BULL. AUSTRAL. MATH. SOC. 54A05, 54C10
VOL. 20 (1979), 447-466. (54D20)

Perfect maps on convergence spaces

Robert A. Herrmann

The concept of the perfect map on a convergence space (X, q), where q is a convergence function, is introduced and investigated. Such maps are not assumed to be either continuous or surjective. Some nontrivial examples of well known mappings between topological spaces, nontopological pretopological spaces and nonpseudotopological convergence spaces are shown to be perfect in this new sense. Among the numerous results obtained is a covering property for perfectness and the result that such maps are closed, compact, and for surjections almost-compact. Sufficient conditions are given for a compact (respectively almost-compact) map to be perfect. In the final section, a major result shows that if $f : (X, q) \to (Y, p)$ is perfect and $g : (X, q) \to (Z, s)$ is weakly-continuous into Hausdorff Z, then $(f, g) : (X, q) \to (Y \times Z, p \times s)$ is perfect. This result is given numerous applications.

1. Introduction

Throughout this paper, we adhere to the following notational conventions. For a set X, $F(X)$ (respectively $U(X)$) denotes the set of all filters (respectively ultrafilters) on X. If $\mathcal{F} \in F(X)$, then $U(\mathcal{F})$ is the set of all ultrafilters on X finer than $\mathcal{F}$. If $A \subset P(X)$, the power set of X, and A has the finite intersection property, then $[A]$ denotes the filter on X generated by A. If $A \in P(X)$, $A \neq \emptyset$, then $[A]$ is the principal filter generated by A. In [6], Kent defines a *convergence function on* X to be a map

Received 23 April 1979. This research was partially supported by a grant from the United States Naval Academy Research Council.

$q : F(X) \to P(X)$ such that

(A) for each $x \in X$, $x \in q([\{x\}])$;

(B) if $F, G \in F(X)$ and $F \subset G$, then $q(F) \subset q(G)$.

A pair (X, q), where q is a convergence function on X, is a *convergence space*. In [7], Kent calls a convergence function which satisfies

(C) if $x \in q(F)$, $F \in F(X)$, then $x \in q(F \wedge \{x\})$,

a *convergence structure*. In recent years, the convergence space definition has required q to be a convergence structure [8], [9], [10], [11]. Throughout this paper, we shall have no need for the additional axiom (C). Thus all spaces we consider are convergence spaces as we have defined them and all undefined concepts are the usual ones associated with convergence space theory and which do not require axiom (C) in their formulation.

The symbols (X, q), (Y, p), (Z, s) denote convergence spaces and (X, τ), (Y, T) topological spaces with topologies τ and T, respectively. For (X, q), a filter $F \in F(X)$ is said to *q-converge to* $x \in X$ if $x \in q(F)$, and we sometimes denote this by $F \xrightarrow[q]{} x$ or simply $F \to x$. For $F \in F(X)$, and $A \subset X$, $F \cap A$ means that $F \cap A \neq 0$ for each $F \in F$. Let $f : X \to Y$. Then for $F \in F(X)$ (respectively $F \in F(Y)$ and $F \cap f[X]$), $f(F) = [\{f[F] \mid F \in F\}]$ (respectively $f^{-1}(F) = \{f^{-1}[F] \mid F \in F\}$).

In [10], Kent and Richardson, assuming axiom (C), say that a continuous surjection $f : (X, q) \to (Y, p)$ is *proper* if

(1) whenever $U \in U(Y)$ and $U \to y \in Y$, then for each $V \in U(X)$ such that $f(V) = U$, there exists some $x \in f^{-1}(y)$ such that $V \to x$.

If f is also a quotient map, then it is called *perfect* in [13].

A natural question which arises is whether or not it is desirable to retain the "continuity" or "surjective" portions of the definition for proper maps, or whether there are interesting convergence spaces and maps which satisfy condition (1), but which are not continuous or not surjective. In [4], a not necessarily continuous nor surjective map $f : (X, \tau) \to (Y, T)$

is called perfect if it is closed and has compact point inverses. It follows from Whyburn's results in [16] that for (X, τ) and (Y, T) considered as convergence spaces, perfect as defined by Iliadis and Fomin [4] is equivalent to condition (1).

We make the following two important observations for nonsurjective maps and condition (1). If $f : (X, q) \to (Y, p)$ is nonsurjective, $\mathcal{U} \in U(Y)$ and $f[X] \notin \mathcal{U}$, then there does not exist a $\mathcal{V} \in U(X)$ such that $f(\mathcal{V}) = \mathcal{U}$. Hence for all such ultrafilters condition (1) holds vacuously. Therefore, in order to establish that condition (1) holds, we need only consider those $\mathcal{U} \in U(Y)$ such that $f[X] \in \mathcal{U}$. Also, if f satisfies condition (1), $\mathcal{U} \in U(Y)$ such that $f[X] \in \mathcal{U}$ and $\mathcal{U} \to y \in Y$, then $y \in f[X]$. For if we assume that $y \notin f[X]$ then we have the contradiction that $f^{-1}(y) = \emptyset$ and there exists some $\mathcal{V} \in U(X)$ such that $f(\mathcal{V}) = \mathcal{U}$.

EXAMPLE 1.1. This is an example of a map between convergence spaces which satisfies condition (1) but which is not continuous. Let (X, τ) be a topological Hausdorff and nonregular space. Let $\omega(X)$ be the absolute of X. Then Iliadis and Fomin [4] show that there exists a noncontinuous map $f : \omega(X) \to (X, \tau)$ which satisfies condition (1).

EXAMPLE 1.2. This is an example of a simple convergence space generated by a topological space (X, τ) for which the identity map satisfies condition (1) for a slightly weaker topological space but which does not satisfy (1) for (X, τ). Let (X, τ) be a nonsemiregular space and τ_s the semiregular topology generated on X by the set of all regular-open subsets of X. The identity $I : (X, \tau_s) \to (X, \tau_s)$ trivially satisfies (1) as a convergence space map, but since (X, τ) is not topologically homeomorphic to (X, τ_s), then $I : (X, \tau) \to (X, \tau_s)$ does not satisfy (1).

EXAMPLE 1.3. This is an example of a nontopological pretopology generated by (X, τ) which is of considerable interest to topologists [1], [2], [4], and for which the identity satisfies (1) but does not satisfy (1) for (X, τ). Let (X, τ) be a topological Hausdorff non-Urysohn, non-semiregular space, where a space X is Urysohn if distinct $x, y \in X$ have disjoint closed neighborhoods. Define the convergence function $\theta : F(X) \to P(X)$ as follows: for each $\mathcal{F} \in F(X)$, $x \in \theta(\mathcal{F})$ if for each

https://doi.org/10.1017/S0004972700011151 Published online by Cambridge University Press

$G \in \tau$ such that $x \in G$ there exists some $F \in \mathcal{F}$ such that $F \subset \mathrm{cl}_X G$. Then (X, θ) is a pretopological space. We show elsewhere [2], that the θ-closure is idempotent if and only if X is almost-regular. Thus Theorem 4 [1] implies that θ is not topological. The identity $I : (X, \tau) \to (X, \tau_s)$ is continuous and does not satisfy (1). However, $I : (X, \theta) \to (X, \theta_s)$, where θ_s is the θ-convergence function defined by τ_s, satisfies (1) and is often said to be θ-*perfect* [1], [2].

