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A NOTE ON NILPOTENT JORDAN RINGS

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ABSTRACT. Let R be a 2-torsion free associative ring with involution. It is shown that if the set S of symmetric elements is nilpotent as a Jordan ring then R is nilpotent.

Let R be an associative ring with involution which is 2-torsion free. The set S of symmetric elements is closed under the Jordan product $x \circ y = xy + yx$ and the set K of skew elements is closed under $[x, y] = xy - yx$. We define $S^{(1)} = S$, $S^{(k+1)} = S^{(k)} \circ S$ and say that S is Jordan nilpotent (of degree n) if $S^{(n)} = 0$ (with n minimal). This definition is equivalent to the condition that there exists an m such that any Jordan product of m elements of S, no matter how associated, is equal to 0 (see, e.g., [2], p. 18, Theorem 2.4). In this note we prove the following result.

THEOREM. *If S is Jordan nilpotent of degree n then R is nilpotent of degree* $\leqq 3 \cdot 5^{n-1}$.

Our interest in this matter stems from the following question posed in [1] (p. 195, Question 5.4):

Let G be a finite group of Jordan automorphisms of a ring such that R has no additive G-torsion. If the Jordan ring R^G of fixed elements is Jordan nilpotent, must R be nilpotent as an associative ring?

The question in general remains open and appears difficult. However, our theorem does answer in the affirmative a special case of the question, namely when R is a ring with involution * and $G = \{1, *\}$. Here, of course, R^G coincides with the symmetric elements S. We have subsequently learned from I. P. Shestakov that O. N. Smirnov (Novosibirsk) in a paper submitted to the Siberian Math J., has in fact proved more general results, still in the special case $G = \{1, *\}$:

(i) If R is associative and S in solvable (i.e., $S^{[n]} = 0$ where $S^{[k+1]} = S^{[k]} \circ S^{[k]}$) then R is nilpotent.

(ii) If R is alternative and S is solvable, then R is solvable.

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(iii) If R is alternative and S is nilpotent, then R is nilpotent if S generates R but in general R need not be nilpotent.

Shestakov has also pointed out the following results of A. P. Semjonov (Jakustsk), which have been submitted for publication:

If R is a Jordan algebra and G is a finite group of automorphisms of R then:

(i) R^G P.I. implies R P.I.
(ii) R^G nil of bounded index implies R nil of bounded index.
(iii) R^G solvable and char. $R = 0$ implies R solvable.

Before proceeding to the proof of the theorem we make some remarks and fix some notation.

First we indicate those properties of S and K which are needed for the theorem, namely, S and K are additive subgroups of R such that

(1) $S \circ S \subseteq S$.
(2) S is closed under tetrads $\{x_1x_2x_3x_4\} = x_1x_2x_3x_4 + x_4x_3x_2x_1$.
(3) $[S, K] \subseteq S$.
(4) $2K^3 \subseteq S + SK + KS$ (just consider $2abc = (abc - cba) + c(b \circ a) - (c \circ a)b + a(c \circ b)$ for $a, b, c \in K$).
(5) $2R \subseteq S + K$.

Furthermore, since R may be localized at the powers of 2 in view of R being 2-torsion free, we may assume without loss of generality that 2 is a bijection on R and accordingly we may replace (4) and (5) by

(4)′ $K^3 \subseteq S + SK + KS$.
(5)′ $R = S + K$.

We will also find it useful to define the following sets:

P_r (resp. Q_r) = the span of products of elements of $S \cup K$

which contain at least r factors from S (resp., whose first r factors lie in S).

$U_{k,p}$ (resp. $V_{k,p}$) = the span of products of elements of $S^{(k)}$

which contain at least p factors from $S^{(k+1)}$ (resp., whose first p factors lie in $S^{(k+1)}$).

LEMMA. $(S^{(k)})^{5m} \subseteq (S^{(k+1)})^m R$.

PROOF. For $x_1, x_2, x_3, x_4, x_5 \in S^{(k)}$ we have

$$\begin{aligned} x_1x_2x_3x_4x_5 &= x_1 \circ \{x_2x_3x_4x_5\} - x_2x_3x_4x_5x_1 - x_5x_4x_3x_2x_1 - x_1x_5x_4x_3x_2 \\ &\equiv -3x_1x_2x_3x_4x_5 \pmod{U_{k,1}} \end{aligned}$$

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where (2) was invoked and repeated use was made of $st \equiv -ts \pmod{S^{(k+1)}}$ for $s \in S^{(k)}$, $t \in S$. Therefore $x_1x_2x_3x_4x_5 \in U_{k,1}$, i.e., $(S^{(k)})^5 \subseteq U_{k,1}$, and consequently $(S^{(k)})^{5m} \subseteq U_{k,m}$. Repeated use of the relation $st \equiv -ts$ (mod $S^{(k+1)}$), $s \in S^{(k)}$, $t \in S^{(k+1)}$ shows that $U_{k,m} \subseteq V_{k,m} \subseteq (S^{(k+1)})^m R$ and the lemma is proved.

PROOF OF THE THEOREM. We are given that $S^{(n)} = 0$. Iteration of the lemma then yields $S^{(n-k)5^k} \subseteq S^{(n)}R = 0$, and so for $k = n - 1$ we have

(6) $S^p = 0, p = 5^{n-1}$.

Since $R = S + K$ it suffices to show that $x_1x_2 \ldots x_{3p} = 0$ where each $x_i \in S \cup K$. Now (4)′ assures us that $x_1x_2 \ldots x_{3p} \in P_p$, and after repeated application of $as \equiv sa$ (mod S), $s \in S$, $a \in K$ we see that $x_1x_2 \ldots x_{3p} \in Q_p$. Therefore (6) forces $R^{3p} = 0$ and the proof is complete.

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