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A Characterization of the Quantum Cohomology Ring of G/B and Applications

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Abstract. We observe that the small quantum product of the generalized flag manifold G/B is a product operation $\star$ on $H^*(G/B) \otimes \mathbb{R}[q_1,\dots,q_l]$ uniquely determined by the facts that it is a deformation of the cup product on $H^*(G/B)$; it is commutative, associative, and graded with respect to $\deg(q_i) = 4$; it satisfies a certain relation (of degree two); and the corresponding Dubrovin connection is flat. Previously, we proved that these properties alone imply the presentation of the ring $(H^*(G/B) \otimes \mathbb{R}[q_1,\dots,q_l], \star)$ in terms of generators and relations. In this paper we use the above observations to give conceptually new proofs of other fundamental results of the quantum Schubert calculus for G/B: the quantum Chevalley formula of D. Peterson (see also Fulton and Woodward) and the "quantization by standard monomials" formula of Fomin, Gelfand, and Postnikov for $G = SL(n,\mathbb{C})$. The main idea of the proofs is the same as in Amarzaya–Guest: from the quantum $\mathcal{D}$-module of G/B one can decode all information about the quantum cohomology of this space.

1 Introduction

Let us consider the complex flag manifold G/B, where G is a connected, simply connected, simple, complex Lie group and $B \subset G$ a Borel subgroup. Let $\mathfrak{t}$ be the Lie algebra of a maximal torus of a compact real form of G and $\Phi \subset \mathfrak{t}^*$ the corresponding set of roots. Consider an arbitrary W-invariant inner product $\langle\, ,\rangle$ on $\mathfrak{t}$. To any root α corresponds the coroot

$$\alpha^\vee := \frac{2\alpha}{\langle \alpha, \alpha\rangle}$$

which is an element of $\mathfrak{t}$ (by using the identification of $\mathfrak{t}$ and $\mathfrak{t}^*$ induced by $\langle\, ,\rangle$). If $\{\alpha_1,\dots,\alpha_l\}$ is a system of simple roots, then $\{\alpha_1^\vee,\dots,\alpha_l^\vee\}$ is a system of simple coroots. Consider $\{\lambda_1,\dots,\lambda_l\} \subset \mathfrak{t}^*$, the corresponding system of fundamental weights defined by $\lambda_i(\alpha_j^\vee) = \delta_{ij}$. The Weyl group W is the subgroup of $O(\mathfrak{t}, \langle\, ,\rangle)$ generated by the reflections about the hyperplanes $\ker\alpha$, $\alpha \in \Phi^+$. It can be shown that W is in fact generated by the *simple reflections* $s_1 = s_{\alpha_1},\dots,s_l = s_{\alpha_l}$ about the hyperplanes $\ker\alpha_1,\dots,\ker\alpha_l$. The *length* $l(w)$ of w is the minimal number of factors in a decomposition of w as a product of simple reflections.

Let $B^- \subset G$ denote the Borel subgroup opposite to B. To each $w \in W$ we assign the *Schubert variety* $X_w = \overline{B^-.w}$. The Poincaré dual of $[X_w]$ is an element of[1]

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[1] All homology and cohomology groups in this paper will be with coefficients in $\mathbb{R}$ (unless otherwise specified).

$H^{2l(w)}(G/B)$, which is called the *Schubert class*. The set $\{\sigma_w \mid w \in W\}$ is a basis of $H^*(G/B) = H^*(G/B, \mathbb{R})$, hence $\{\sigma_{s_1}, \dots, \sigma_{s_l}\}$ is a basis of $H^2(G/B)$. A theorem of Borel [Bo] says that the map

$$H^*(G/B) \to S(\mathfrak{t}^*)/S(\mathfrak{t}^*)^W = \mathbb{R}[\{\lambda_i\}]/I_W \tag{1.1}$$

described by $\sigma_{s_i} \mapsto [\lambda_i]$, $1 \le i \le l$, is a ring isomorphism (we are denoting by $S(\mathfrak{t}^*)^W = I_W$ the ideal of $S(\mathfrak{t}^*) = \mathbb{R}[\{\lambda_i\}]$ generated by the non-constant W-invariant polynomials).

To any l-tuple $d = (d_1, \dots, d_l)$ with $d_i \in \mathbb{Z}$, $d_i \ge 0$ corresponds a *Gromov-Witten invariant* $\langle \cdot | \cdot | \cdot \rangle_d$. To define it, we make the identification $H_2(G/B, \mathbb{Z}) = \mathbb{Z}^l$ via the basis consisting of the two-dimensional Schubert classes, that is, the classes whose Poincaré duals are $\sigma_{w_0 s_1}, \dots, \sigma_{w_0 s_l}$, where w_0 denotes the longest element of W. We denote by

$$(\cdot\,,\cdot): H^*(G/B) \times H^*(G/B) \to \mathbb{R}$$

the Poincaré pairing of G/B. To any three Schubert classes $\sigma_u, \sigma_v, \sigma_w$ one assigns the number, denoted by $\langle \sigma_u | \sigma_v | \sigma_w \rangle_d$, that counts the holomorphic curves $\varphi\colon \mathbb{C}P^1 \to G/B$ such that $\varphi_*([\mathbb{C}P^1]) = d$ in $H_2(G/B)$ and $\varphi(0)$, $\varphi(1)$ and $\varphi(\infty)$ are in general translates of the Schubert varieties dual to σ_u, σ_v, respectively σ_w. Let us consider the variables $q_1, \dots, q_l$. The *quantum cohomology ring* of G/B is the space $H^*(G/B) \otimes \mathbb{R}[\{q_i\}]$ equipped with the product $\circ$ which is $\mathbb{R}[\{q_i\}]$-linear and for any two Schubert classes σ_u, σ_v, $u, v \in W$ we have

$$\sigma_u \circ \sigma_v = \sum_{d=(d_1,\dots,d_l)\ge 0} q^d \sum_{w \in W} (\sigma_u \circ \sigma_v)_d \sigma_w,$$

$u, v \in W$. Here q^d denotes $q_1^{d_1} \cdots q_l^{d_l}$, and the cohomology class $(\sigma_u \circ \sigma_v)_d$ is determined by

$$\big((\sigma_u \circ \sigma_v)_d, \sigma_w\big) = \langle \sigma_u | \sigma_v | \sigma_w \rangle_d, \tag{1.2}$$

for any $w \in W$. It turns out that the product $\circ$ is commutative, associative and it is a deformation of the cup product (by which we mean that if we formally set $q_1 = \cdots = q_l = 0$, then $\circ$ becomes the same as the cup product). If we assign

$$\deg q_i = 4, \quad 1 \le i \le l,$$

then we also have the grading condition $\deg(a \circ b) = \deg a + \deg b$ for any two homogeneous elements a, b of $H^*(G/B) \otimes \mathbb{R}[\{q_i\}]$. For more details about quantum cohomology we refer the reader to Fulton and Pandharipande [FP].

The first goal of our paper is to prove the following characterization of $\circ$.

