C_r^*(\mathbb{F}_\infty)$ as an example of subexponential dynamical growth.

2 Entropy and Embeddings of ℓ_1^n into M_k

We begin by recalling the definitions of topological entropy and Voiculescu-Brown entropy. Let X be a compact metric space and $T\colon X \to X$ a homeomorphism. For a finite open cover $\mathcal{U}$ we set

$$h_{\text{top}}(T, \mathcal{U}) = \lim_{n\to\infty} \frac{1}{n} \log N(\mathcal{U} \vee T^{-1}\mathcal{U} \vee \cdots \vee T^{-(n-1)}\mathcal{U})$$

where $N(\cdot)$ denotes the smallest cardinality of a subcover and the join $\mathcal{U}_1 \vee \cdots \vee \mathcal{U}_m$ of a finite collection $\mathcal{U}_1, \ldots, \mathcal{U}_m$ of open covers is the set of all non-empty intersections $U_1 \cap \cdots \cap U_m$ with $U_i \in \mathcal{U}_i$ for each $i = 1, \ldots, m$. The *topological entropy* of T is defined by

$$h_{\text{top}}(T) = \sup_{\mathcal{U}} h_{\text{top}}(T, \mathcal{U})$$

where the supremum is taken over all open covers $\mathcal{U}$. We can alternatively express the entropy in terms of separated and spanning sets. A set $E \subset X$ is said to be (n, ε)-*separated (with respect to T)* if for every $x, y \in E$ with $x \neq y$ there is a $0 \leq k \leq n-1$ such that $d(T^k x, T^k y) > \varepsilon$, and (n, ε)-*spanning (with respect to T)* if for every $x \in X$ there is a $y \in E$ such that $d(T^k x, T^k y) \leq \varepsilon$ for each $k = 0, \ldots, n-1$. We write $\operatorname{sep}_n(T, \varepsilon)$ and $\operatorname{spn}_n(T, \varepsilon)$ to denote the largest cardinality of an (n, ε)-separated set and the smallest cardinality of an (n, ε)-spanning set, respectively. We then have

$$h_{\text{top}}(T) = \sup_{\varepsilon>0} \limsup_{n\to\infty} \frac{1}{n} \log \operatorname{sep}_n(T, \varepsilon) = \sup_{\varepsilon>0} \limsup_{n\to\infty} \frac{1}{n} \log \operatorname{spn}_n(T, \varepsilon).$$

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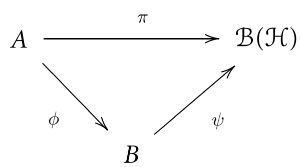
The fundamental prototypical example is the shift on $\{1, \dots, d\}^{\mathbb{Z}}$, with entropy $\log d$. For general references on topological entropy see [9, 15].

Turning now to the noncommutative domain, we let A be an exact C^*-algebra and $\pi\colon A \to \mathcal{B}(\mathcal{H})$ a faithful representation. By [21] exactness is equivalent to nuclear embeddability, and the latter guarantees, for every finite $\Omega \subset A$ and $\delta > 0$, the non-emptiness of the collection $\mathrm{CPA}(\pi, \Omega, \delta)$ of triples (ϕ, ψ, B) where B is a finite-dimensional C^*-algebra and $\phi\colon A \to B$ and $\psi\colon B \to \mathcal{B}(\mathcal{H})$ are completely positive contractive linear maps such that the diagram



