BULLETIN **13** (1), 1970

FINITE PROJECTIVE DISTRIBUTIVE LATTICES

BY

G. GRÄTZER[1] AND B. WOLK

The theorem stated below is due to R. Balbes[2]. The present proof is direct; it uses only the following two well-known facts: (i) Let K be a category of algebras, and let free algebras exist in K; then an algebra is projective if and only if it is a retract of a free algebra. (ii) Let F be a free distributive lattice with basis $\{x_i \mid i \in I\}$; then $\bigwedge (x_i \mid i \in J_0) \leq \bigvee (x_i \mid i \in J_1)$ implies $J_0 \cap J_1 \neq \phi$. Note that (ii) implies (iii): If for $J_0 \subseteq I$, $a, b \in F$, $\bigwedge (x_i \mid i \in J_0) \leq a \vee b$, then $\bigwedge (x_i \mid i \in J_0) \leq a$ or b.

THEOREM. *A finite distributive lattice L is projective in the category of (finite) distributive lattices if and only if the join of two meet-irreducible elements is again meet-irreducible.*

Proof. Let L be projective, so by (i) we can assume that $L \subseteq F$, where F is free on x_i, $i \in I$, and ρ is a retraction: $F \to L$. Let a and b be meet-irreducible, and let $a \vee b$ be meet-reducible, that is, for some $c, d \in L$, $a \vee b \geq c \wedge d$, $a \vee b \ngeq c$, $a \vee b \ngeq d$. Let $c = \bigvee (\bigwedge C_k \mid k)$, $d = \bigvee (\bigwedge D_l \mid l)$, where C_k and D_l are finite subsets of $\{x_i \mid i \in I\}$. Then $c = c\rho = \bigvee (\bigwedge C_k\rho \mid k)$, $d = d\rho = \bigvee (\bigwedge D_l\rho \mid l)$, and both are $\nleq a \vee b$. Therefore there exist k and l such that $\bigwedge C_k\rho$ and $\bigwedge D_l\rho \nleq a \vee b$. At the same time $\bigwedge C_k \wedge \bigwedge D_l \leq c \wedge d \leq a \vee b$, therefore $\bigwedge C_k \wedge \bigwedge D_l \leq a$ or b by (iii). Applying ρ we get $\bigwedge C_k\rho \wedge \bigwedge D_l\rho \leq a$ or b, which means that either a or b is not meet-irreducible, a contradiction.

Conversely, let L satisfy the condition of the theorem, and let m_i, $i \in I$ be the meet-irreducible elements of L. Let F be the free distributive lattice on x_i, $i \in I$, and let α be the homomorphism of F onto L extending $x_i \to m_i$, $i \in I$. Let G be the join-subsemilattice of F generated by the x_i, $i \in I$. Now we define a map[3] $\varphi: L \to F$ by

$$a\varphi = \bigwedge (x \mid x \in G, x\alpha \geq a) \quad \text{for } a \in L.$$

Intuitively, $a\varphi$ is the smallest element in the subset $a\alpha^{-1}$ of F.

Distributivity shows that $(a \vee b)\varphi = a\varphi \vee b\varphi$, since any x in G such that $x\alpha \geq a \vee b$ is of the form $x = y \vee z$, $y \in G$, $y\alpha \geq a$, $z \in G$, $z\alpha \geq b$, and conversely. To show that φ preserves meets, note that, by the assumptions of the theorem, $x\alpha$ is meet-irreducible

---

[1] Research supported by the National Research Council of Canada.

[2] Pacific J. Math. **21** (1967), 405–420.

[3] The map φ is by necessity the same as in R. Balbes, loc. cit.

for all x in G. Thus $x\alpha \geq a \wedge b$ is equivalent to the condition: $x\alpha \geq a$ or $x\alpha \geq b$. This proves that φ is a homomorphism of L into F.

Finally, we have $a = \bigwedge (m_i \mid m_i \geq a)$ for every element a of L. Thus $a\varphi\alpha = a$, and so L is a retract of F, and thus projective.

UNIVERSITY OF MANITOBA,
WINNIPEG, MANITOBA