Theorem 1.1 *Let $\star$ be a product on the space $H^*(G/B) \otimes \mathbb{R}[\{q_i\}]$ which is commutative, associative, is a deformation of the cup product (in the sense defined above), satisfies the condition $\deg(a \star b) = \deg a + \deg b$ for a, b homogeneous elements of $H^*(G/B) \otimes \mathbb{R}[\{q_i\}]$ with respect to the grading $\deg q_i = 4$.*

Further assume:

(i) *The connection $\nabla^\hbar$ on the trivial vector bundle*

$$H^*(G/B) \times H^2(G/B) \to H^2(G/B)$$

given by $\nabla^\hbar = d + \frac{1}{\hbar}\omega$, where $\omega(X, Y) = X \star Y$, $X \in H^2(G/B), Y \in H^*(G/B)$, is flat for all $\hbar \neq 0$. Equivalently, if ω_k is the matrix of the $\mathbb{R}[\{q_i\}]$-linear endomorphism $\sigma_{s_k}\star$ of $H^*(G/B) \otimes \mathbb{R}[\{q_i\}]$ with respect to the Schubert basis, then we have

$$\frac{\partial}{\partial t_i}\omega_j = \frac{\partial}{\partial t_j}\omega_i$$

for all $1 \leq i, j \leq l$ (the convention $q_i = e^{t_i}$ is in force).

(ii) *We have*

$$\sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle \sigma_{s_i} \star \sigma_{s_j} = \sum_{i=1}^{l} \langle \alpha_i^\vee, \alpha_i^\vee \rangle q_i.$$

Then $\star$ is the quantum product $\circ$.

It is known that the conditions (i) and (ii) are satisfied by the quantum product $\circ$. The connection $\nabla^\hbar$ corresponding to $\circ$ is known as the *Dubrovin connection*, after B. Dubrovin, who introduced it and proved that it is flat (see [Du]). As for (ii), a proof of it can be found in [Ki]. For the reader's convenience, we will include proofs of (i) and (ii) for the product $\circ$ in the appendix. It is interesting to note that both properties follow easily from the so-called *divisor property* of the three-point Gromov–Witten invariants.

Remarks (i) The proof of Theorem 1.1 will be given in Section 2. The main tool we will be using is the notion of $\mathcal{D}$-module, in the spirit of B. Kim [Ki], Guest [Gu], Amarzaya and Guest [AG], and Iritani [Ir]. Here is a brief outline of the proof: $\mathcal{D}$ denotes the differential operator algebra generated by $e^{t_1}, \dots, e^{t_l}, \hbar\frac{\partial}{\partial t_1}, \dots, \hbar\frac{\partial}{\partial t_l}$. We will show that the $\mathcal{D}$-modules associated in Iritani's manner to the products $\circ$ and $\star$ are isomorphic by using techniques developed by B. Kim (actually a result we have proved in our previous paper [Ma3]). More precisely, we obtain the *quantum Toda* $\mathcal{D}$-module, determined by the integrals of motion of the quantum Toda lattice integrable system. Amarzaya and Guest [AG] have found a concrete method of decoding the quantum cohomology of G/B out of the latter $\mathcal{D}$-module by solving a certain PDE system. At the last step of our proof we will be applying their method.

(ii) Theorem 1.1 (more precisely, its hypotheses) can be considered as an alternative definition of the (small) quantum cohomology ring of G/B. The reader will decide whether this is more convenient than the original definition, given in terms of rational curves (see [FP]). The following question arises: can one prove the main results of the quantum Schubert calculus for G/B starting from the new definition? We have already proved in [Ma3] that if $\star$ is a product as in Theorem 1.1, then the ring $(H^*(G/B) \otimes \mathbb{R}[\{q_i\}], \star)$ has the expected presentation in terms of generators and relations, namely the one determined by Kim [Ki]. We will explain in what follows (see the remaining part of this section) how one can prove the quantum Chevalley and quantum Giambelli formulas for the abstract ring $(H^*(G/B) \otimes \mathbb{R}[\{q_i\}], \star)$. An important ingredient of the proof is the combinatorial quantum cohomology ring of G/B, which is a purely combinatorial object defined and investigated by us [Ma4]. Then in Section 3 we will address the case $G = SL(n, \mathbb{C})$ and give a direct proof of the "quantization via standard monomials" formulas of Fomin, Gelfand and Postnikov [FGP], but this time without using the combinatorial quantum cohomology ring of [Ma4]. It is important to note that in this way we obtain conceptually new proofs of all the main results of quantum Schubert calculus for G/B (simply because the actual quantum product $\circ$ satisfies the hypotheses of Theorem 1.1, as we explained above).

The second main goal of our paper is to give new proofs of the quantum Chevalley, quantum Giambelli, and the "quantization via standard monomials" formulas. To this end, we need a characterization of the quantum Giambelli polynomials in terms of the flatness of the Dubrovin connection. Let us denote by $QH^*(G/B)$ the quotient ring $\mathbb{R}[\{\lambda_i\}, \{q_i\}]/\langle R_1, \dots, R_l\rangle$, where $R_1, \dots, R_l$ are the quantum deformations in the quantum cohomology ring $(H^*(G/B) \otimes \mathbb{R}[\{q_i\}], \circ)$ of the fundamental homogeneous generators of $S(\mathfrak{t}^*)^W$ ($R_1, \dots, R_l$ have been determined explicitly by B. Kim [Ki]; we will present in Section 2 a few more details about that). For any $c \in \mathbb{R}[\{\lambda_i\}, \{q_i\}]$ we denote by $[c]_q$ the coset of c in $QH^*(G/B)$. The map $\sigma_{s_i} \mapsto [\lambda_i]_q$ induces a tautological isomorphism

$$(H^*(G/B) \otimes \mathbb{R}[\{q_i\}], \circ) \simeq QH^*(G/B). \tag{1.3}$$

Finding for each $w \in W$ a polynomial $\hat{c}_w \in \mathbb{R}[\{\lambda_i\}, \{q_i\}]$ whose coset in $QH^*(G/B)$ is the image of σ_w, in other words, solving the quantum Giambelli problem, would lead to a complete knowledge of the quantum cohomology of G/B. We are looking for conditions which determine the polynomials $\hat{c}_w$. First of all, let us consider for each $w \in W$ a polynomial[2] $c_w \in \mathbb{R}[\{\lambda_i\}]$ whose coset corresponds to σ_w via the isomorphism (1.1). There are two natural conditions that we impose on the polynomials $\hat{c}_w$:

$$\deg \hat{c}_w = \deg c_w \tag{1.4}$$

with respect to the grading $\deg \lambda_i = 2$, $\deg q_i = 4$, and

$$\hat{c}_w|_{(\text{ all } q_i = 0)} = c_w. \tag{1.5}$$

Whenever the conditions (1.4) and (1.5) are satisified, the cosets $[\hat{c}_w]_q$, $w \in W$, are a basis of $QH^*(G/B)$ over $\mathbb{R}[\{q_i\}]$. Consider the 1-form

$$\omega = \sum_{i=1}^{l} \omega_i dt_i,$$

where ω_i is the matrix of multiplication of $QH^*(G/B)$ by $[\lambda_i]_q$ with respect to the latter basis. We can prove the following.

[2]These are solutions of the classical Giambelli problem for G/B. Such polynomials have been constructed, for instance, by Bernstein, I. M. Gelfand and S. I. Gelfand [BGG].

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Corollary 1.2 *Let $\hat{c}_w$, $w \in W$, be polynomials in $\mathbb{R}[\{\lambda_i\},\{q_i\}]$ which satisfy the properties* (1.4) *and* (1.5)*. Then the image of σ_w by the isomorphism* (1.3) *is $[\hat{c}_w]_q$ for all $w \in W$ if and only if the connection*

$$\nabla^\hbar = d + \frac{1}{\hbar}\omega$$

is flat for all $\hbar \in \mathbb{R} \setminus \{0\}$. The latter condition reads

$$\frac{\partial}{\partial t_i}\omega_j = \frac{\partial}{\partial t_j}\omega_i,$$

for all $1 \le i, j \le l$.