approximately commutes to within δ on Ω, *i.e.,* $\|(\psi \circ \phi)(x) - \pi(x)\| < \delta$ for all $x \in \Omega$. As shown in the proof of [5, Prop.1.3], the infimum of rank B over all $(\phi, \psi, B) \in \mathrm{CPA}(\pi, \Omega, \delta)$ (with rank referring to the dimension of a maximal commutative C^*-subalgebra) does not depend on the particular faithful representation π; we denote this quantity by $\mathrm{rcp}(\Omega, \delta)$. We point out that the above C^*-algebras B may in fact be taken to be full matrix algebras, since a finite-dimensional C^*-algebra B can be embedded in the matrix algebra M_k where $k = \operatorname{rank} B$, in which case the identity map on B factors as the composition of the inclusion $B \hookrightarrow M_k$ with a conditional expectation $M_k \to B$. We also note that if A is nuclear (in particular, if A is commutative) we can alternatively define $\mathrm{rcp}(\Omega, \delta)$ by substituting the identity map on A for π in the above and taking the corresponding infimum. Furthermore, if A is unital then in both this nuclear reformulation and the more general exact setting we will obtain the same value of entropy in the last line of the display below if we assume that the ϕ and ψ in our approximately commuting diagrams are unital completely positive maps, although $\mathrm{rcp}(\Omega, \delta)$ as we have defined it may not always be equal to its unital version (see [5]). For an automorphism α of A we then set

$$\begin{aligned} ht(\alpha, \Omega, \delta) &= \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{rcp}(\Omega \cup \alpha\Omega \cup \cdots \cup \alpha^{n-1}\Omega, \delta), \\ ht(\alpha, \Omega) &= \sup_{\delta>0} ht(\alpha, \Omega, \delta), \\ ht(\alpha) &= \sup_{\Omega} ht(\alpha, \Omega), \end{aligned}$$

with the last supremum taken over all finite sets $\Omega \subset A$. We call $ht(\alpha)$ the *Voiculescu-Brown entropy* of α. For some computations see [36, 8, 3, 11, 12] and for a survey see [33].

Postponing momentarily the presentation of examples (to which we will turn at natural points as our discussion evolves), we first make the general remark that it is not at all clear from the definition of Voiculescu-Brown entropy how exponential growth (*i.e.,* positive values) might be produced, even for commutative systems.

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This is in striking contrast to topological entropy, for which the mechanism behind exponential growth is manifest in an example like the shift on $\{1, \ldots, d\}^{\mathbb{Z}}$.

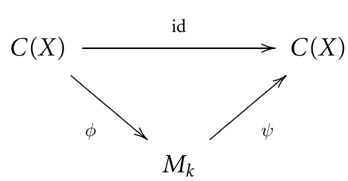
In fact, in every case to date in which positive Voiculescu-Brown entropy has been established, the argument has ultimately hinged on the use of some measure-theoretic entropy, whether it has involved an appeal to Connes-Narnhofer-Thirring entropy or Sauvageot-Thouvenot entropy or to the equality of Voiculescu-Brown entropy and the topological entropy on the pure state space in the separable unital commutative setting (whose only known proof relies on properties of Connes-Narnhofer-Thirring entropy and the classical variational principle [36, Prop. 4.8]). It is thus natural to ask if we can avoid measure-theoretic entropies altogether and obtain a direct geometric explanation for the production of exponential growth.

To approach this problem, let us first examine how the kind of mixing that produces positive topological entropy is reflected geometrically at the C^*-algebra level. Consider the right shift T on $X = \{-1, 1\}^{\mathbb{Z}}$ and the associated automorphism α of the C^*-algebra $C(X)$ of complex-valued functions on X given by $\alpha(f) = f \circ T$ for all $f \in C(X)$. Define the function $f \in C(X)$ by

$$f((a_k)_{k\in\mathbb{Z}}) = a_0$$

for all $(a_k)_{k\in\mathbb{Z}} \in X$. Now given any $n \in \mathbb{N}$, for every $\gamma \in \{-1, 1\}^{\{0,\ldots,n-1\}}$ there is some $x \in X$ such that, for each $k = 0, \ldots, n-1$, the function $\alpha^k(f)$ takes the value $\gamma(k)$ at x. This implies that any map sending the standard basis of ℓ_1^n to $\{f, \alpha(f), \ldots, \alpha^{n-1}(f)\}$ extends linearly to an isometric isomorphism of the real linear spans. This simple example illustrates that, in general, a high degree of dynamical mixing will produce real linear subspaces which are isometrically isomorphic to ℓ_1^n in a canonical way with respect to the iterates of one or more suitably chosen real-valued functions.