REMARK 1.1. In Section 2, we show that the identity map onto any nonpseudotopological convergence space is perfect from its pseudotopological modification but not continuous. Moreover, in Section 3, we show that there exist many well known noncontinuous and nonsurjective maps between convergence spaces, which satisfy (1).

2. Perfect maps

The previous examples and remark indicate that a foundational investigation into the nature of maps which satisfy condition (1) should be of interest to convergence space theorists. Throughout the remainder of this paper, no map will be assumed continuous nor surjective. Moreover, the concept of "perfect" as we now define it does not correspond to perfect as used in [13] even though it does correspond to the definition given in [4] for topological spaces as well as that found in [1] for the θ-convergence spaces.

DEFINITION 2.1. A map $f : (X, q) \to (Y, p)$ is perfect if whenever $\mathcal{U} \in U(Y)$ and $\mathcal{U} \to y \in Y$, then for each $\mathcal{V} \in U(X)$ such that $f(\mathcal{V}) = \mathcal{U}$ there exists some $x \in f^{-1}(y)$ such that $\mathcal{V} \to x$.

Prior to showing that perfect maps may be characterized by a covering property, we introduce the following notation and terminology, where X is a convergence space. For $A \subset X$, let

$$\mathcal{U}(A) = \{x \mid [x \in U(X)] \wedge \exists y[y \in A] \wedge [x \to y]\} .$$

We call a set $A \subset X$, *B-compact* if $B \subset X$ and for each $\mathcal{U} \in U(X)$ such that $A \in \mathcal{U}$ there exists some $b \in B$ such that $\mathcal{U} \to b$. Observe that the empty set is B-compact for each $B \subset X$, and if $B = \emptyset$, then A is B-compact if and only if $A = \emptyset$. For $A \subset X$, a filter base $\mathcal{F}$ on X is

https://doi.org/10.1017/S0004972700011151 Published online by Cambridge University Press

A-compact if for each $\mathcal{U} \in \mathcal{U}([\mathcal{F}])$ there exists some $x \in A$ such that $\mathcal{U} \to x$. Notice that $A \subset X$ is A-compact if and only if it is compact in the usual convergence space sense and A is *almost-compact* [15] or *absolutely bounded* [5] if and only if it is X-compact. For $A \subset X$, let $S(\mathcal{U}(A))$ be the set of all choice sets generated by $\mathcal{U}(A)$. Also notice that if $\emptyset \neq A \subset X$, then $\mathcal{U}(A) \neq \emptyset$. The next result relates the concept of A-compact filters to finite covers.

THEOREM 2.1. *For a space* (X, q) *and nonempty* $A \subset X$ *a filter base* $\mathcal{F}$ *on* X *is* A*-compact if and only if for each* $S \in S(\mathcal{U}(A))$ *there exists a nonempty finite subset* $S_f \subset S$ *and some* $F \in \mathcal{F}$ *such that* $F \subset \cup S_f$.

Proof. For the necessity, assume that $\mathcal{F}$ is A-compact. For some fixed $S \in S(\mathcal{U}(A))$ and some $G \in S$, if $F \in \mathcal{F}$ and $F - G = \emptyset$, then the result follows. Hence assume that for each $F \in \mathcal{F}$, $F - G \neq \emptyset$. The set $G_G = \{F - G \mid F \in [\mathcal{F}]\}$ is a filter base on X and $[\mathcal{F}] \subset [G_G]$. Observe that $G \notin [G_G]$. Assume that $\vee\{[G_G] \mid G \in S\} = \mathcal{G} \neq P(X)$. Then there exists $\mathcal{U} \in \mathcal{U}([\mathcal{F}])$ such that $\mathcal{G} \subset \mathcal{U}$. Since $\mathcal{F}$ is A-compact, then $\mathcal{U} \in \mathcal{U}(A)$. Thus there exists $G' \in S$ such that $G' \in \mathcal{U}$. However, $[G_{G'}] \subset \mathcal{G}$ implies that $\mathcal{U} \in \mathcal{U}([G_{G'}])$. Since $X - G' \in [G_{G'}]$, we have the contradiction that $G' \notin \mathcal{U}$. Thus $\vee\{[G_G] \mid G \in S\} = P(X)$ implies that there exists a natural number n such that $\cap\{F_i - G_i \mid i \in n\} = \emptyset$, $G_i \in \mathcal{F}$ and $G_i \in S$ for each $i \in n$. Let $F \in \mathcal{F}$ such that $F \subset \cap\{F_i \mid i \in n\}$. Then $\cap\{F - G_i \mid i \in n\} = \emptyset$ implies that $F \subset \cup\{G_i \mid i \in n\}$.

For the sufficiency, let $\mathcal{U} \in \mathcal{U}([\mathcal{F}])$ and assume that $\mathcal{U} \not\to x$ for any $x \in A$. Then $\mathcal{U} \notin \mathcal{U}(A)$. Thus for each $\mathcal{V} \in \mathcal{U}(A)$ there exists some $V \in \mathcal{V}$ and $U \in \mathcal{U}$ such that $V \cap U = \emptyset$. Now consider the set

$$K = \{(V, U) \mid \exists \mathcal{V}[[\mathcal{V} \in \mathcal{U}(A)] \wedge [V \in \mathcal{V}] \wedge [U \in \mathcal{U}] \wedge [V \cap U] = \emptyset]\}.$$

Then the first projection $P_1(K) \in S(\mathcal{U}(A))$. Hence there exists an $F \in \mathcal{F}$ and $\{V_1, \ldots, V_n\} \subset P_1(K)$ such that $F \subset V_1 \cup \ldots \cup V_n$. Now $U \cap F \neq \emptyset$ for each $U \in \mathcal{U}$. Consider $\{(V_1, U_1), \ldots, (V_n, U_n)\} \subset K$. Then $(U_1 \cap \ldots \cap U_n) \cap F \neq \emptyset$ implies that

https://doi.org/10.1017/S0004972700011151 Published online by Cambridge University Press

$$\emptyset \neq (V_1 \cup \ldots \cup V_n) \cap (U_1 \cap \ldots \cap U_n) \subset \cup\{V_i \cap U_i \mid i = 1, \ldots, n\} = \emptyset .$$

The sufficiency follows from this contradiction and the proof is complete.

REMARK 2.1. We have proved Theorem 2.1 for filter bases rather than filters since it is applicable to these useful objects. For the remainder of this paper, we concentrate upon filters even though most of the following results may be shown to hold, at least in part, for filter bases.

The next result characterizes perfectness in terms of A-compactness.