Proof Consider the $\mathbb{R}[\{q_i\}]$-linear isomorphism[3]

$$\delta\colon QH^*(G/B) \to H^*(G/B) \otimes \mathbb{R}[\{q_i\}] = \mathbb{R}[\{\lambda_i\},\{q_i\}]/(I_W \otimes \mathbb{R}[\{q_i\}])$$

determined by

$$\delta[\hat{c}_w]_q = [c_w], \tag{1.6}$$

for all $w \in W$. Define the product $\star$ on $H^*(G/B) \otimes \mathbb{R}[\{q_i\}]$ by

$$x \star y = \delta(\delta^{-1}(x)\delta^{-1}(y)),$$

$x, y \in H^*(G/B) \otimes \mathbb{R}[\{q_i\}]$. The product is commutative and associative; it is a deformation of the cup product on $H^*(G/B)$; and it satisfies $\deg(a \star b) = \deg a + \deg b$, where $a, b \in H^*(G/B) \otimes \mathbb{R}[\{q_i\}]$ are homogeneous elements. The map δ is obviously a ring isomorphism between $QH^*(G/B)$ and $(H^*(G/B) \otimes \mathbb{R}[\{q_i\}], \star)$. In particular, the following degree two relation holds:

$$\sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle [\lambda_i] \star [\lambda_j] = \sum_{i=1}^{l} \langle \alpha_i^\vee, \alpha_i^\vee \rangle q_i.$$

Moreover, the matrix of $[\lambda_i]\star$ on $H^*(G/B) \otimes \mathbb{R}[q_1, \dots, q_l]$ with respect to the Schubert basis $\{[c_w] : w \in W\}$ is just ω_i. So if the connection $\nabla^\hbar$ is flat for all $\hbar$, then by Theorem 1.1 the products $\star$ and $\circ$ are the same. This implies that δ is just the isomorphism (1.3). The conclusion follows from the definition (1.6) of δ. ■

Corollary 1.2 will be used in Section 3 in order to recover the "quantization via standard monomials" theorem of Fomin, Gelfand, and Postnikov for $G = SL(n, \mathbb{C})$ (see [FGP, Theorem 1.1]). It is important to note that the proof does not make use of the combinatorial quantum cohomology ring, as in the case of the quantum Chevalley formula (see below).

[3] This is what Amarzaya and Guest [AG] call a *quantum evaluation map*.

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Our strategy for proving the quantum Chevalley formula involves using the combinatorial quantum product, which has been constructed in [Ma4]. By definition, this is a product $\star$ on $H^*(G/B) \otimes \mathbb{R}[\{q_i\}]$ which does satisfy the *quantum Chevalley formula*, namely:

$$\sigma_{s_i} \star \sigma_w = \sigma_{s_i}\sigma_w + \sum \lambda_i(\alpha^\vee)\sigma_{ws_\alpha} q^{\alpha^\vee},$$

for $1 \leq i \leq l$, $w \in W$. Here the sum runs over all positive roots α with the property that $l(ws_\alpha) = l(w) - 2\,\mathrm{height}(\alpha^\vee) + 1$, where we consider the expansion $\alpha^\vee = m_1\alpha_1^\vee + \cdots + m_l\alpha_l^\vee$, $m_j \in \mathbb{Z}$, $m_j \geq 0$ and denote

$$\mathrm{height}(\alpha^\vee) = m_1 + \cdots + m_l, \quad q^{\alpha^\vee} = q_1^{m_1} \cdots q_l^{m_l}.$$

We have also shown [Ma4] that $\star$ satisfies all hypotheses of Theorem 1.1. We deduce the following.

Corollary 1.3 *The combinatorial and actual quantum products coincide. Consequently, the actual quantum product $\circ$ satisfies the quantum Chevalley formula:*

$$\sigma_{s_i} \circ \sigma_w = \sigma_{s_i}\sigma_w + \sum_{l(ws_\alpha)=l(w)-2\,\mathrm{height}(\alpha^\vee)+1} \lambda_i(\alpha^\vee)\sigma_{ws_\alpha} q^{\alpha^\vee}, \tag{1.7}$$

for $1 \leq i \leq l$, $w \in W$.

Remark Formula (1.7) plays a crucial role in the study of the quantum cohomology algebra of G/B, as this is generated over $\mathbb{R}[q_1, \dots, q_l]$ by the degree 2 Schubert classes $\sigma_{s_1}, \dots, \sigma_{s_l}$. The formula was announced by D. Peterson. A rigorous intersection-theoretic proof has been given by W. Fulton and C. Woodward [FW]. Our proof of this formula is conceptually different from theirs.

A quantum Giambelli formula, *i.e.,* a formula for representatives of Schubert classes via the isomorphism (1.3), for the combinatorial quantum product has been proved in [Ma4]. Consequently, the same formula holds true for the actual quantum product $\circ$.

2 $\mathcal{D}$-Modules and Quantum Cohomology

The goal of this section is to give a proof of Theorem 1.1.

We denote by $\mathcal{D}$ the Heisenberg algebra, by which we mean the associative $\mathbb{R}[\hbar]$-algebra generated by $Q_1, \dots, Q_l, P_1, \dots, P_l$, subject to the relations

$$[Q_i, Q_j] = [P_i, P_j] = 0, \quad [P_i, Q_j] = \delta_{ij}\hbar Q_j, \tag{2.1}$$

$1 \leq i, j \leq l$. It becomes a graded algebra with respect to the assignments

$$\deg Q_i = 4, \quad \deg P_i = \deg \hbar = 2. \tag{2.2}$$

Note that any element D of $\mathcal{D}$ can be written uniquely as an $\mathbb{R}[\hbar]$-linear combination of monomials of type $Q^I P^J$.

A concrete realization of $\mathcal{D}$ can be obtained by putting $Q_i = e^{t_i}$ and $P_i = \hbar\frac{\partial}{\partial t_i}$, $1 \leq i \leq l$. We will be interested in certain elements of $\mathcal{D}$ which arise in connection with the Hamiltonian system of Toda lattice type corresponding to the coroots of G, namely the first quantum integrals of motion of this system. Those are homogeneous elements $D_k = D_k(\{Q_i\}, \{P_i\}, \hbar)$ of $\mathcal{D}$, $1 \leq k \leq l$, which commute with

$$D_1 = \sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle P_i P_j - \sum_{i=1}^{l} \langle \alpha_i^\vee, \alpha_i^\vee \rangle Q_i$$

and also satisfy the property that $D_k(\{0\}, \{\lambda_i\}, 0)$, $1 \leq k \leq l$, are just the fundamental homogeneous W-invariant polynomials (for more details concerning the differential operators $D_1, \dots, D_l$ we address the reader to [Ma3]). We will denote by $\mathcal{I}$ the left-sided ideal of $\mathcal{D}$ generated by $D_1, \dots, D_l$.