Now suppose that in our shift example we have a matrix algebra M_k and completely positive contractions $\phi\colon C(X) \to M_k$ and $\psi\colon M_k \to C(X)$ such that the diagram



approximately commutes to within δ on $\Omega_n = \{f, \alpha(f), \ldots, \alpha^{n-1}(f)\}$. Then this diagram also approximately commutes to within δ on the entire unit ball of the real linear span X of Ω_n by virtue of the fact that Ω_n forms a standard basis for a copy of ℓ_1^n. It follows that if $\delta < 1$ then $\phi(X)$ is $(1-\delta)^{-1}$-isomorphic to ℓ_1^n, *i.e.*, the Banach-Mazur distance

$$d(\phi(X), \ell_1^n) = \inf\{\|\Gamma\|\|\Gamma^{-1}\|\colon \Gamma\colon \phi(X) \to \ell_1^n \text{ is an isomorphism}\}$$

is at most $(1-\delta)^{-1}$. Now we know that the Voiculescu-Brown entropy agrees with the topological entropy on the pure state space, whose positive value of $\log 2$ is captured

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in the dynamical mixing that produces ℓ_1^n via the iterates of f. Thus, taking a Banach space viewpoint, we might suspect that, in general, for a fixed $K \geq 1$, the presence of a subspace K-isomorphic to ℓ_1^n within the real linear space of self-adjoint elements of M_k implies an asymptotic exponential dependence of k on n. This is indeed the case, as demonstrated by the following key proposition, for which I am grateful to Nicole Tomczak-Jaegermann. For simplicity, in the proposition statement and thenceforth (with the exception of Proposition 2.4 below) we will take our spaces to be over the complex numbers, as this will only affect our statements up to a fixed isomorphism factor; for example, in the above situation the $(1-\delta)^{-1}$-isomorphism between $\phi(X)$ and ℓ_1^n extends to a $2(1-\delta)^{-1}$-isomorphism between the complex linear span of $\phi(X)$ and the complex scalar version of ℓ_1^n, where by a *D-isomorphism* we mean an isomorphism $\Gamma \colon Y \to Z$ between Banach spaces which satisfies $\|\Gamma\|\|\Gamma^{-1}\| \leq D$.

Proposition 2.1 *Let X be an n-dimensional subspace of M_k (with the C^*-algebra norm) which is D-isomorphic to ℓ_1^n. Then*

$$n \leq aD^2 \log k$$

where $a > 0$ is a universal constant.

The proof, which can be found in [18], is based on a comparison of the type 2 (Rademacher) constants of the spaces involved. The required estimate on the type 2 constant of M_k, in particular, can be obtained using bounds on the type 2 constants of the Schatten p-classes which follow from Tomczak-Jaegermann's work in [35].

With Proposition 2.1 at hand we can now make the following general statement yielding positive Voiculescu-Brown entropy as a conclusion.

Proposition 2.2 *Let α be an automorphism of an exact C^*-algebra A. Suppose there exist a finite subset $\Omega \subset A$, a $D \geq 1$, and subsets $I_n \subset \{0,\ldots,n-1\} \times \Omega$ satisfying $\limsup_{n\to\infty} |I_n|/n > 0$ such that for each $n \in \mathbb{N}$ some (equivalently, any) map from the standard basis of $\ell_1^{I_n}$ to $\{\alpha^k(x) : (k,x) \in I_n\}$ linearly extends to a D-isomorphism. Then $ht(\alpha) > 0$.*

