

LIMITING DISTRIBUTIONS FOR A CLASS OF DIMINISHING URN MODELS

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Abstract

In this work we analyze a class of 2×2 Pólya–Eggenberger urn models with ball replacement matrix $M = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix}$, $a, d \in \mathbb{N}$ and $c = pa$ with $p \in \mathbb{N}_0$. We determine limiting distributions by obtaining a precise recursive description of the moments of the considered random variables, which allows us to deduce asymptotic expansions of the moments. In particular, we obtain limiting distributions for the pills problem $a = c = d = 1$, originally proposed by Knuth and McCarthy. Furthermore, we also obtain limiting distributions for the well-known sampling without replacement urn, $a = d = 1$ and $c = 0$, and generalizations of it to arbitrary $a, d \in \mathbb{N}$ and $c = 0$. Moreover, we obtain a recursive description of the moment sequence for a generalized problem.

Keywords: Pólya–Eggenberger urn model; pills problem; sampling without replacement; diminishing urn

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1. Introduction

1.1. Pólya–Eggenberger urn models

Pólya–Eggenberger urn models are defined in the following way. We start with an urn containing n white balls and m black balls. The evolution of the urn occurs in discrete time steps. At every step a ball is drawn at random from the urn. The color of the ball is inspected and then the ball is returned to the urn. According to the observed color of the ball there are added/removed balls due to the following rules. If a white ball has been drawn, we put into the urn a white balls and b black balls, but if a black ball has been drawn, we put into the urn c white balls and d black balls. The values $a, b, c, d \in \mathbb{Z}$ are fixed integer values and the urn model is specified by the 2×2 ball replacement matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This definition extends naturally also to higher dimensions. Urn models are simple, useful mathematical tools for describing many evolutionary processes in diverse fields of application, such as analysis of algorithms and data structures, statistics, and genetics. Owing to their importance in applications, there is a huge literature on the stochastic behavior of urn models;

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see, for example, [12] and [18]. Recently, a few different approaches have been proposed, which yield deep and far-reaching results for very general urn models; see [5], [6], [10], and [11]. Most papers in the literature impose the so-called tenability condition on the ball replacement matrix, so that the process can be continued ad infinitum, or no balls of a given color being completely removed. However, in some applications, there are urn models with a very different nature, which we will refer to as diminishing urn models. We refer the reader to [9] for a detailed description of diminishing urn models.

A well-known example of a diminishing urn model is the classical sampling without replacement urn model with transition matrix

$$M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this model, balls are drawn at random one after another from an urn containing balls of two different colors and not replaced. What is the probability that k balls of one color remain when balls of the other color are all removed? Another famous diminishing urn model is the so-called OK Corral urn, which serves as a mathematical model of the historic gun fight at the OK Corral. The ball transition matrix of the OK Corral urn model is given by

$$M = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

This problem was introduced by Williams and McIlroy [23], and can be viewed as a mathematical model for warfare and conflicts; see [14] and [15]. It was studied by several authors using different approaches, leading to very deep and interesting results; see, for example, Stadjé [21], Kingman [13] and [14], and Kingman and Volkov [15], or the recent works of Puyhaubert [20], Flajolet *et al.* [5], and Turner [22]. A vivid interpretation is as follows. Two groups of gunmen, group A and group B (with n and m gunmen, respectively), face each other. At every discrete time step, one gunman is chosen uniformly at random, who then shoots and kills exactly one gunman of the other group. The gunfight ends when one group is completely ‘eliminated’. Several questions are of interest: what is the probability that group A (group B) survives, and what is the probability that the gunfight ends with k survivors of group A (group B)? Moreover, one is also interested in the total number of survivors, regardless of the group. It turns out that the limit laws arising in the OK Corral urn model are of a different nature compared to the limit laws arising in the sampling without replacement urn model, which can easily be seen by comparing the limit laws given in [5], [13], [14], [15], [20], and [21], basically normal distributions or related distributions, with the limit laws—beta distributions, exponential distributions, and geometric distributions—arising from the sampling without replacement urn model:

$$P\{X_{m,n} = k\} = \binom{m-1+n-k}{m-1} / \binom{m+n}{m}, \quad 0 \leq k \leq n.$$

This explicit formula can be proven in various ways, e.g. via lattice path counting arguments or generating functions [9]. Here $X_{m,n}$ denotes the random variable representing the number of white balls, when all black balls have been drawn, starting with n white and m black balls, in the sampling without replacement urn model.