Let $\star$ be a product on $H^*(G/B) \otimes \mathbb{R}[\{q_i\}]$ which satisfies the hypotheses of Theorem 1.1. Let us denote by E the $\mathcal{D}$-module (*i.e.,* vector space with an action of the algebra $\mathcal{D}$) $H^*(G/B) \otimes \mathbb{R}[\{q_i\}, \hbar]$ defined by

$$Q_i.a = q_i a, \quad P_i.a = \sigma_{s_i} \star a + \hbar q_i \frac{\partial}{\partial q_i} a,$$

$1 \leq i \leq l$, $a \in H^*(G/B) \otimes \mathbb{R}[\{q_i\}, \hbar]$. The isomorphism type of the $\mathcal{D}$-module E corresponding to $\star$ is uniquely determined by the hypotheses of Theorem 1.1, as the following proposition shows.

Proposition 2.1 *If $\star$ is a product with the properties stated in Theorem* 1.1*, then the map $\phi\colon \mathcal{D} \to H^*(G/B) \otimes \mathbb{R}[\{q_i\}, \hbar]$ given by*

$$f(\{Q_i\}, \{P_i\}, \hbar) \overset{\phi}{\mapsto} f(\{Q_i\}, \{P_i\}, \hbar) \cdot 1 = f\Big(\{q_i\}, \Big\{\sigma_{s_i} \star + \hbar q_i \frac{\partial}{\partial q_i}\Big\}, \hbar\Big) \cdot 1$$

is surjective and induces an isomorphism of $\mathcal{D}$-modules

$$\mathcal{D}/\mathcal{I} \simeq E, \tag{2.3}$$

where $\mathcal{I}$ is the left-sided ideal of $\mathcal{D}$ generated by the quantum integrals of motion of the Toda lattice (see above).

Proof We will use the grading on $H^*(G/B) \otimes \mathbb{R}[\{q_i\}, \hbar]$ induced by the usual grading on $H^*(G/B)$, $\deg q_i = 4$ and $\deg \hbar = 2$. Combined with the grading defined by (2.2), this makes ϕ into a degree preserving map (more precisely, it maps a homogeneous element of $\mathcal{D}$ to a homogeneous element of the same degree in $H^*(G/B) \otimes \mathbb{R}[\{q_i\}, \hbar]$).

Let us prove first the surjectivity stated in our theorem. It is sufficient to show that any homogeneous element $a \in H^*(G/B) \otimes \mathbb{R}[\{q_i\}, \hbar]$ can be written as

$$f(\{Q_i\}, \{P_i\}, \hbar) \cdot 1.$$

We proceed by induction on $\deg a$. If $\deg a = 0$, everything is clear. Now consider $a \in H^*(G/B) \otimes \mathbb{R}[\{q_i\}, \hbar]$, a homogeneous element of degree at least 2. By a result of Siebert and Tian [ST], we can express $a = g(\{q_i\}, \{\sigma_{s_i} \star\}, \hbar)$ for a certain polynomial g. We have

$$a - g(\{Q_i\}, \{P_i\}, \hbar) \cdot 1 = a - g(\{q_i\}, \left\{ \sigma_{s_i} \star + \hbar q_i \frac{\partial}{\partial q_i} \right\}, \hbar) \cdot 1 = \hbar b,$$

where $b \in H^*(G/B) \otimes \mathbb{R}[\{q_i\}, \hbar]$ is homogeneous of degree $\deg a - 2$ or it is zero. We use the induction hypothesis for b.

We proved [Ma3, proof of Lemma 4.5] that the generators $D_k = D_k(\{Q_i\}, \{P_i\}, \hbar)$, $1 \leq k \leq l$, of the ideal $\mathcal{I}$ satisfy

$$D_k(\{Q_i\}, \{P_i\}, \hbar) \cdot 1 = 0. \tag{2.4}$$

If we let $\hbar$ approach 0 in (2.4), we obtain the relations

$$D_k(\{q_i\}, \{\sigma_{s_i} \star\}, 0) = 0, \tag{2.5}$$

$1 \leq k \leq l$. They generate the whole ideal of relations in the ring $(H^*(G/B) \otimes \mathbb{R}[\{q_i\}], \star)$.

We need to show that if D is an element of $\mathcal{D}$ with the property that

$$D(\{Q_i\}, \{P_i\}, \hbar) \cdot 1 = 0, \tag{2.6}$$

then $D \in \mathcal{I}$. Because the map ϕ is degree preserving, we may assume that D is homogeneous and proceed by induction on $\deg D$. If $\deg D = 0$, *i.e.,* D is constant, then (2.6) implies $D = 0$, hence $D \in \mathcal{I}$. It now follows the induction step. From

$$D.1 = D(\{q_i\}, \left\{ \sigma_{s_i} \star + \hbar q_i \frac{\partial}{\partial q_i} \right\}, \hbar) \cdot 1 = 0,$$

for all $\hbar$, we deduce the relation $D(\{q_i\}, \{\sigma_{s_i} \star\}, 0) = 0$ in the ring $(H^*(G/B) \otimes \mathbb{R}[\{q_i\}], \star)$. Consequently we have the following polynomial identity

$$D(\{q_i\}, \{\lambda_i\}, 0) = \sum_k f_k(\{q_i\}, \{\lambda_i\}) D_k(\{q_i\}, \{\lambda_i\}, 0),$$

for certain polynomials f_k. By using the commutation relations (2.1), we obtain the following identity in $\mathcal{D}$:

$$\begin{aligned} D(\{Q_i\}, \{P_i\}, 0) &\equiv \sum_k f_k(\{Q_i\}, \{P_i\}) D_k(\{Q_i\}, \{P_i\}, 0) \bmod \hbar \\ &\equiv \sum_k f_k(\{Q_i\}, \{P_i\}) D_k(\{Q_i\}, \{P_i\}, \hbar) \bmod \hbar. \end{aligned}$$

In other words,

$$D(\{Q_i\},\{P_i\},\hbar)=\sum_k f_k(\{Q_i\},\{P_i\})D_k(\{Q_i\},\{P_i\},\hbar)+\hbar D'(\{Q_i\},\{P_i\},\hbar),$$

for a certain $D' \in \mathcal{D}$, with $\deg D' < \deg D$. From (2.5) and (2.6) we deduce that

$$D'(\{Q_i\},\{P_i\},\hbar)\cdot 1 = 0.$$

Since $\deg D' < \deg D$, we only have to use the induction hypothesis for D' and get to the desired conclusion. ■

Note that (2.3) is also an isomorphism of $\mathbb{R}[\{Q_i\},\hbar]$-modules. Since the actual quantum product $\circ$ satisfies the hypotheses of Theorem 1.1, we deduce that the dimension of $\mathcal{D}/\mathcal{I}$ as an $\mathbb{R}[\{Q_i\},\hbar]$-module equals $|W|$. Let us consider the "standard monomial basis" $\{[C_w] : w \in W\}$ of $\mathcal{D}/\mathcal{I}$ over $\mathbb{R}[\{Q_i\},\hbar]$ with respect to a choice of a Gröbner basis of the ideal $\mathcal{I}$ (for more details, see Guest [Gu, §1] and the references therein). Any C_w is a monomial in $P_1,\dots,P_l$ and the cosets of the monomials $c_w = C_w(\lambda_1,\dots,\lambda_l)$, $w \in W$ in $H^*(G/B) = S(\mathfrak{t}^*)/S(\mathfrak{t}^*)^W = \mathbb{R}[\{\lambda_i\}]/I_W$ are a basis. We will need the following result.