Proposition 2.2 is a direct consequence of Proposition 2.1, as the latter shows that if

$$\limsup_{n\to\infty} |I_n|/n \geq \mu > 0$$

then for any $0 < \delta < D^{-1}$ we have

$$ht(\alpha, \Omega, \delta) \geq \mu a^{-1} D^{-2} (1 - D\delta)^2 > 0.$$

Note that if the hypotheses of Proposition 2.2 hold for an automorphism α, then they also hold for any automorphism β of a C^*-algebra B such that there is a surjective *-homomorphism $\gamma \colon B \to A$ satisfying $\gamma \circ \beta = \alpha \circ \gamma$ (as is easily checked using the contractivity of γ), so that every such C^*-dynamical extension β of α has positive entropy. We thus obtain some information about the behaviour of Voiculescu-Brown entropy undertaking extensions, which in general has remained a mystery.

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As a concrete illustration of Proposition 2.2, consider the following operator-theoretic examples from [18]. We start with a sequence $\gamma \in \{-1,0,1\}^{\mathbb{Z}}$ in which every finite string of -1's and 1's occurs. Setting $E_i = \{k \in \mathbb{Z} : \gamma(k) = i\}$ for each $i = -1, 0, 1$, we define the operator $x \in \mathcal{B}(\ell_2(E_{-1} \cup E_1))$ by specifying

$$x\xi_k = \gamma(k)\xi_k$$

on the set $\{\xi_k \colon k \in E_{-1} \cup E_1\}$ of standard basis elements of $\ell_2(\mathbb{Z})$. Next take the direct sum of x with any self-adjoint operator of norm at most one on $\mathcal{B}(\ell_2(E_0))$, which yields an operator y on $\mathcal{B}(\ell_2(\mathbb{Z})) \supset \mathcal{B}(\ell_2(E_{-1} \cup E_1)) \oplus \mathcal{B}(\ell_2(E_0))$. Now if α is an automorphism of an exact C^*-subalgebra A of $\mathcal{B}(\ell_2(\mathbb{Z}))$ which restricts to the inner automorphism of $\mathcal{B}(\ell_2(\mathbb{Z}))$ arising from the canonical bilateral shift on $\ell_2(\mathbb{Z})$ as applied to y and its iterates, then $ht(\alpha) > 0$ by Proposition 2.2, since it is readily seen from our choice of sequence γ that for every $n \in \mathbb{N}$ any map from the standard basis of ℓ_1^n to $\{y, \alpha(y), \dots, \alpha^{n-1}(y)\}$ linearly extends over $\mathbb{R}$ to an isometric isomorphism, and hence over $\mathbb{C}$ to a 2-isomorphism. Without addressing here the general question of the existence of exact C^*-algebras admitting an automorphism α as above, we point out in particular that if a is a diagonal operator then we obtain from the shift on $\ell_2(\mathbb{Z})$ a positive entropy automorphism of the unital commutative C^*-algebra A generated by a and its iterates. Note than we can thus obtain positive topological entropy without having a topological description of the system at hand.

Problem 2.3 Under what conditions does the converse of Proposition 2.2 hold?

For homeomorphisms of the Cantor set we can show that the converse of Proposition 2.2 is valid, and even in a stronger form:

Proposition 2.4 *Let $T \colon X \to X$ be a homeomorphism of the Cantor set. Then $h_{\text{top}}(T) > 0$ if and only if there is a continuous function $f \colon X \to \mathbb{R}$ and subsets $I_n \subset \{0, \dots, n-1\}$ with $\limsup_{n\to\infty} |I_n|/n > 0$ such that for each $n \in \mathbb{N}$ the set $\{f \circ T^k : k \in I_n\}$ forms a standard basis for a copy of $\ell_1^{I_n}$ inside the real Banach space $C(X, \mathbb{R})$,* i.e., *any map from the standard basis elements of $\ell_1^{I_n}$ to $\{f \circ T^k : k \in I_n\}$ extends linearly over $\mathbb{R}$ to an isometric isomorphism.*