In this work we will analyze diminishing Pólya–Eggenberger urn models with ball replacement matrix M given by

$$M = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix}, \quad \text{with } a, d \in \mathbb{N} \text{ and } c = pa, \quad p \in \mathbb{N}_0. \tag{1}$$

Throughout this work we use the notation $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \{0, 1, 2, \dots\}$. We are interested in the distribution of the random variable $X_{m,n}$, representing the number of white balls, when all black balls have been drawn, starting with n white and m black balls. We assume that the initial number of white balls is a multiple of a and that the initial number of black balls is a multiple of d ; equivalently, we consider the random variables $X_{dm,an}$ with $m, n \in \mathbb{N}_0$. The distribution of the random variable $X_{dm,an}$ in the context of the evolution of an urn, with ball replacement matrix given by M , may be described as follows. We have a state space $\mathcal{S} := \{(dm, an) \mid m, n \in \mathbb{N}_0\}$, where the evolution of the urn takes place. The evolution stops at *absorbing states* $\mathcal{A} := \{(0, an) \mid n \in \mathbb{N}_0\}$. The question is then to determine the probability $P\{X_{dm,an} = k\}$ that a certain state $k \in \mathcal{A}$ is reached, starting with an white balls and dm black balls. The aim of this work is the derivation of limiting distributions of the random variables $X_{dm,an}$ for diminishing urn models, when the urn evolves according to a ball replacement matrix M given by (1). We will see that different limiting distributions arise according to the growth of m and n . Note that, when starting with $an + \alpha$ white balls, where $1 \leq \alpha < a$, the urn model is no longer well defined. It may happen that at some stage only α white balls are left, but we are forced to remove a white balls, when choosing a white ball. The same problem occurs when the parameter $c \neq pa$ is not a multiple of the parameter a in the definition of the ball replacement matrix.

1.2. Motivation

Our studies of the class of diminishing urns with a ball replacement matrix given by (1) is motivated by the following problems.

1.2.1. *The pills problem.* The pills problem was originally proposed by Knuth and McCarthy [17, p. 264]; the solution appeared in [16]. A vivid interpretation of the pills problem is the following. In a bottle there are m large pills and n small pills. A large pill is equivalent to two small pills. Every day a person chooses a pill at random. If a small pill is chosen, it is swallowed, but if a large pill is chosen, it is broken into two halves, one half is swallowed and the other half, which is now considered a small pill, is returned to the bottle. The problem, proposed in [17], was to find the expected number of small pills remaining when there are no more large pills left in the bottle. Brennan and Prodinger revisited this problem in [2], where they showed how one can derive the exact moments of the pills problem (at least in principle), and computed them up to the third moment. Furthermore, they also considered variations of the problem, assuming, e.g. that a large pill is equal to p small pills, where they also succeeded in computing the expected value. The pills problem corresponds to the derivation of the expected value of $X_{m,n}$ for a diminishing urn model with ball replacement matrix

$$M = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}.$$

In the recent work of Hwang *et al.* [9] the limiting distributions of the pills problem and a related model, namely

$$M = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix},$$

were obtained by using generating functions. It was shown that the limiting distributions significantly differ for these two problems. The generating functions approach of [9] has the benefit that one not only obtains the limiting distributions, but also the exact distribution of $X_{m,n}$ for the two considered urn models. However, it seems difficult to extend the generating

function approach to study the class with a ball replacement given by (1) in full generality. Hence, the results of [9] motivated us to analyze the class of urns with replacement matrix