THEOREM 2.2. *Let* $f : (X, q) \to (Y, p)$. *Then* f *is perfect if and only if whenever* $U \in U(Y)$, $f[X] \in U$ *and* $U \to y \in Y$, *then* $f^{-1}(U)$ *is* $f^{-1}(y)$*-compact.*

Proof. Let $f[X] \in U \in U(Y)$. If $V \in U(f^{-1}(U))$, then $f^{-1}(U) \subset V$ implies that $U \subset U(f^{-1}(U)) \subset f(V)$. Hence $U = f(V)$. On the other hand, if $V \in U(X)$ such that $f(V) = U$, then $f^{-1}(U) \subset V$ implies that $V \in U(f^{-1}(U))$. Thus $V \in U(X)$ such that $f(V) = U$ if and only if $V \in U(f^{-1}(U))$. The result now follows from the definition of $f^{-1}(y)$-compactness and the observation that we need only consider those $U \in U(X)$ such that $f[X] \in U$.

REMARK 2.2. If we include in $F(X)$ the trivial "filter" $P(X)$, then the requirement that $f[X] \in U$ in Theorem 2.2 can be removed. For if $f[X] \notin U$, then $f^{-1}(U) = P(X)$ and $U(f^{-1}(U)) = \emptyset$.

We now extend Theorem 2.2 to include all filters on Y which intersect $f[X]$.

THEOREM 2.3. *Let* $f : (X, q) \to (Y, p)$. *Then* f *is perfect if and only if whenever* $F \in F(X)$, $F \cap f[X]$ *and* $F \to y \in Y$, *then* $f^{-1}(F)$ *is* $f^{-1}(y)$*-compact.*

Proof. The sufficiency is apparent from Theorem 2.2. For the necessity, assume that f is perfect, $F \in F(X)$, $F \cap f[X]$ and $F \to y \in Y$. Notice that if $U \in U(F)$, then $f[X] \in U$. It is easy to verify that $U(f^{-1}(F)) = \cup\{U(f^{-1}(U)) \mid U \in U(F)\}$. Hence let

https://doi.org/10.1017/S0004972700011151 Published online by Cambridge University Press

$V \in U(f^{-1}(F))$. Then there exists $U \in U(F)$ such that $f(V) = U$. Since $U \to y$, then perfectness implies that for some $x \in f^{-1}(y)$, $V \to x$. Consequently, $f^{-1}(F)$ is $f^{-1}(y)$-compact and the proof is complete.

Recall that for each $A \subset X$, the *q-closure* (or simply *closure*) of A is

$$\mathrm{cl}_q(A) = \{x \mid [x \in X] \wedge \exists y[[y \in U(X)] \wedge [A \in y] \wedge [y \to x]]\} .$$

The q-closure is often denoted by Γ_q [1], [2], [9], [10]. A set $A \subset X$ is *q-closed* (or simply *closed*) if $A = \mathrm{cl}_q(A)$. Kent [10] defines $\mathrm{cl}_q^0(A) = A$, $\mathrm{cl}_q^1(A) = \mathrm{cl}_q(A)$; if α is an ordinal number and $\alpha - 1$ exists, then $\mathrm{cl}_q^{\alpha}(A) = \mathrm{cl}_q\left(\mathrm{cl}_q^{\alpha-1}(A)\right)$ and if α is a limit ordinal, then $\mathrm{cl}_q^{\alpha}(A) = \cup\left\{\mathrm{cl}_q^{\beta}(A) \mid \beta < \alpha\right\}$. The smallest ordinal γ such that $\mathrm{cl}_q^{\gamma}(A) = \mathrm{cl}_q^{\gamma+1}(A)$ for each $A \subset X$ is denoted by γ_q and is *the length of the decomposition series for* (X, q) [10]. For a filter base F on X , the *q-adherence* (or simply *adherence*) of F is

$$a_q(F) = \{x \mid [x \in X] \wedge \exists y[[y \in U(X)] \wedge [[F] \subset y] \wedge [y \to x]]\} .$$

Note that for $A \subset X$, $\mathrm{cl}_q(A) = a_q(\{A\})$. Given (X, q) we define the *pseudotopological modification* (X, q^*) as follows: $x \in q^*(F)$ if and only if $x \in q(U)$ for each $U \in U(F)$ [6]. Observe that $U_q(X) = U_{q^*}(X)$. We now characterize A-compactness in terms of q-adherence and give a major characterization for perfectness.

THEOREM 2.4. *For* (X, q) *, let* $F \in F(X)$ *. Then for each* $G \in F(X)$ *such that* $F \subset G$ *, it follows that* $a_q(G) \cap A \neq \emptyset$ *if and only if* F *is* *A-compact.*

Proof. For the sufficiency, assume that F is A-compact and $F \subset G \in F(X)$. Let $U \in U(G)$. Then $U \in U(F)$ implies that $U \to x$ for some $x \in A$. Hence $x \in a_q(G) \cap A$.

For the necessity, assume that F is not A-compact. Then there exists some $U \in U(F)$ such that $U \not\to x$ for any $x \in A$. Since $F \subset G$

and $a_q(U) \cap A \neq \emptyset$, then for some $x' \in A$, $U \to x'$. This contradiction completes the proof.

Notice that if $F, G \in F(X)$, $F \subset G$, $A \subset X$, and F is A-compact, then G is A-compact.

THEOREM 2.5. *A map* $f : (X, q) \to (Y, p)$ *is perfect if and only if for each* $F \in F(X)$ *, it follows that* $a_p(f(F)) \subset f[a_q(F)]$.

Proof. For the necessity, let $y \in a_p(f(F))$, $F \in F(X)$. Then there exists $U \in U(Y)$ such that $f(F) \subset U$ and $U \to y \in Y$. Clearly, we have that $f[X] \in U$, $f^{-1}(F) \vee U = U$ and $f^{-1}(U) \vee F \neq P(X)$. Now

$$f^{-1}(U) \vee f^{-1}(f(F)) \subset f^{-1}(U) \vee F ,$$

$$f^{-1}(U \vee f(F)) = f^{-1}(U) \vee f^{-1}(f(F)) ,$$

and f being perfect imply that the filter $f^{-1}(U) \vee F$ is $f^{-1}(y)$-compact. Since $U(f^{-1}(U) \vee F) \neq \emptyset$ and $U(f^{-1}(U) \vee F) \subset U(F)$, it follows that there exists $V \in U(X)$ such that $F \subset V$ and $V \to x$ for some $x \in f^{-1}(y)$. Hence $x \in a_q(F)$ and $f(x) = y$ imply that $y \in f[a_q(F)]$.

For the sufficiency, let $F \in F(X)$, $F \cap f[X]$, $F \to y$, $G \in F(X)$, and $f^{-1}(F) \subset G$. Since $F \subset f(f^{-1}(F)) \subset f(G)$, then $f(G) \to y$. Thus $y \in a_p(f(G))$ implies that $y \in f[a_q(G)]$. Hence $a_q(G) \cap f^{-1}(y) \neq \emptyset$. Since G is an arbitrary filter finer than $f^{-1}(F)$, Theorem 2.4 implies that $f^{-1}(F)$ is $f^{-1}(y)$-compact. The sufficiency follows from Theorem 2.3.