Proof In view of Proposition 2.2 we need only prove the "only if" implication. Suppose then that $h_{\text{top}}(T) > 0$. We will begin by showing the existence of a 2-element clopen partition $\mathcal{U}$ of X such that $h_{\text{top}}(T, \mathcal{U}) > 0$ (*cf.* the first part of the proof of Proposition 1 in [2]). Since the topology of X is generated by the clopen sets, there is a finite clopen cover $\mathcal{V} = \{V_1, \dots V_n\}$ of X such that $h_{\text{top}}(T, \mathcal{V}) > 0$. We may assume that $\mathcal{V}$ is in fact a partition of X by suitably refining it if necessary. For each $k = 1, \dots, n$ denote by $\mathcal{V}_k$ the clopen partition $\{V_k, X \setminus V_k\}$. Since $\mathcal{V}_1 \vee \cdots \vee \mathcal{V}_n$ is a refinement of $\mathcal{V}$ we then have

$$0 < h_{\text{top}}(T, \mathcal{V}) \le h_{\text{top}}(T, \mathcal{V}_1 \vee \cdots \vee \mathcal{V}_n) \le \sum_{k=1}^{n} h_{\text{top}}(T, \mathcal{V}_k).$$

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Thus for some $k = 1, \dots, n$ the 2-element clopen partition $\mathcal{V}_k$ satisfies $h_{\mathrm{top}}(T, \mathcal{V}_k) > 0$, as desired. Rewrite this clopen partition as $\mathcal{U} = \{U_1, U_{-1}\}$.

Next set $f = \chi_{U_1} - \chi_{U_{-1}} \in C(X, \mathbb{R})$ where χ_{U_1} and $\chi_{U_{-1}}$ are the characteristic functions of U_1 and U_{-1}, respectively. For each $n \in \mathbb{N}$ denote by $\mathcal{W}_n$ the clopen partition $\mathcal{U} \vee T^{-1}\mathcal{U} \vee \cdots \vee T^{-(n-1)}\mathcal{U}$ and by E_n the set of all $\gamma \in \{-1, 1\}^{\{0,\dots,n-1\}}$ such that $\bigcap_{k=0}^{n-1} T^{-k} U_{\gamma(k)} \in \mathcal{W}_n$. By the Sauer-Perles-Shelah lemma (or more specifically the consequence thereof formulated as Lemma 2.2 in [13]) there are a $c > 0$ and an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ there is a subset $I_n \subset \{0, 1, \dots, n-1\}$ satisfying $|I_n| \geq cn$ and $E_n|I_n = \{-1, 1\}^{I_n}$. Now if $n \geq n_0$ then for every $\gamma \in E_n|I_n$ there is a point $x \in X$ such that $(f \circ T^k)(x) = \gamma(k)$ for every $k \in I_n$. As in the example of the shift on $\{-1, 1\}^{\mathbb{Z}}$, this implies that any map sending the standard basis of $\ell_1^{I_n}$ to $\{f \circ T^k : k \in I_n\}$ extends linearly over $\mathbb{R}$ to an isometric isomorphism, and thus since $\limsup_{n\to\infty} |I_n|/n \geq c > 0$ we are done. ■

In this section we have used only the Banach space structure of the C^*-algebras in question. For a stark illustration of how the operator space structure can come into play, consider the shift $u_k \mapsto u_{k+1}$ on the full group C^*-algebra $C^*(\mathbb{F}_\infty)$ of the free group on a countable set of generators with associated unitaries $\{u_k\}_{k\in\mathbb{Z}}$. For each $n \in \mathbb{N}$ the set $\{u_1, u_2, \dots, u_n\}$ forms a standard basis for a copy of ℓ_1^n, but the C^*-algebra $C^*(\mathbb{F}_\infty)$ is not exact and for $n \geq 2$ any isomorphism between $\mathrm{span}\{u_1, u_2, \dots, u_n\}$ and a subspace of a matrix algebra has completely bounded isomorphism constant at least $n(2\sqrt{n-1})^{-1}$ (see [30]). In the next section we will discuss the analogous shift on the reduced group C^*-algebra $C_r^*(\mathbb{F}_\infty)$, which from a geometric viewpoint is drastically different.