$$M = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix}.$$

1.2.2. *Sampling without replacement.* This classical urn model corresponds to the urn with ball replacement matrix

$$M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The distribution of the types of balls after t draws is very well known (see, e.g. [5]), but here we will focus on the limiting distributions of $X_{m,n}$. Note that this problem is often treated by introducing two absorbing axes, i.e. $\{(0, n) : n \geq 0\} \cup \{(m, 0) : m \geq 0\}$, but we rather simply use the absorbing axis $\mathcal{A} = \{(0, n) : n \geq 0\}$, which is fully sufficient. We will also derive limiting distributions for the generalizations

$$M = \begin{pmatrix} -a & 0 \\ 0 & -d \end{pmatrix}.$$

1.3. Weighted lattice paths

It is useful to describe and visualize the evolution of an urn with ball replacement matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

by weighted paths, which is described here in the case of urns with two types of balls. If the urn contains m black balls and n white balls, and we pick a white ball, which appears with probability $n/(m+n)$, this corresponds to a step $(m, n) \rightarrow (m+a, n+b)$, which has weight $n/(m+n)$, and if we pick a black ball, this corresponds to a step $(m, n) \rightarrow (m+c, n+d)$, which appears with probability $m/(m+n)$ and thus has weight $m/(m+n)$. The weight of a path after t successive draws consists of the product of the weights of every step. For a diminishing urn, we find that the sum of the weights of all possible paths starting at state (m, n) and ending at the absorbing state $(i, j) \in \mathcal{A}$ (which did not pass another absorbing state earlier) gives the required probability that, when starting at (m, n) , we end at (i, j) .

Unfortunately, the weighted path approach is in general not effective for studying the behavior of urn models. An example for a weighted path corresponding to the evolution of a diminishing urn is given in Figure 1. The steps associated with a ball replacement matrix

$$M = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix}$$

are visualized in Figure 2.

1.4. Goal

We will determine the structure of the moments of $X_{dm,an}$ for urn models with ball replacement matrix

$$M = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix}, \quad a, d \in \mathbb{N} \text{ and } c = pa \text{ with } p \in \mathbb{N}_0,$$

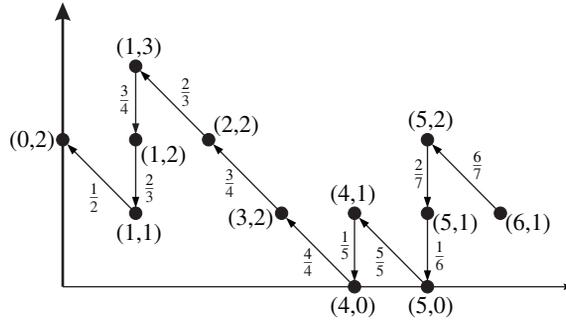


FIGURE 1: An example of a weighted path from $(6, 1)$ to the absorbing state $(0, 2)$ for the so-called pills problem with ball replacement matrix $M = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$ and the vertical absorbing axis $\mathcal{A} = \{(0, n) : n \geq 0\}$. The illustrated path has weight $\frac{6}{7} \frac{2}{7} \frac{1}{6} \frac{1}{5} \frac{4}{4} \frac{3}{4} \frac{2}{3} \frac{2}{3} \frac{1}{2} = \frac{3}{3920}$.

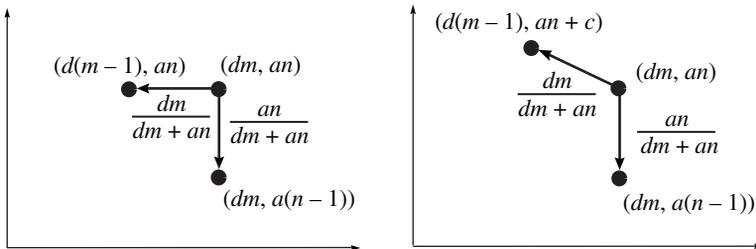


FIGURE 2: The steps associated with the transition matrix $M = \begin{pmatrix} -a & 0 \\ c & -a \end{pmatrix}$ for $c = 0$ (left) and $c > 0$ (right).

as well as providing explicit formulae for the expectation and variance of $X_{dm,an}$. Moreover, we will determine limiting distributions of the random variable $X_{dm,an}$ with replacement matrix M as given in (1). As a byproduct we (re)obtain limiting distributions for the pills problem, and also for generalizations of it.