Proposition 2.2 *There exists a unique basis $\{[\bar{C}_w] : w \in W\}$ of $\mathcal{D}/\mathcal{I}$ over $\mathbb{R}[\{Q_i\},\hbar]$ with the following properties:*

(i) *For all $w \in W$ the element $\bar{C}_w = \bar{C}_w(\{Q_i\},\{P_i\},\hbar)$ of $\mathcal{D}$ is homogeneous of degree $2\deg c_w$ with respect to the grading defined by* (2.2).

(ii) *For all $w \in W$ we have*

$$\bar{C}_w(\{0\},\{\lambda_i\},\hbar) \equiv c_w \bmod I_W;$$

in particular $\bar{C}_w(\{0\},\{\lambda_i\},\hbar) \bmod I_W$ is independent of $\hbar$.

(iii) *The elements $(\bar{\Omega}^i_{vw})^{1\le i\le l}_{v,w\in W}$ of $\mathbb{R}[Q_1,\dots,Q_l,\hbar]$ determined by*

$$P_i[\bar{C}_w] = \sum_{v\in W} \bar{\Omega}^i_{vw}[\bar{C}_v],$$

are independent of $\hbar$.

Proof In order to show that such a basis exists, we consider the isomorphism

$$\phi\colon \mathcal{D}/\mathcal{I} \to H^*(G/B)\otimes\mathbb{R}[\{q_i\},\hbar]$$

induced by the actual quantum product $\circ$ via Proposition 2.1. The basis $\{[c_w] : w \in W\}$ of the right-hand side induces the basis $\{[\bar{C}_w] = \phi^{-1}([c_w]) : w \in W\}$ of $\mathcal{D}/\mathcal{I}$ over $\mathbb{R}[\{Q_i\},\hbar]$. It is obvious that the latter basis satisfies (i) and (iii). In order to show that it also satisfies (ii), we consider the following commutative diagram:

$$\begin{tikzcd} \mathcal{D}/\mathcal{I} \arrow[rr, "\phi"] \arrow[dr, "\psi_1"'] & & H^*(G/B)\otimes\mathbb{R}[\{q_i\},\hbar] \arrow[dl, "\psi_2"] \\ & H^*(G/B)\otimes\mathbb{R}[\hbar] & \end{tikzcd}$$

where ψ_2 is the canonical projection and

$$\psi_1 \colon \mathcal{D}/\mathcal{J} \to H^*(G/B) \otimes \mathbb{R}[\hbar] = (\mathbb{R}[\{\lambda_i\}]/I_W) \otimes \mathbb{R}[\hbar]$$

is given by $[D(\{Q_i\}, \{P_i\}, \hbar)] \mapsto [D(\{0\}, \{\lambda_i\}, \hbar)]$. Note that ψ_1 is well defined, as for any $k = 1, 2, \ldots, l$, the polynomial $D_k(\{0\}, \{\lambda_i\}, \hbar)$ is independent of $\hbar$, being equal to u_k, the k-th fundamental W-invariant polynomial (see [Ma2, §3]). We observe that $[\bar{C}_w(\{0\}, \{\lambda_i\}, \hbar)] = \psi_1[\bar{C}_w] = \psi_2[c_w] = [c_w]$, hence condition (ii) is satisfied.

In order to show that there exists *at most one* such basis, one can use the method of [AG, §2]. More precisely, we only need to note that the PDE system presented there has at most one "admissible" solution. ■

Now we can prove our main result.

Proof of Theorem 1.1 Let $\star$ be a product with the properties stated in Theorem 1.1. Consider the isomorphism of $\mathcal{D}$-modules $\phi \colon \mathcal{D}/\mathcal{J} \to H^*(G/B) \otimes \mathbb{R}[\{q_i\}, \hbar]$ given by Proposition 2.1. The basis $\{[c_w] : w \in W\}$ of the right-hand side induces the basis $\{[\bar{C}_w] = \phi^{-1}([c_w]) : w \in W\}$ of $\mathcal{D}/\mathcal{J}$ over $\mathbb{R}[\{Q_i\}, h]$. It is obvious that the latter satisfies hypotheses (i) and (iii) of Proposition 2.2. We show that it also satisfies (ii) by using the argument already employed in the first part of the proof of Proposition 2.2. Now from Proposition 2.2, we deduce that $[\bar{C}_w] = [\hat{C}_w]$, for $w \in W$, where the basis $\{[\hat{C}_w] : w \in W\}$ is induced by the actual quantum product $\circ$. Now, since ϕ is an isomorphism of $\mathcal{D}$-modules, $\phi([\bar{C}_w]) = [c_w]$ and $\phi(P_i) = [\lambda_i]$, we deduce that the matrix of $[\lambda_i]\star$ with respect to the basis $\{[c_w] : w \in W\}$ is the same as the matrix of P_i with respect to the basis $\{[\bar{C}_w] : w \in W\}$. Consequently we have $[\lambda_i] \star a = [\lambda_i] \circ a$ for all $a \in H^*(G/B) \otimes \mathbb{R}[q_1, \ldots, q_l]$. Hence the products $\star$ and $\circ$ are the same. ■

3 Quantization Map for Fl_n

In the case $G = SL(n, \mathbb{C})$, the resulting flag manifold is Fl_n, which is the space of all complete flags in $\mathbb{C}^n$. Borel's presentation (see (1.1)) in this case reads

$$H^*(Fl_n) = \mathbb{R}[\lambda_1, \ldots, \lambda_{n-1}]/(I_n)_{\geq 2},$$

where $(I_n)_{\geq 2}$ denotes the ideal generated by the nonconstant symmetric polynomials of degree at least 2 in the variables

$$x_1 := \lambda_1, \quad x_2 := \lambda_2 - \lambda_1, \quad \ldots \quad x_{n-1} := \lambda_{n-1} - \lambda_{n-2}, \quad x_n := -\lambda_{n-1}.$$

Equivalently, we have $H^*(Fl_n) = \mathbb{R}[x_1, \ldots, x_n]/I_n$ where I_n denotes the ideal generated by the nonconstant symmetric polynomials of degree at least 1 in the variables $x_1, \ldots, x_n$. For any $k \in \{0, 1, \ldots, n\}$ we consider the polynomials $e_0^k, \ldots, e_k^k$ in the variables $x_1, \ldots, x_k$ which can be described by

$$\det \left[\begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & x_k \end{pmatrix} + \mu I_k \right] = \sum_{i=0}^{n} e_i^k \mu^{k-i}.$$

For $i_1, \ldots, i_{n-1} \in \mathbb{Z}$ such that $0 \leq i_j \leq j$, we define

$$e_{i_1 \cdots i_{n-1}} = e^1_{i_1} \cdots e^{n-1}_{i_{n-1}}.$$

These are called the *standard elementary monomials.* It is known (see [FGP, Proposition 3.4]) that the set $\{[e_{i_1 \cdots i_{n-1}}] : 0 \leq i_j \leq j\}$ is a basis of $H^*(Fl_n)$.