3 The Free Shift

Here we apply our geometric perspective to the example of the free shift on the reduced group C^*-algebra $C_r^*(\mathbb{F}_\infty)$ of the free group on a countable set $\{g_k\}_{k\in\mathbb{Z}}$ of generators. This automorphism, which we will simply refer to as the free shift, arises from the shift $k \mapsto k+1$ on the index set $\mathbb{Z}$. It can be regarded as a quantized version of the shift $k \mapsto k+1$ on $\mathbb{Z}$ compactified with a fixed point at infinity, whereby orthogonality in ℓ_∞^n (for which the characteristic functions of the singletons $\{1\}, \dots, \{n\} \subset \mathbb{Z}$ form a standard basis) is replaced by orthogonality in ℓ_2^n (for which the unitaries associated to the elements $g_1, \dots, g_n \in \mathbb{F}_\infty$ form a standard basis up to 2-isomorphism). While this analogy makes little sense from the perspective of free probability (in terms of which reduced free products actually exhibit parallels with tensor products [37]), it is the appropriate one for our dynamical context, as we will see. In the noncommutative case, however, we must also take the operator space structure into account. Indeed our "quantization" is not unique and we could also consider for example the analogous free shift on the Cuntz algebra $\mathcal{O}_\infty$ [6] (see the second paragraph below).

It was shown in both [11] and [6], via different arguments, that the free shift has zero Voiculescu-Brown entropy. Correspondingly, the compactified shift on $\mathbb{Z}$ has zero topological entropy, as is readily computed directly from the definition.

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In [33] Størmer describes the free shift as the "most noncommutative" situation and accordingly asserts that highly noncommutative systems tend to have small entropy. The phenomenon underlying this statement appears in fact to have less to do with noncommutativity per se than with the relation of orthogonality, which is being considered here in its quantized Hilbert space sense (cf. the discussion in the introduction of [34]) but is equally well associated to zero entropy (in fact, arithmetic dynamical growth) in the classical commutative situation. It is hardly a coincidence that the matrix algebra M_n accommodates both ℓ_∞^n (down the diagonal) and ℓ_2^n (across any row or down any column), giving us a hint that the dynamical growth is subexponential, and hence that the entropy is zero, for both the compactified shift on $\mathbb{Z}$ and the free shift. This hint leads directly to a proof of zero Voiculescu-Brown entropy for the compactified shift on $\mathbb{Z}$ at the C^*-algebra level. On the other hand, in the noncommutative case we have to be more careful as a result of the operator space structures involved, and indeed the situation for the free shift is subtler and more sophisticated arguments are required [11, 6]. In fact the closed subspace spanned by the unitaries in $C_r^*(\mathbb{F}_\infty)$ corresponding to the generators of $\mathbb{F}_\infty$ is completely isomorphic to the closed subspace spanned by the elements $e_{1i} \oplus e_{i1}$ in the direct sum $R \oplus C$ of the row and column Hilbert spaces in $\mathcal{B}(\ell_2)$ [14] (see also Section 8.3 of [28]). Actually, if we take the operator space structure of a matrix algebra into consideration then the geometric hint from above applies precisely and directly in the noncommutative case if we switch to the nuclear setting of the Cuntz algebra $\mathcal{O}_\infty$ (which can be viewed as an infinite reduced free product of Toeplitz algebras—see Chapter 1 of [38]) and consider the automorphism defined by shifting the index on the canonical isometries $\{s_k\}_{k\in\mathbb{Z}}$ [6], for then arithmetic dynamical growth and hence zero Voiculescu-Brown entropy, at least at the local level of the canonical isometries, is readily seen by combining the fact that the closed subspace spanned by $\{s_k\}_{k\in\mathbb{Z}}$ is canonically completely isometric to the column Hilbert space in $\mathcal{B}(\ell_2)$ (see Section 1 of [29]) with a result of Pop and Smith that permits us to use general completely contractive linear maps in the definition of Voiculescu-Brown entropy [31] and an appeal to Wittstock's extension theorem which permits us to extend completely contractive linear maps into $\mathcal{B}(\mathcal{H})$ for any Hilbert space $\mathcal{H}$ (see [27]).