For fixed m and n tending to ∞ , we can show that $X_{dm,an}/(an)$ tends to a so-called Kumaraswamy-distributed random variable. Furthermore, we show that, for m tending to ∞ , the limiting distribution for $c \geq 0$ changes according to the quotient a/d , with $a, d \in \mathbb{N}$, and the proportion of m and n . We will also encounter Weibull distributions as limiting distributions.

1.5. Notation

We denote by $X_n \xrightarrow{\mathcal{L}} X$ the weak convergence, i.e. the convergence in distribution, of the sequence of random variables X_n to a random variable X . We use the notation $X \stackrel{\mathcal{L}}{=} Y$ for the equality in distribution of the random variables X and Y . Furthermore, we define $H_n := \sum_{k=1}^n 1/k$ to be the n th harmonic number and $H_n^{(2)} := \sum_{k=1}^n 1/k^2$ to be the n th second-order harmonic number. We denote by $[n_k]$ the unsigned Stirling number of the first kind, and by $\{n_k\}$ the Stirling numbers of the second kind; see, e.g. [7]. Furthermore, throughout this work, we use the Pochhammer symbol for the falling factorial $(x)_\ell := x(x-1) \cdots (x-\ell+1)$. Note that in combinatorics $(x)_\ell$ is used for the falling factorial, whereas in the theory of special functions the same notation is used for the rising factorial. Alternate notation for the falling factorials include $x^{\underline{\ell}}$, as propagated by Graham *et al.* [7]. Moreover, we use standard asymptotic notation, such as the big-O notation, small-o notation, and also the asymptotic equivalence of functions $f \sim g \Leftrightarrow \lim(f/g) = 1$.

1.6. Plan of the paper

The main results of this work are the characterization of the limiting distributions of $X_{dm,an}$ depending on the ball replacement matrix M and the initial states, which are contained in the next section. In Section 4 we give a recursive description of the moments of $X_{dm,an}$ together with the derivation of the expectation and the variance. Section 5 is devoted to the proofs of the limiting distribution results. A generalization of the considered urn model is then discussed in Section 6.

2. Preliminaries

In the following we collect some basic facts about important probability distributions appearing later in our analysis.

The Kumaraswamy distribution with parameters $\alpha, \beta > 0$ is the distribution of a random variable $K = K(\alpha, \beta)$ with density $f(t) = f_K(t)$ and distribution function given by

$$f(t) = \alpha\beta t^{\alpha-1}(1-t^\alpha)^{\beta-1} \quad \text{and} \quad P\{K \leq x\} = 1 - (1-x^\alpha)^\beta, \quad x \in [0, 1]. \quad (2)$$

The s th moment of a Kumaraswamy-distributed random variable $K = K(\alpha, \beta)$ is given by

$$E(K^s) = \frac{\Gamma(\beta + 1)\Gamma(1 + s/\alpha)}{\Gamma(1 + \beta + s/\alpha)}, \quad s \geq 1, \quad (3)$$

and the Kumaraswamy distribution is uniquely determined by its sequence of moments $(E(K^s))_{s \in \mathbb{N}}$. The Kumaraswamy distribution is closely related to the beta distribution: a Kumaraswamy-distributed random variable $K = K(\alpha, \beta)$ can be expressed in terms of a beta-distributed random variable $B \stackrel{\mathcal{L}}{=} B(1, \beta)$ with parameters 1 and β as follows: $K \stackrel{\mathcal{L}}{=} B^{1/\alpha}$.