We also consider the polynomials[4] $\hat{e}^k_0, \ldots, \hat{e}^k_k$ in the variables $x_1, \ldots, x_k, q_1, \ldots, q_{k-1}$, which are described by

$$\det \left[\begin{pmatrix} x_1 & q_1 & 0 & \cdots & 0 \\ -1 & x_2 & q_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & x_{k-1} & q_{k-1} \\ 0 & \cdots & 0 & -1 & x_k \end{pmatrix} + \mu I_k \right] = \sum_{i=0}^{k} \hat{e}^k_i \mu^{k-i}.$$

For $i_1, \ldots, i_{n-1}$ such that $0 \leq i_j \leq j$, we define the *quantum standard elementary monomials*

$$\hat{e}_{i_1 \cdots i_{n-1}} = \hat{e}^1_{i_1} \cdots \hat{e}^{n-1}_{i_{n-1}}.$$

By a theorem of Ciocan-Fontanine [Ci] (in fact Kim's theorem for $G = SL(n, \mathbb{C})$, see Section 1), we have the following isomorphism of $\mathbb{R}[q_1, \ldots, q_{n-1}]$-algebras

$$(3.1) \quad (H^*(Fl_n) \otimes \mathbb{R}[q_1, \ldots, q_{n-1}], \circ) \simeq QH^*(Fl_n) := \mathbb{R}[x_1, \ldots, x_n, q_1, \ldots, q_{n-1}]/\langle \hat{e}^n_1, \ldots, \hat{e}^n_n \rangle,$$

which is canonical in the sense that $[x_i]$ is mapped to $[x_i]_q$. According to [FGP], we will call this the *quantization map.* Since the conditions (1.4) and (1.5) are satisfied, we deduce that $\{[\hat{e}_{i_1 \cdots i_{n-1}}]_q : 0 \leq i_j \leq j\}$ is a basis of $QH^*(Fl_n)$ over $\mathbb{R}[q_1, \ldots, q_{n-1}]$. We also point out the obvious fact that $\{[e_{i_1 \cdots i_{n-1}}] : 0 \leq i_j \leq j\}$ is a basis of $H^*(Fl_n) \otimes \mathbb{R}[q_1, \ldots, q_{n-1}]$ over $\mathbb{R}[q_1, \ldots, q_{n-1}]$. The goal of this section is to give a different proof to the following theorem of Fomin, Gelfand, and Postnikov.

Theorem 3.1 **([FGP, Theorem 1.1])** *The quantization map described by equation* (3.1) *sends* $[e_{i_1 \ldots i_{n-1}}]$ *to* $[\hat{e}_{i_1 \ldots i_{n-1}}]_q$.

The main instrument of our proof is the $\mathcal{D}$-module $\mathcal{D}/\mathcal{J}$ defined in Section 2. In this case (*i.e.,* $G = SL(n, \mathbb{C})$) we can describe it explicitly, as follows: $\mathcal{D}$ is the (noncommutative) Heisenberg algebra defined at the beginning of Section 2 where $l = n - 1$. The left ideal $\mathcal{J}$ of $\mathcal{D}$ is generated by $\mathcal{E}^n_1, \ldots, \mathcal{E}^n_{n-1}$, where

$$\det \left[\begin{pmatrix} P_1 & Q_1 & 0 & \cdots & 0 \\ -1 & P_2 - P_1 & Q_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & P_{n-1} - P_{n-2} & Q_{n-1} \\ 0 & \cdots & 0 & -1 & -P_{n-1} \end{pmatrix} + \mu I_n \right] = \sum_{i=0}^{n} \mathcal{E}^n_i \mu^{n-i}.$$

[4]These are the polynomials E^k_i of [FGP].

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In fact we will need more general elements of $\mathcal{D}$, namely, for each $k \in \{1, \dots, n-1\}$, we consider the elements $\mathcal{E}_i^k$ of $\mathcal{D}$, with $0 \le i \le k$, given by

$$\det\left[\begin{pmatrix} P_1 & Q_1 & 0 & \cdots & 0 \\ -1 & P_2 - P_1 & Q_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & P_{k-1} - P_{k-2} & Q_{k-1} \\ 0 & \cdots & 0 & -1 & P_k - P_{k-1} \end{pmatrix} + \mu I_k\right] = \sum_{i=0}^{k} \mathcal{E}_i^k \mu^{k-i}.$$

One can easily see that when we expand the determinant in the left-hand side of the last equation we will have no occurrence of P_jQ_j or Q_jP_j, $1 \le j \le k-1$. This means that the lack of commutativity of Q_j and P_j creates no ambiguity in the definition of $\mathcal{E}_1^n, \dots, \mathcal{E}_{n-1}^n$. We can also deduce that each of $\mathcal{E}_1^k, \dots, \mathcal{E}_k^k$ is a linear combination of monomials in the variables $\{P_1, \dots, P_k, Q_1, \dots, Q_{k-1}\}$, with no ocurrence of P_jQ_j or Q_jP_j (*i.e.*, the order of factors in each monomial is not important). As a consequence, the following recurrence formula [FGP, (3.5)] still holds:

$$\mathcal{E}_i^k = \mathcal{E}_i^{k-1} + X_k \mathcal{E}_{i-1}^{k-1} + Q_{k-1}\mathcal{E}_{i-2}^{k-2}, \tag{3.2}$$

where X_k stands for $P_k - P_{k-1}$ and, by convention, $\mathcal{E}_j^k = 0$, unless $0 \le j \le k$. It is worth mentioning the following commutation relations, which will be used later:

$$[X_k, \mathcal{E}_j^l] = 0, \quad [Q_k, \mathcal{E}_j^l] = 0, \tag{3.3}$$

whenever $l \le k-1$. We also note that $\mathcal{E}_0^k = 1$ and $\mathcal{E}_1^k = P_k$ (where P_n is by convention equal to 0). We will prove the following result.

Lemma 3.2 *The elements $\mathcal{E}_1^k, \dots, \mathcal{E}_{k-1}^k$ of $\mathcal{D}$ commute with each other.*

Proof Consider the coordinates $s_0, \dots, s_{k-1}$ on $\mathbb{R}^k$. Following [KJ], we consider the differential operators $D_j(\hbar\frac{\partial}{\partial s_0}, \dots, \hbar\frac{\partial}{\partial s_{k-1}}, e^{s_1-s_0}, \dots, e^{s_{k-1}-s_{k-2}})$ given by

$$\det\left[\begin{pmatrix} \hbar\frac{\partial}{\partial s_0} & e^{s_1-s_0} & 0 & \cdots & 0 \\ -1 & \hbar\frac{\partial}{\partial s_1} & e^{s_2-s_1} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & \hbar\frac{\partial}{\partial s_{k-2}} & e^{s_{k-1}-s_{k-2}} \\ 0 & \cdots & 0 & -1 & \hbar\frac{\partial}{\partial s_{k-1}} \end{pmatrix} + \mu I_k\right] = \sum_{i=0}^{k} D_i^k \mu^{k-i}.$$

By [KJ, Proposition 1], we have $[D_i^k, D_j^k] = 0$ for all $0 \le i, j \le k$. In order to prove our lemma, it is sufficient to note that if we make the change of coordinates

$$s_1 - s_0 = t_1, \cdots, s_{k-1} - s_{k-2} = t_{k-1}, -s_{k-1} = t_k,$$

we obtain

$$\hbar\frac{\partial}{\partial s_0} = -\hbar\frac{\partial}{\partial t_1} = -P_1, \quad \hbar\frac{\partial}{\partial s_1} = \hbar\frac{\partial}{\partial t_1} - \hbar\frac{\partial}{\partial t_2} = P_1 - P_2, \dots,$$
$$h\frac{\partial}{\partial s_{k-1}} = \hbar\frac{\partial}{\partial t_{k-1}} - \hbar\frac{\partial}{\partial t_k} = P_{k-1} - P_k,$$

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where we have used the presentation of $\mathcal{D}$ given by $P_i = \hbar\frac{\partial}{\partial t_i}, Q_i = e^{t_i}$, $1 \leq i \leq n-1$. ■

The following technical result will be needed later.