COROLLARY 2.5.1. *If* $f : (X, q) \to (Y, p)$ *is perfect,* α *is an ordinal number, and* $B \subset X$ *, then* $\mathrm{cl}_q(f[B]) \subset f\left[\mathrm{cl}_q^{\alpha}(B)\right]$.

Proof. This follows from a slight modification of the proof of Proposition 3.2 in [10].

COROLLARY 2.5.2. *If* $f : (X, q) \to (Y, p)$ *is perfect and* $B \subset X$ *is closed, then* $f[B]$ *is closed.*

https://doi.org/10.1017/S0004972700011151 Published online by Cambridge University Press

A map $f : (X, q) \to (Y, p)$ is *weakly-continuous* (respectively *continuous*) if for each $\mathcal{U} \in U(X)$ (respectively $F \in F(X)$) such that $\mathcal{U} \to x$ (respectively $F \to x$), $f(\mathcal{U})$ (respectively $f(F)$) p-converges to $f(x)$. Clearly, continuity implies weak-continuity. However, if the space (X, q) is a non-pseudotopological convergence space (that is $q \neq q^*$), then there exists some $F \in F(X)$ such that $F \notin U(X)$ and F q^*-converges to some $x \in X$ but does not q-converge to x. Then, in this case, the identity $I : (X, q^*) \to (X, q)$ is weakly-continuous, but not continuous. Observe that I is also perfect. We now characterize weak-continuity.

THEOREM 2.6. *If $f : (X, q) \to (Y, p)$ is weakly-continuous, then for each $F \in F(X)$, $f[a_q(F)] \subset a_p(f(F))$.*

Proof. Obvious.

THEOREM 2.7. *If $f : (X, q) \to (Y, p)$ and for each $\mathcal{U} \in U(Y)$ such that $f[X] \in \mathcal{U}$, $f[a_p(\mathcal{U})] \subset a_p(f(\mathcal{U}))$, then f is weakly-continuous.*

Proof. This follows easily by contradiction.

COROLLARY 2.7.1. *A map $f : (X, q) \to (Y, p)$ is weakly-continuous if and only if for each $F \in F(X)$, $f[a_q(F)] \subset a_p(f(F))$.*

COROLLARY 2.7.2. *If $f : (X, q) \to (Y, p)$ is weakly-continuous, then for each ordinal number α and $B \subset X$, $f\left[\mathrm{cl}_q^\alpha(B)\right] \subset \mathrm{cl}_p(f[B])$.*

Proof. This follows by a straightforward transfinite induction proof.

COROLLARY 2.7.3. *If $f : (X, q) \to (Y, p)$ is weakly-continuous and perfect, then for each ordinal α and $B \subset X$, $f\left[\mathrm{cl}_q^\alpha(B)\right] = \mathrm{cl}_p^\alpha(f[B])$.*

For each $F \in F(X)$, let $\mathrm{cl}_q^0(F) = F$. For an ordinal number α, if $\alpha - 1$ exists, let $\mathrm{cl}_q^\alpha(F) = \left[\left\{\mathrm{cl}_q^\alpha(F) \mid F \in F\right\}\right]$. If α is a limit ordinal, let $\mathrm{cl}_q^\alpha(F) = \wedge\left\{\mathrm{cl}_q^\beta(F) \mid \beta < \alpha\right\}$. For a space (X, q), define the *pretopological modification*, $\hat{q}$, as follows: $x \in \hat{q}(F)$ if and only if $N_q(x) = \wedge\{\mathcal{U} \mid \mathcal{U} \in U(\{x\})\} \subset F$. The *topological modification*, $\lambda(q)$, is the topology generated by the set of complements of the q-closed subsets

https://doi.org/10.1017/S0004972700011151 Published online by Cambridge University Press

of X .

THEOREM 2.8. *Let* $f : (X, q) \to (Y, p)$.

(i) If f *is weakly-continuous, then for each* $F \in F(X)$ *and ordinal* α , $\mathrm{cl}_p^{\alpha}(f(F)) \subset f\left[\mathrm{cl}_q^{\alpha}(F)\right]$.

(ii) If f *is perfect, then for each* $F \in F(X)$ *and ordinal* α , $f\left[\mathrm{cl}_q^{\alpha}(F)\right] \subset \mathrm{cl}_p^{\alpha}(f(F))$.

(iii) If f *is a weakly-continuous and perfect surjection, then* $\gamma_p \le \gamma_q$.

(iv) If f *is a weakly-continuous and perfect surjection and* $\hat{q}$ *is a topology, then* $\hat{p}$ *is a topology.*

Proof. *(i)* and *(ii)* follow from Corollaries 2.5.1 and 2.7.2. *(iii)* and *(iv)* follow as in the proof of Proposition 3.3 [*10*] and by Theorem 2.4 in [*6*].

REMARK 2.3. If $f : (X, q) \to (Y, p)$ and $F \in F(X)$, then it follows that $U(f(F)) = \{f(U) \mid U \in U(F)\}$. This yields that if f is weakly-continuous, then $f : (X, q^*) \to (Y, s)$ is continuous for each $s \in \{p^*, \hat{p}, \lambda(p)\}$. Moreover, following Kent [*8*] it is not difficult to show that $f : (X, \hat{q}) \to (Y, s)$ is continuous for each $s \in \{\hat{p}\ \ \lambda(p)\}$. Indeed, let $A \subset Y$ be p-closed. Then assuming that f is weakly-continuous, we have that

$$f^{-1}[A] \subset \mathrm{cl}_q(f^{-1}[A]) \subset f^{-1}\left[\mathrm{cl}_p\left(f[f^{-1}[B]]\right)\right] \subset f^{-1}[\mathrm{cl}_p(A)] = f^{-1}[A]$$

implies that $f : (X, \lambda(q)) \to (Y, \lambda(p))$ is continuous.

COROLLARY 2.8.1. *If* $\hat{q} = \lambda(q)$ *and* $\hat{p} \ne \lambda(p)$, *then there does not exist a weakly-continuous and perfect surjection from* (X, q) *onto* (Y, p) .