Størmer furthermore expresses surprise in [33] that the free shift has zero entropy in view of the fact that it is extremely ergodic, in the sense that its extension to the weak operator closure admits no proper globally invariant injective von Neumann subalgebra except for the scalars [32]. However, extreme ergodicity in topological dynamics (*i.e.,* the non-existence, at the function level, of proper globally invariant unital C^*-subalgebras besides the scalars) is associated with an extreme lack of recurrence, which results in arithmetic dynamical growth and hence zero entropy. Moreover, this behaviour is manifested precisely in our classical example of the compactified shift on $\mathbb{Z}$. Indeed if $f \in C(\mathbb{Z} \cup \{\infty\})$ belongs to a proper unital C^*-subalgebra of $C(\mathbb{Z} \cup \{\infty\})$ which is globally invariant under the induced C^*-algebra automorphism, then with T denoting the compactified shift we obtain from the Stone-Weierstrass theorem the existence of two distinct points $m, n \in \mathbb{Z} \cup \{\infty\}$ such that $f(T^j(m)) = f(T^j(n))$ for all $j \in \mathbb{Z}$, from which we infer by the continuity of f that $f(k) = f(\infty)$ for all $k \in \mathbb{Z}$, *i.e.,* f is constant. Thus the only globally invariant unital C^*-subalgebras are the scalars and $C(\mathbb{Z} \cup \{\infty\})$ itself.

By viewing the free shift as a quantization of the compactified shift on $\mathbb{Z}$ it therefore seems natural to expect zero Voiculescu-Brown entropy in light of the above geometric and topological considerations. We will not delve here into a rigorous explanation for zero entropy, referring again to [11, 6] for proofs.

Along the same lines, we might also regard a reduced free product of C^*-algebra automorphisms as a quantized version of a disjoint union of homeomorphisms or of a direct sum of C^*-algebra automorphisms. Indeed by [7] we have the formula

$$ht(\alpha * \beta) = \max(ht(\alpha), ht(\beta))$$

for reduced free products with amalgamation over a finite-dimensional C^*-algebra, paralleling the behaviour of disjoint unions with respect to topological entropy or direct sums with respect to Voiculescu-Brown entropy.

Thus the world of freeness, while inexhaustibly rich from a free probability viewpoint (see [38, 16]), is associated with a high degree of determinism in C^*-dynamics.

The discussion of this section vividly illustrates the idea, promoted by Weaver in [39], that the notion of quantization is at essence about Hilbert space, with noncommutativity appearing as a technical consequence.

4 Induced Dynamics on State Spaces

We will indicate in this section how Proposition 2.1 can be applied in a more systematic way to obtain a noncommutative analogue of Glasner and Weiss's result (Theorem 1.1) that zero topological entropy implies zero entropy on the space of probability measures. Full details can be found in [18].

The proof of Proposition 2.1 in [13], which involves a functional-analytic application of the combinatorial Sauer-Perles-Shelah lemma, can be adapted to establish the following result. Here C_1^n denotes the space of $n \times n$ matrices with the trace class norm.