Lemma 3.3 *We have* $[\mathcal{E}_{j+1}^{k+1}, \mathcal{E}_i^k] = [\mathcal{E}_{i+1}^{k+1}, \mathcal{E}_j^k]$.

Proof We prove this by induction on $k \geq 0$. For $k = 0$, the equation is obvious (by the convention made above, we have $\mathcal{E}_0^j = 0$). It follows the induction step. We use the recurrence formula (3.2). This gives

$$[\mathcal{E}_{j+1}^{k+1}, \mathcal{E}_i^k] = [\mathcal{E}_{j+1}^k + X_{k+1}\mathcal{E}_j^k + Q_k\mathcal{E}_{j-1}^{k-1}, \mathcal{E}_i^k] = [Q_k\mathcal{E}_{j-1}^{k-1}, \mathcal{E}_i^k].$$

We continue by using again equation (3.2) and obtain

$$\begin{aligned}
&[Q_k\mathcal{E}_{j-1}^{k-1}, \mathcal{E}_i^{k-1} + X_k\mathcal{E}_{i-1}^{k-1} + Q_{k-1}\mathcal{E}_{i-2}^{k-2}] \\
&\qquad = [Q_k, X_k]\mathcal{E}_{i-1}^{k-1}\mathcal{E}_{j-1}^{k-1} + [Q_k\mathcal{E}_{j-1}^{k-1}, Q_{k-1}\mathcal{E}_{i-2}^{k-2}] \\
&\qquad = [Q_k, X_k]\mathcal{E}_{i-1}^{k-1}\mathcal{E}_{j-1}^{k-1} + Q_k[\mathcal{E}_{j-1}^{k-1}, Q_{k-1}\mathcal{E}_{i-2}^{k-2}] \\
&\qquad = [Q_k, X_k]\mathcal{E}_{i-1}^{k-1}\mathcal{E}_{j-1}^{k-1} + Q_k[\mathcal{E}_{j-1}^{k-1}, \mathcal{E}_i^k - \mathcal{E}_i^{k-1} - X_k\mathcal{E}_{i-1}^{k-1}] \\
&\qquad = [Q_k, X_k]\mathcal{E}_{i-1}^{k-1}\mathcal{E}_{j-1}^{k-1} + Q_k([\mathcal{E}_{j-1}^{k-1}, \mathcal{E}_i^k] - [\mathcal{E}_{j-1}^{k-1}, X_k\mathcal{E}_{i-1}^{k-1}]) \\
&\qquad = [Q_k, X_k]\mathcal{E}_{i-1}^{k-1}\mathcal{E}_{j-1}^{k-1} + Q_k[\mathcal{E}_{j-1}^{k-1}, \mathcal{E}_i^k].
\end{aligned}$$

Here we have used the commutation relations (3.3) several times. Similarly, we obtain

$$[\mathcal{E}_{i+1}^{k+1}, \mathcal{E}_j^k] = [Q_k, X_k]\mathcal{E}_{j-1}^{k-1}\mathcal{E}_{i-1}^{k-1} + Q_k[\mathcal{E}_{i-1}^{k-1}, \mathcal{E}_j^k].$$

We use the induction hypothesis to finish the proof. ■

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1 Let ω_k denote the matrix of multiplication by $[\lambda_k]_q$ with respect to the basis $\{[\hat{e}_{i_1\cdots i_{n-1}}]_q : 0 \leq i_j \leq j\}$ of $QH^*(Fl_n)$ (see equation (3.1)). More precisely, the entries of ω_i are polynomials in $q_1, \dots, q_{n-1}$, determined by

$$[\lambda_k]_q[\hat{e}_{i_1\cdots i_{n-1}}]_q = \sum_{l_1,\dots,l_{n-1}} \omega_k^{i_1\cdots i_{n-1}, l_1\cdots l_{n-1}}[\hat{e}_{l_1\dots l_{n-1}}]_q.$$

According to Corollary 1.2, it is sufficient to show that

$$\frac{\partial}{\partial t_i}\omega_j = \frac{\partial}{\partial t_j}\omega_i, \tag{3.4}$$

for $1 \leq i, j \leq n-1$, where as usual, we use the convention $q_i = e^{t_i}$. For $i_1, \dots, i_{n-1}$ such that $0 \leq i_j \leq j$, we consider

$$\mathcal{E}_{i_1\cdots i_{n-1}} := \mathcal{E}_{i_1}^1\mathcal{E}_{i_2}^2\cdots\mathcal{E}_{i_{n-1}}^{n-1}.$$

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In order to prove equation (3.4), it is sufficient to prove the following claim.

Claim In $\mathcal{D}/\mathcal{I}$ we have

$$(3.5)\qquad [P_k][\mathcal{E}_{i_1\cdots i_{n-1}}] = \sum_{l_1,\dots,l_{n-1}} \Omega_k^{i_1\cdots i_{n-1},l_1\cdots l_{n-1}}[\mathcal{E}_{l_1\cdots l_{n-1}}],$$

where each $\Omega_k^{i_1\cdots i_{n-1},l_1\cdots l_{n-1}}$ is obtained from $\omega_k^{i_1\cdots i_{n-1},l_1\cdots l_{n-1}}$ by the modification $Q_i \mapsto q_i$.

Indeed, if we make the usual identifications $P_k = \hbar\frac{\partial}{\partial t_k}$, $Q_k = e^{t_k}$, $1 \le k \le n-1$, then (3.5) implies that the connection

$$d + \sum_{k=1}^{n-1} \frac{1}{\hbar}\Omega_k dt_k$$

is flat (see [Gu, Proposition 1.1]) for all values of $\hbar$, which implies (3.4). The proof of the claim relies on a noncommutative version of the quantum straightening algorithm of Fomin, Gelfand, and Postnikov [FGP]. The key equation is the following.

$$(3.6)\qquad \mathcal{E}_i^k\mathcal{E}_{j+1}^{k+1} + \mathcal{E}_{i+1}^k\mathcal{E}_j^k + Q_k\mathcal{E}_{i-1}^{k-1}\mathcal{E}_j^k = \mathcal{E}_j^k\mathcal{E}_{i+1}^{k+1} + \mathcal{E}_{j+1}^k\mathcal{E}_i^k + Q_k\mathcal{E}_{j-1}^{k-1}\mathcal{E}_i^k.$$

We note that this is the same as [FGP, (3.6)]. The difference is that here we work in the algebra $\mathcal{D}$, which is not commutative, so it is not *a priori* clear that (3.6) still holds. In order to prove it, we use equation (3.2) twice and obtain:

$$(\mathcal{E}_{j+1}^{k+1} - \mathcal{E}_{j+1}^k)\mathcal{E}_i^k = (X_{k+1}\mathcal{E}_j^k + Q_k\mathcal{E}_{j-1}^{k-1})\mathcal{E}_i^k,$$

and

$$(\mathcal{E}_{i+1}^{k+1} - \mathcal{E}_{i+1}^k)\mathcal{E}_j^k = (X_{k+1}\mathcal{E}_i^k + Q_k\mathcal{E}_{i-1}^{k-1})\mathcal{E}_j^k.$$