Recall that $B \subset X$ is *compact* if B is compact in the convergence function q' *induced on* B . The function q' is defined as follows: $F \in F(B)$ is q'-convergent to $x \in B$ if $[F]$ is q-convergent to x . Thus $B \subset X$ is compact if and only if for each $F \in F(B)$, $a_{q'}(F) \ne \emptyset$ if and only if for each $F \in F(B)$, $a_q([F]) \cap B \ne \emptyset$ if and only if for each

https://doi.org/10.1017/S0004972700011151 Published online by Cambridge University Press

$U \in U(B)$, U is q'-convergent to some $x \in B$ if and only if for each $U \in U(X)$ such that $B \in U$, U is q'-convergent to some $x \in B$ if and only if for each $U \in U(X)$ such that $B \in U$, U is q-convergent to some $x \in B$. A set $B \subset X$ is *almost-compact* [5], [15], if and only if for each $U \in U(X)$ such that $B \in U$, U is q-convergent to some $x \in X$ if and only if B is X-compact. The set of all almost-compact subsets of X forms a *bornology* [3]. Obviously, if $A, B \subset X$ are almost-compact, then $A \cup B$ is almost-compact. If $A \subset B \subset X$ and B is almost-compact, then A is almost-compact. A map $f : (X, q) \to (Y, p)$ is *compact* (respectively *almost-compact*) if for each compact (respectively almost-compact) $B \subset Y$, $f^{-1}[B]$ is compact (respectively almost-compact). It is a strightforward exercise to show that the weakly-continuous image of a compact (respectively almost-compact) set is compact (respectively almost-compact).

THEOREM 2.9. *If $B \subset Y$ is compact (respectively almost-compact) and $f : (X, q) \to (Y, p)$ is perfect (respectively and surjective), then $f^{-1}[B]$ is compact (respectively almost-compact).*

Proof. For the compact case, assume that $f^{-1}[B] \neq \emptyset$. Let $F \in F(f^{-1}[B])$ and G be the filter on B generated by $\{f[F] \mid F \in F\}$. Then $a_p(G) \cap B \neq \emptyset$ implies that there exists $U \in U(X)$ finer than G , such that $U \to y \in B$. Hence $y \in a_p(f(F))$. Theorem 2.5 yields that $y \in f[a_q(F)]$. Therefore, there is an $x \in a_q(F)$ such that $y = f(x)$ and $f(x) \in B$. Consequently, $x \in a_q(F) \cap f^{-1}[B]$ implies that $f^{-1}[B]$ is compact.

For the almost-compact case, let $U \in U(X)$ such that $f^{-1}[B] \in U$. Then $f[f^{-1}[B]] = B \in f(U) \in U(Y)$. Hence $f(U) \to y \in Y$. Perfectness implies that $f^{-1}(f(U))$ is $f^{-1}(y)$-compact. Thus if $V \in U(f^{-1}(f(U)))$, then $V \to x$ for. some $x \in f^{-1}(y)$. Since $U \in U(f^{-1}(f(U)))$, then $U \to x$ for some $x \in f^{-1}(y)$ implies that $f^{-1}[B]$ is almost-compact.

COROLLARY 2.9.1. *If $f : (X, q) \to (Y, p)$ is perfect, then f is a closed map with compact point inverses.*

https://doi.org/10.1017/S0004972700011151 Published online by Cambridge University Press

COROLLARY 2.9.2. *If* $f : (X, q) \to (Y, p)$ *is perfect (respectively and surjective), then* f *is a compact (respectively almost-compact) map.*

The various convergence functions definable on X may be ordered as follows: $q_1 \le q_2$ if and only if $q_2(F) \subset q_1(F)$ for each $F \in F(X)$. If $q_1 \le q_2$, then we say that q_1 is *coarser* than q_2 or that q_2 is *finer* than q_1. The modifications of (X, q) have the ordering $\lambda(q) \le \hat{q} \le q^* \le q$ [6]. It follows from the definition that if $f : (X, t) \to (Y, s)$ is perfect, where $t \le q$ and $s \le p$, then $f : (X, t_1) \to (Y, s_1)$ is perfect, where $t_1 \le t$ and $s \le s_1 \le p$. The next result exhibits an interesting phenomenon associated with this ordered behavior and shows that perfect maps have an additional strength.

THEOREM 2.10. *If* $f : (X, t) \to (Y, s)$ *is perfect, where* $\lambda(q) \le t \le q$ *and* $s = \lambda(p)$ *or* $\hat{p} \le s \le p$, *then* $f : (X, \lambda(q)) \to (Y, s_1)$, $\lambda(p) \le s_1 \le p$, *is perfect.*

Proof. Since for the space (Y, p), $A \subset Y$ is $\lambda(p)$-closed if and only if $\mathrm{cl}_p(A) = A$, $\mathrm{cl}_{\hat{p}}(A) = \mathrm{cl}_s(A) = \mathrm{cl}_p(A)$ and $f : (X, \lambda(q)) \to (Y, s)$ is perfect, then it follows that $f : (X, \lambda(q)) \to (Y, \lambda(p))$ is a closed map with compact point inverses. Using Whyburn's classical result [16], this implies that f is topologically perfect. The proof is completed by application of the observation preceding this theorem.

COROLLARY 2.10.1. *If* $f : (X, q) \to (Y, p)$ *is a weakly-continuous and perfect surjection with* $\hat{q} = \lambda(q)$, *then* $f : (X, \hat{q}) \to (Y, \hat{p})$ *is a topological continuous and perfect surjection.*

Proof. Theorem 2.8 part *(iv)* implies that $\hat{p} = \lambda(p)$. By Remark 2.3 we have that $f : (X, \hat{q}) \to (Y, \hat{p})$ is continuous.

In the proof of Theorem 2.10 we use Whyburn's classical result that a closed map with compact point inverses is perfect for topological spaces. The following example shows that this is not the case for convergence spaces. Hence a future investigation into conditions under which a closed map with compact point inverses is perfect should prove profitable.

EXAMPLE 2.1. Let (X, q) be a pseudotopological space which is not pretopological. Since (X, q) and $(X, \hat{q})$ have the same closed sets and each $x \in X$ is q and $\hat{q}$-compact, then the identity map $I : (X, q) \to (X, \hat{q})$ is closed and has compact point inverses. However, there exists some $\mathcal{U} \in U(X)$ and $x \in X$ such that $\mathcal{U}$ is $\hat{q}$-convergent to x, but not q-convergent to x. Consequently, $a_{\hat{q}}(I(\mathcal{U})) \not\subset I(a_q(\mathcal{U}))$ implies that I is not perfect.

Whyburn [16] gives an example of a compact map between topological spaces which is not perfect.

A space (X, q) is *locally* (respectively *almost-locally*) *compact* if each convergent ultrafilter contains a compact (respectively almost-compact) set. We note that there exist in the literature other distinct definitions for locally compact spaces. For example, a Hausdorff space, assuming axiom (C), may be called locally compact if it is open in each Hausdorff compactification [12]. In [15], Vinod-Kumar apparently shows that this latter type of local compactness is equivalent to almost-local compactness. Recall that a space (X, q) is *Hausdorff* if for each $\mathcal{F} \in F(X)$, $q(\mathcal{F})$ contains at most one element. A surjection $f : (X, q) \to (Y, p)$ is *biquotient* if $\mathcal{U} \in U(Y)$ and $\mathcal{U} \to y$, then there exists some $x \in f^{-1}(y)$ and $\mathcal{V} \in U(X)$ such that $\mathcal{V} \to x$. Clearly, a perfect surjection is biquotient, and if (X, q) is locally compact (respectively almost-locally compact), $f : (X, q) \to (Y, p)$ is a weakly-continuous biquotient map, then Y is locally compact (respectively almost-locally compact). Moreover, Corollary 2.9.2 implies that if $f : (X, q) \to (Y, p)$ is a weakly-continuous perfect surjection and (Y, p) is locally compact (respectively almost-locally compact), then X is locally compact (respectively almost-locally compact). We now give a sufficient condition for a compact (respectively almost-compact) map to be perfect.