Lemma 4.1 *Given $\varepsilon > 0$ and $\lambda > 0$ there exist $n_0 \in \mathbb{N}$ and $\mu > 0$ such that, for all $n \geq n_0$, if $\phi\colon C_1^{r_n} \to \ell_\infty^n$ is a *-linear map of norm at most 1 such that the image of the unit ball of $C_1^{r_n}$ under ϕ contains an ε-separated set of self-adjoint elements of cardinality at least $e^{\lambda n}$, then $r_n \geq e^{\mu n}$.*

Whereas Glasner and Weiss apply information about the possible size of subspaces of ℓ_∞^n and ℓ_1^n which are approximately isometric to ℓ_2^k, for the proof of Lemma 4.1 we must alternatively appeal to Proposition 2.1, which addresses the matrix situation.

Now let A be an separable exact C^*-algebra and α an automorphism of A, and denote by T_α the induced homeomorphism $\omega \mapsto \omega \circ \alpha$ of the quasi-state space $Q(A)$ (*i.e.,* the convex set of positive linear functionals ϕ on A with $\|\phi\| \leq 1$, equipped with the weak* topology, under which it is compact). Suppose that T_α has positive topological entropy (this actually implies that the entropy is infinite—see [18]—but this is not of consequence for the present discussion). Then by metrizing $Q(A)$ by taking the supremum of the absolute values of the differences of two given elements under evaluation on a compact and total subset K of self-adjoint elements of A, we

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can apply the separated set definition of topological entropy and push everything down to the level of the matrix algebras involved in the definition of Voiculescu-Brown entropy to permit an application of Lemma 4.1, with the required *-linear map from $C_1^{r_n}$ to ℓ_∞^n constructed by evaluating a suitable finite subset of K on the relevant elements of $Q(A)$ as modeled at the matrix level. Lemma 4.1 guarantees exponential growth in the rank of the matrix algebras, and so we obtain the following result.

Theorem 4.2 *Let A be a separable exact C^*-algebra and α an automorphism of A. If α has zero Voiculescu-Brown entropy, then T_α has zero topological entropy.*

If A is unital then in the statement of the thereom we can also replace T_α with its restriction to the state space.

Since topological entropy is non-decreasing under passing to subsystems, Theorem 4.2 shows in particular that if the induced homeomorphim T_α on the quasi-state space has positive topological entropy, then every C^*-dynamical extension of α has positive Voiculescu-Brown entropy (cf. the second paragraph after the statement of Proposition 2.2). More generally:

Corollary 4.3 *Let A and B be separable exact C^*-algebras and $\alpha\colon A \to A$ and $\beta\colon B \to B$ automorphisms. Suppose that the homeomorphism of the quasi-state space of A has positive topological entropy, and suppose that there exists a surjective positive contractive linear map $\gamma\colon B \to A$ such that $\alpha\circ\gamma = \gamma\circ\beta$, or an injective positive contractive linear map $\rho\colon A \to B$ such that $\beta \circ \rho = \rho \circ \alpha$. Then β has positive Voiculescu-Brown entropy. More generally, the same conclusion holds whenever α can be obtained from β through a finite chain of intermediary automorphisms intertwined in succession by maps of the same form as γ or ρ.*

In the other direction, we can draw from Theorem 4.2 some topological-dynamical conclusions such as the following proposition, which holds in view of the fact that the free shift on $C_r^*(\mathbb{F}_\infty)$ (see Section 3) has zero Voiculescu-Brown entropy [11, 6].

Proposition 4.4 *The homeomorphism of the state space of $C_r^*(\mathbb{F}_\infty)$ induced by the free shift has zero topological entropy.*

Finally, we ask:

Question 4.5 Does the converse of Theorem 4.2 hold?

Possible candidates for counterexamples to the converse of Theorem 4.2 can be found among the collection of automorphisms of the rotation C^*-algebras A_θ associated to a given matrix $S \in SL(2,\mathbb{Z})$ with eigenvalues off the unit circle. For every θ the automorphism of A_θ defined via S has positive Voiculescu-Brown entropy [17], but we have not been able to determine the topological entropy on the state space for irrational θ. See [18] for details.

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