If we subtract the second equation from the first one, we obtain:

$$\mathcal{E}_{i+1}^{k+1}\mathcal{E}_j^k - \mathcal{E}_{j+1}^{k+1}\mathcal{E}_i^k = \mathcal{E}_{i+1}^k\mathcal{E}_j^k - \mathcal{E}_{j+1}^k\mathcal{E}_i^k + Q_k(\mathcal{E}_{i-1}^{k-1}\mathcal{E}_j^k - \mathcal{E}_{j-1}^{k-1}\mathcal{E}_i^k).$$

Now the left-hand side can be written as

$$\mathcal{E}_j^k\mathcal{E}_{i+1}^{k+1} - \mathcal{E}_i^k\mathcal{E}_{j+1}^{k+1} + [\mathcal{E}_{i+1}^{k+1}, \mathcal{E}_j^k] - [\mathcal{E}_{j+1}^{k+1}, \mathcal{E}_i^k] = \mathcal{E}_j^k\mathcal{E}_{i+1}^{k+1} - \mathcal{E}_i^k\mathcal{E}_{j+1}^{k+1},$$

where we have used Lemma 3.3. Equation (3.6) has been proved. Now we can use it exactly as in the commutative situation, described in [FGP], in order to obtain the expansion of the product of $P_k = \mathcal{E}_1^k$ and $\mathcal{E}_{i_1\dots i_{n-1}} = \mathcal{E}_{i_1}^1 \dots \mathcal{E}_{i_{n-1}}^{n-1}$. More precisely, we begin with

$$P_k\mathcal{E}_{i_1\cdots i_{n-1}} = \mathcal{E}_{i_1}^1 \cdots \mathcal{E}_{i_{k-1}}^{k-1}P_k\mathcal{E}_{i_k}^k\mathcal{E}_{i_{k+1}}^{k+1} \cdots \mathcal{E}_{i_{n-1}}^{n-1} = \mathcal{E}_{i_1}^1 \cdots \mathcal{E}_{i_{k-1}}^{k-1}\mathcal{E}_1^k\mathcal{E}_{i_k}^k\mathcal{E}_{i_{k+1}}^{k+1} \cdots \mathcal{E}_{i_{n-1}}^{n-1},$$

and then we use (3.6) repeatedly. The resulting coefficients in the final expansion will be the same as in the commutative situation. This finishes the proof of the claim, and also of Theorem 3.1. ■

A Appendix

We will give simple proofs of properties (i) and (ii) in Theorem 1.1 for the actual quantum product $\circ$. They are both straightforward consequences of the following "divisor property" (see [FP, (40)] for a more general version of this formula):

$$\langle \sigma_{s_j} | \sigma_w | \sigma_v \rangle_d = d_j \langle \sigma_w | \sigma_v \rangle_d, \tag{A.1}$$

for any $1 \le j \le l$, $d = (d_1, \dots, d_l) \in H_2(G/B, \mathbb{Z})$, and $v, w \in W$. Here $\langle \sigma_w | \sigma_v \rangle_d$ is the *two-point Gromov–Witten invariant*, which represents the number of holomorphic maps $\varphi \colon \mathbb{P}^1 \to G/B$ with $\varphi_*([\mathbb{P}^1]) = d$ in $H_2(G/B)$ and such that $\varphi(\mathbb{P}^1)$ intersects general translates of the Schubert varieties dual to σ_w and σ_v, modulo $PSL(2, \mathbb{C})$ (the latter group acts on φ by reparametrizing it).

First we prove condition (i), *i.e.,* the flatness of the Dubrovin connection. This is

$$\nabla^\hbar = d + \frac{1}{\hbar}\omega,$$

where ω is the 1-form on $H^2(G/B)$ with values in $\mathrm{End}(H^*(G/B))$ given by

$$\omega_t(X, Y) = X \circ Y,$$

for $t = (t_1, \dots, t_l) \in H^2(G/B)$, $X \in H^2(G/B)$ and $Y \in H^*(G/B)$. Here the convention $q_i = e^{t_i}$, $1 \le i \le l$ is in force. Note that the ω can be expressed as

$$\omega = \sum_{i=1}^{l} \omega_i dt_i,$$

where ω_i denotes the matrix of the operator $\sigma_{s_i} \circ$ on $H^*(G/B)$ with respect to the basis consisting of the Schubert classes.

Lemma A.1 *The Dubrovin connection $\nabla^\hbar$ is flat for any $\hbar \in \mathbb{R} \setminus \{0\}$,* i.e., *we have*

$$d\omega = \omega \wedge \omega = 0.$$

Proof The fact that $d\omega = 0$ amounts to $\frac{\partial}{\partial t_i}\omega_j = \frac{\partial}{\partial t_j}\omega_i$, which is equivalent to $d_i(\sigma_{s_j} \circ \sigma_w)_d = d_j(\sigma_{s_i} \circ \sigma_w)_d$ for any $w \in W$ and any $d = (d_1, \dots, d_l)$, hence, by (1.2), to $d_i \langle \sigma_{s_j} | \sigma_w | \sigma_v \rangle_d = d_j \langle \sigma_{s_i} | \sigma_w | \sigma_v \rangle_d$. The latter equation is an obvious consequence of the divisor rule (A.1). The equality $\omega \wedge \omega = 0$ is equivalent to $\omega_i \omega_j = \omega_j \omega_i$, $1 \le i, j \le l$; this follows immediately from the fact that the product $\circ$ is commutative and associative. ■

Next we turn to property (ii).

Lemma A.2 *We have*

$$\sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle \sigma_{s_i} \circ \sigma_{s_j} = \sum_{i=1}^{l} \langle \alpha_i^\vee, \alpha_i^\vee \rangle q_i. \tag{A.2}$$

Proof The crucial point of the proof is the following equation.

$$\sigma_{s_i} \circ \sigma_{s_j} = \sigma_{s_i}\sigma_{s_j} + \delta_{ij}q_j. \tag{A.3}$$

In turn, by (1.2) this amounts to the fact that if $e_k := (0, \dots, 0, 1, 0, \dots, 0)$ (where 1 is in the k-th position), then the homology class $(\sigma_{s_i} \circ \sigma_{s_j})_{e_k} \in H^0(G/B)$ satisfies

$$((\sigma_{s_i} \circ \sigma_{s_j})_{e_k}, PD[\mathrm{pt}]) = \delta_{ijk},$$

where, by definition, δ_{ijk} is 1 if $i = j = k$ and 0 otherwise, and $PD[\mathrm{pt}]$ denotes the (top-dimensional) cohomology class which is Poincaré dual to a point. By (1.2) and the divisor rule (A.1), we only need to prove that

$$\langle \sigma_{s_i} | PD[\mathrm{pt}] \rangle_{e_i} = 1.$$

But this follows immediately from the fact that the Poincaré dual of the homology class e_j is $\sigma_{w_0 s_i}$, and the intersection pairing of the latter with σ_{s_i} equals 1.

Now (A.3) implies (A.2), because

$$\sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle \sigma_{s_i}\sigma_{s_j} = 0,$$

which in turn follows from the fact that the polynomial $\sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle \lambda_i \lambda_j \in S(\mathfrak{t}^*)$ is W-invariant (being just the squared norm on $\mathfrak{t}$). ■

Remark Another proof of the last lemma can be found in [Ma1, §3].

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