THEOREM 2.11. *Let $\mathcal{B} \subset P(Y)$ be such that for each $\mathcal{U} \in U(Y)$ there exists some $B \in \mathcal{B}$ such that $B \in \mathcal{U}$. If $f : (X, q) \to (Y, p)$ is weakly-continuous and for each $B \in \mathcal{B}$, $f^{-1}[B]$ is almost-compact and Y is Hausdorff, then f is perfect.*

Proof. Let $\mathcal{F} \in F(X)$ and $y \in a_p(f(\mathcal{F}))$. Then there exists $\mathcal{U} \in U(Y)$ such that $\mathcal{U} \to y$ and $f(\mathcal{F}) \subset \mathcal{U}$. Also there exists some $B \in \mathcal{B}$

https://doi.org/10.1017/S0004972700011151 Published online by Cambridge University Press

such that $B \in \mathcal{U}$ and $f^{-1}[B]$ is almost-compact. Let $U \in \mathcal{U}$ and $F \in \mathcal{F}$. Then $f[F] \cap U \neq \emptyset$ implies that $F \cap f^{-1}(\mathcal{F})$. Thus $\mathcal{G} = f^{-1}(\mathcal{U}) \vee \mathcal{F} \neq P(X)$. Since $B \in \mathcal{U}$, then $f^{-1}[B] \in \mathcal{G}$. Almost-compactness implies that there exists $\mathcal{V} \in U(X)$ such that $\mathcal{V} \to x \in X$ and $f^{-1}(\mathcal{U}) \subset \mathcal{V}$. Weak-continuity implies that $f(\mathcal{V}) \to f(x)$. However, $\mathcal{U} \subset f\left(f^{-1}(\mathcal{U})\right) \subset f(\mathcal{V})$ and Hausdorff imply that $\mathcal{U} = f(\mathcal{V})$ and $f(x) = y$. Since $\mathcal{F} \subset \mathcal{V}$, then $x \in a_q(\mathcal{F})$. Thus $y \in f\left[a_q(\mathcal{F})\right]$. Application of Theorem 2.5 completes the proof.

COROLLARY 2.11.1. *Let (Y, p) be locally compact (respectively almost-locally compact) and Hausdorff. If $f : (X, q) \to (Y, p)$ is weakly-continuous and compact (respectively almost-compact), then f is perfect.*

REMARK 2.4. Further investigations into the relation between compact, almost-compact, and perfect maps should be a useful exercise.

Theorem 2.5 has other immediate and interesting applications. For example, if $f : (X, q) \to (Y, p)$ is a perfect surjection and X is Hausdorff, then Y is Hausdorff. Moreover, it immediately implies that the composition of perfect maps is a perfect map. We conclude this section with a sufficient condition for a map to be perfect and show that a perfect map can not be extended to a proper extension of its domain such that it is weakly-continuous.

THEOREM 2.12. *Let $f : (X, q) \to (Y, p)$ be weakly-continuous, $f[X]$ closed in Y and Y Hausdorff. If for each $\mathcal{F} \in F(X)$, $f(\mathcal{F})$ converges to $y \in Y$ implies $a_q(\mathcal{F}) \neq \emptyset$, then f is perfect.*

Proof. Let $\mathcal{F} \in F(Y)$, $\mathcal{F} \to y$, and $\mathcal{F} \cap f[X]$. Now $y \in f[X]$ since $f[X]$ is closed in Y and $f^{-1}(\mathcal{F}) \in F(X)$. Assume that $\mathcal{G} \in F(X)$ and $f^{-1}(\mathcal{F}) \subset \mathcal{G}$. Then $\mathcal{F} \subset f\left(f^{-1}(\mathcal{F})\right) \subset f(\mathcal{G})$ implies that $f(\mathcal{G}) \to y$. Hence $y \in a_p\left(f(\mathcal{G})\right)$. Now since Y is Hausdorff, $a_p\left(f(\mathcal{G})\right) = \{y\}$. However $a_p(\mathcal{G}) \subset f^{-1}\left(a_p\left(f(\mathcal{G})\right)\right) = f^{-1}(y)$ by weak-continuity. Since $a_p(\mathcal{G}) \neq \emptyset$, then $a_q(\mathcal{G}) \cap f^{-1}(y) \neq \emptyset$ implies by Theorem 2.4 that $f^{-1}(\mathcal{F})$ is $f^{-1}(y)$-compact. Hence Theorem 2.3 yields that f is perfect.

COROLLARY 2.12.1. *Let* $f : (X, q) \to (Y, p)$ *be weakly-continuous,* X *compact,* Y *Hausdorff, and* $f[X]$ *closed in* Y . *Then* f *is a perfect map.*

REMARK 2.5. Observe that $f : (X, q) \to (Y, p)$ is perfect if and only if $f : (X, q^*) \to (Y, p^*)$ is perfect.

A subspace (X, q') of (Z, q) is *dense* if for each $z \in Z$ there exists $\mathcal{U} \in \mathcal{U}(Z)$ such that $\mathcal{U} \to z$ and $\mathcal{U} \cap X$. If (X, q') is a subspace of (Z, q) , then $Z - X$ *is separated from* X if $\mathcal{U} \in \mathcal{U}(Z)$ and $\mathcal{U}$ q-converges to $r \in Z-X$, then $\mathcal{U}$ does not q-converge to any $x \in X$. Observe that if Z is Hausdorff, then for any subspace X it follows that $Z - X$ is separated from X . For a topological space, if Z is Hausdorff except for X , then $Z - X$ is separated from X .

THEOREM 2.13. *Let* (X, q') *be a proper dense subspace of* (Z, q) *and* $Z - X$ *be separated from* X . *If* $f : (X, q') \to (Y, p)$ *is perfect, then there does not exist an extension* F *of* f *onto* Z *such that* $F : (Z, q) \to (Y, p)$ *is weakly-continuous at any* $r \in Z-X$.

Proof. Let $r \in Z-X$, $\mathcal{U} \in \mathcal{U}(Z)$ q-converge to r , $\mathcal{U} \cap X$. Now let $\mathcal{F} = \mathcal{U}_X$ be the trace ultrafilter on X . Then since $Z - X$ is separated from X , $a_{q'}(\mathcal{F}) = \emptyset$. Now f being perfect implies that $a_p(f(\mathcal{F})) = \emptyset$. Assume that there exists $F : (Z, q) \to (Y, p)$ such that $F|X = f$ and that F is weakly-continuous at r . Then $F(\mathcal{U}) \to F(r)$. However, $f(\mathcal{F}) = F(\mathcal{U})$ and $F(\mathcal{U}) \in \mathcal{U}(Y)$ imply that $F(r) \in a_p(f(\mathcal{F}))$. This contradiction completes the proof.

3. A product result

In this final section, we establish a product space result for perfect maps which has numerous interesting applications. For spaces (X, q) , (Y, p) the *product convergences function* $r = q \times p$ on $X \times Y$ is defined as follows: for each $\mathcal{F} \in F(X \times Y)$, $(x, y) \in r(\mathcal{F})$ if and only if $x \in q(P_1(\mathcal{F}))$ and $y \in p(P_2(\mathcal{F}))$, where P_1 and P_2 are the projections of $X \times Y$ onto X and Y , respectively. For two maps $f : (X, q) \to (Y, p)$ and $g : (X, q) \to (Z, s)$, we define the map $(f, g) : (X, q) \to (Y \times Z, p \times s)$ by $(f, g)(x) = (f(x), g(x))$ for each $x \in X$.

https://doi.org/10.1017/S0004972700011151 Published online by Cambridge University Press

THEOREM 3.1. *Let* $f : (X, q) \to (Y, p)$ *and* $g : (X, q) \to (Z, s)$ *be continuous (respectively weakly-continuous). Then* (f, g) *is continuous (respectively weakly-continuous).*

Proof. For the weakly-continuous case, let $\mathcal{U} \in U(X)$ be q-convergent to $x \in X$. Then $f(\mathcal{U}) \to f(x)$ and $g(\mathcal{U}) \to g(x)$. Now from the definition it follows that $(f, g)(\mathcal{U}) \in U(Y \times Z)$. Notice that $P_1((f, g)(\mathcal{U})) = f(\mathcal{U})$ and $P_2((f, g)(\mathcal{U})) = g(\mathcal{U})$. Thus $(f, g)(\mathcal{U}) \to (f(x), g(x))$.

For continuity, it is a straightforward argument to show that for each $\mathsf{F} \in F(X)$ such that $\mathsf{F} \to x$, $f(\mathsf{F}) \subset P_1((f, g)(\mathsf{F})) \to f(x)$ and the result is easily established.

THEOREM 3.2. *If* $f : (X, q) \to (Y, p)$ *is perfect and the map* $g : (X, q) \to (Z, s)$ *is weakly-continuous into Hausdorff* Z *, then* $(f, g) : (X, q) \to (Y\times Z, p\times s)$ *is perfect.*

Proof. Let $\mathcal{U} \in U(Y \times Z)$ be such that $(f, g)[X] \in \mathcal{U}$ and $\mathcal{U}$ is r-convergent to (y, z), $r = p \times s$. Now, for each $A \subset Y \times Z$,

$$(f, g)^{-1}[A] \subset f^{-1}[P_1[A]] \cap g^{-1}[P_2[A]] .$$

Thus $f^{-1}(P_1(\mathcal{U})) \vee g^{-1}(P_2(\mathcal{U})) \subset (f, g)^{-1}(\mathcal{U})$. Since f is perfect and $P_1(\mathcal{U}) \to y$, then $f^{-1}(P_1(\mathcal{U}))$ is $f^{-1}(y)$-compact. Hence for each $\mathcal{V} \in U(X)$ such that $f^{-1}(P_1(\mathcal{U})) \subset \mathcal{V}$ there exists some $x_v \in f^{-1}(y)$ such that $\mathcal{V} \to x_v$. Since $f^{-1}(P_1(\mathcal{U})) \subset (f, g)^{-1}(\mathcal{U})$ and $g^{-1}(P_2(\mathcal{U})) \subset (f, g)^{-1}(\mathcal{U})$, then for any $\mathcal{W} \in U((f, g)^{-1}(\mathcal{U}))$ there exists $x_w \in f^{-1}(y)$ such that $\mathcal{W} \to x_w$ and $g^{-1}(P_2(\mathcal{U})) \subset \mathcal{W}$. By weak-continuity, $g(\mathcal{W}) \to g(x_w)$. Now $P_2(\mathcal{U}) \to z$ and $P_2(\mathcal{U}) \subset g\left[g^{-1}(P_2(\mathcal{U}))\right] \subset g(\mathcal{W})$ imply that $g(\mathcal{W}) \to z$. The hausdorffness of Z yields that $g(x) = z$. Consequently, for each $\mathcal{W} \in U((f, g)^{-1}(\mathcal{U}))$ there exists some $x_w \in f^{-1}(y)$ such that $x_w \in (f, g)^{-1}(y, z)$. Thus (f, g) is perfect.

COROLLARY 3.2.1. *If* $f : (X, q) \to (Y, p)$ *is perfect (respectively weakly-) continuous,* $g : (X, q) \to (Z, s)$ *is (respectively weakly-) continuous into Hausdorff* Z *, then* $(f, g) : (X, q) \to (Y \times Z, p \times s)$ *is (respectively weakly-) continuous and perfect.*

COROLLARY 3.2.2. *If* $f : (X, q) \to (Y, p)$ *is perfect and the map* $g : (X, q) \to (Z, s)$ *is weakly-continuous into Hausdorff* Z *, then* $(f, g)[X]$ *is a closed subset of* $Y \times Z$ *.*

COROLLARY 3.2.3. *If* $f : (X, q) \to (Y, p)$ *is weakly-continuous and* Y *is Hausdorff, then the graph of* f *is a closed subspace of* $X \times Y$ *.*

Let (X, q) be compact, Y Hausdorff, $f : (X, q) \to (Y, p)$ a weakly-continuous surjection and $g : (X, q) \to (Z, s)$ weakly-continuous. Then $(f, g) : (X, q) \to (Y \times Z, p \times s)$ is a weakly-continuous and perfect map by application of Corollary 2.12.1 and Theorem 3.2.

A bijection $f : (X, q) \to (Y, p)$ is a (respectively *weak-*) *homeomorphism* if f and f^{-1} are (respectively weakly-) continuous. If the injection $f : (X, q) \to (Y, p)$ is not a surjection, then f is a (respectively *weak-*) *embedding* if $f : (X, q) \to (f[X], p')$ is a (respectively weak-) homeomorphism.

THEOREM 3.3. *If there exists a weakly-continuous and perfect map* $f : (X, q) \to (Y, p)$ *and a weakly-continuous injection* $g : (X, q) \to (Z, s)$ *, where* Z *is Hausdorff, then* (X, q) *is weakly-homeomorphic to a subspace of* $Y \times Z$ *.*

Proof. From Corollary 3.2.3, $(f, g)[X]$ is closed in $Y \times Z$ and Theorem 3.1 implies that (f, g) is a weakly-continuous injection onto $(f, g)[X] = W$. We show that for $r' = (p \times s)'$,

$$(f, g)^{-1} : (W, r') \to (X, q)$$

is weakly-continuous. Let $U \in U(W)$ be such that U is r'-convergent to $w \in W$. Then there exists a unique $x \in X$ such that $(f, g)(x) = w$. From the perfectness of (f, g) , we have that $(f, g)^{-1}([U])$ is $(f, g)^{-1}(w) = \{x\}$-compact by Theorem 2.3. Hence if $V \in U\big((f, g)^{-1}([U])\big)$, then $V \to x$. Since (f, g) is an injection, then $(f, g)^{-1}([U]) = (f, g)^{-1}(U)$ is an ultrafilter on X . Hence

https://doi.org/10.1017/S0004972700011151 Published online by Cambridge University Press

$(f, g)^{-1}(U) \to x$.

COROLLARY 3.3.1. *If there exists a continuous and perfect map* $f : (X, q^*) \to (Y, p)$ *and a continuous injection* $g : (X, q^*) \to (Z, s)$, *then* (X, q^*) *is homeomorphic to a closed subspace of* $Y \times Z$.

We next present some interesting results which are easily verified and which are used to establish the last major proposition in this present investigation.

THEOREM 3.4. *If* $f : (X, q) \to (Y, p)$ *is a weakly-continuous injection, then* $f^{-1} : f([X], p') \to (X, q)$ *is perfect.*

Proof. Assume that $U \in U(X)$ and $U \to x$. We need to show that $(f^{-1})^{-1}(U) = f(U)$ is $(f^{-1})^{-1}(x) = f(x)$-compact. Thus for each ultrafilter V on $f[X]$ such that $f(U) \subset V$, we must show that $V \to f(x)$. However, $f(U) = V$ and weak-continuity imply that $V \to f(x)$.

COROLLARY 3.4.1. *Let* $f : (X, q) \to (Y, p)$ *be a weak-embedding. Then* f *and* $f^{-1} : (f[X], p') \to (X, q)$ *are perfect.*

We extract the following result from the proof of Theorem 3.3.

THEOREM 3.5. *Let* $f : (X, q) \to (Y, p)$ *be a perfect injection. Then* $f^{-1} : (f[X], p') \to (X, q)$ *is weakly-continuous.*

COROLLARY 3.5.1. *A weakly-continuous and perfect injection is a weak-embedding.*

COROLLARY 3.5.2. *Let* $f : (X, q) \to (Y, p)$ *be weakly-continuous and* Y *Hausdorff. Then* $(I, f) : (X, q) \to (X \times Y, q \times p)$ *(respectively* $(I, f) : (X, q^*) \to (X \times Y, (q \times p)^*)$ *) is a weak-embedding (respectively an embedding), where* I *is the identity on* X .

THEOREM 3.6. *Let* $f : (X, q) \to (Y, p)$ *and* $g : (Y, p) \to (Z, s)$ *both be weakly-continuous. If* $gf : (X, q) \to (Z, s)$ *is perfect and* Z *is Hausdorff, then* f *is perfect.*

Proof. Let $h = (f, gf)$. Clearly gf is weakly-continuous. Hence h is perfect. Let I be the identity on Y . Then $(I, g) : (Y, p) \to (Y \times Z, p \times s)$ is a weak-embedding. Hence $(I, g)^{-1} : (G(g), (p \times s)') \to (Y, p)$, where $G(g)$ is the graph of g , is

https://doi.org/10.1017/S0004972700011151 Published online by Cambridge University Press

perfect. Now h is perfect on any subspace of Y. Thus $h : (X, q) \to (G(g), (p \times s)')$ is perfect. The composition $((I, g)^{-1})h = f$ is perfect and this completes the proof.

We leave to the reader other applications of the results from this section. We point out, however, two more useful propositions. If $f : (X, q) \to (f[X], p')$ is perfect and $f[X]$ is p-closed, then $f : (X, q) \to (Y, p)$ is perfect. If $f : (X, q) \to (Y, p)$ is perfect and $A \subset X$ is q-closed, then $f : (A, q') \to (Y, p)$ is perfect. Finally, we mention that many of the results in this paper hold for structures that only satisfy axiom (B).

References

[1] R.F. Dickman, Jr., and Jack R. Porter, "θ-perfect and θ-absolutely closed functions", *Illinois J. Math.* 21 (1977), 42-60.

[2] Robert A. Herrmann, "θ-ridigity and the idempotent θ-closure operator", *Math. Sem. Notes* 6 (1978), 217-220.

[3] H. Hogbe-Nlend, *Théorie des bornologies et applications* (Lecture Notes in Mathematics, 213. Springer-Verlag, Berlin, Heidelberg, New York, 1971).

[4] С. Илиадис, С. Фомин [S. Iliadis, S. Fomin], "Метод центрированных систем в теории топологических пространств" [The method of centred systems in the theory of topological spaces], *Uspehi Mat. Nauk* 21 (1966), no. 4, 47-76; English translation: *Russian Math. Surveys* 21 (1966), no. 4, 37-62.

[5] Shouro Kasahara, "Boundedness in convergence spaces", *Proceedings of the University of Nevada, Reno, Conference on Convergence Spaces*, 1976, 83-92 (Department of Mathematics, University of Nevada, Reno, Nevada, 1976).

[6] D. Kent, "Convergence functions and their related topologies", *Fund. Math.* 54 (1964), 125-133.

[7] D. Kent, "On convergence groups and convergence uniformities", *Fund. Math.* 60 (1967), 213-222.

https://doi.org/10.1017/S0004972700011151 Published online by Cambridge University Press

[8] D. Kent, "Convergence quotient maps", *Fund. Math.* 65 (1969), 197-205.

[9] D. Kent and G.D. Richardson, "Minimal convergence spaces", *Trans. Amer. Math. Soc.* 160 (1971), 487-499.

[10] Darrell C. Kent and Gary D. Richardson, "Open and proper maps between convergence spaces", *Czechoslovak Math. J.* 23 (98) (1973), 15-23.

[11] D.C. Kent and G.D. Richardson, "The regularity series of a convergence space II", *Bull. Austral. Math. Soc.* 15 (1976), 223-243.

[12] C.J.M. Rao, "On smallest compactification for convergence spaces", *Proc. Amer. Math. Soc.* 44 (1974), 225-230.

[13] G.D. Richardson and D.C. Kent, "The regularity series of a convergence space", *Bull. Austral. Math. Soc.* 13 (1975), 21-44.

[14] Vinod-Kumar, "On the largest Hausdorff compactification on a Hausdorff convergence space", *Bull. Austral. Math. Soc.* 16 (1977), 189-197.

[15] Vinod-Kumar, "Compactification of a convergence space", *Proc. Amer. Math. Soc.* 73 (1979), 256-262.

[16] G.T. Whyburn, "Directed families of sets and closedness of functions", *Proc. Nat. Acad. Sci. U.S.A.* 54 (1965), 688-692.

Department of Mathematics,
United States Naval Academy,
Annapolis,
Maryland,
USA.