The Weibull distribution with parameters $k, \lambda > 0$ is the distribution of a random variable $W = W(k, \lambda)$ with support $[0, \infty)$, where the density function and distribution function are given by

$$f(t) = \frac{k}{\lambda} \left(\frac{t}{\lambda}\right)^{k-1} e^{-(t/\lambda)^k}, \quad t \geq 0, \quad P\{W \leq x\} = 1 - e^{-(x/\lambda)^k}, \quad x \geq 0.$$

The s th moment of W is given by

$$E(W^s) = \lambda^s \Gamma\left(1 + \frac{s}{k}\right), \quad s \geq 1,$$

and it is known that, for $k \geq 1$, the Weibull distribution is uniquely determined by its sequence of moments $(E(W^s))_{s \in \mathbb{N}}$. Special instances of the Weibull distribution are the exponential distribution $k = 1$ and the Rayleigh distribution $k = 2$. Note that it is known that the Weibull distribution can be expressed as the $(1/k)$ th power of a standard exponentially distributed random variable $\varepsilon \stackrel{\mathcal{L}}{=} \text{Exp}(1)$ with intensity 1 times λ , i.e. $W \stackrel{\mathcal{L}}{=} \lambda(\varepsilon)^{1/k}$.

For a given parameter $\rho > 0$, there exists a discrete distribution $Y = Y_\rho$, with probability mass function given by

$$P\{Y = \ell\} = \frac{\rho^\ell}{\ell!} \frac{d}{a} \int_0^\infty x^{\ell+d/a-1} e^{-x^{d/a}-\rho x} dx, \quad (4)$$

such that the factorial moments of Y are essentially given by the power moments of a Weibull-distributed random variable W with parameters $k = d/a$ and $\lambda = 1$:

$$E((Y)_s) = \rho^s \Gamma\left(1 + \frac{as}{d}\right), \quad E(Y^s) = \sum_{j=1}^s \left\{ \begin{matrix} s \\ j \end{matrix} \right\} \rho^j \Gamma\left(1 + \frac{aj}{d}\right).$$

Moreover, the moment generating function $\varphi(z) = E(e^{zY})$ of Y is related to the moment generating function $\psi(z) = E(e^{zW})$ of W , if it exists, by

$$\varphi(z) = \psi(\rho(e^z - 1)).$$

This can easily be verified by a direct computation: the s th factorial moment $E((Y)_s)$ satisfies

$$\begin{aligned} E((Y)_s) &= \sum_{\ell \geq 0} (\ell)_s P\{Y = \ell\} \\ &= \sum_{\ell \geq s} (\ell)_s \frac{\rho^\ell d}{\ell! a} \int_0^\infty x^{\ell+d/a-1} e^{-x^{d/a}-\rho x} dx \\ &= \rho^s \sum_{\ell \geq s} \frac{\rho^{\ell-s} d}{(\ell-s)! a} \int_0^\infty x^{\ell+d/a-1} e^{-x^{d/a}-\rho x} dx \\ &= \rho^s \frac{d}{a} \int_0^\infty x^{s+d/a-1} e^{-x^{d/a}-\rho x} \sum_{\ell \geq 0} \frac{\rho^\ell x^\ell}{\ell!} dx \\ &= \rho^s \frac{d}{a} \int_0^\infty x^{s+d/a-1} e^{-x^{d/a}} dx \\ &= \rho^s \Gamma\left(1 + \frac{as}{d}\right). \end{aligned}$$

This implies the stated results for the ordinary moments $E(Y^s) = \sum_{j=1}^s \left\{ \begin{matrix} s \\ j \end{matrix} \right\} E((Y)_j)$. The moment generating function of Y is given by

$$\begin{aligned} \varphi(z) &= E(e^{zY}) \\ &= \sum_{\ell \geq 0} e^{z\ell} P\{Y = \ell\} \\ &= \frac{d}{a} \int_0^\infty x^{d/a-1} e^{-x^{d/a}-\rho x} \sum_{\ell \geq 0} \frac{\rho^\ell}{\ell!} x^\ell e^{z\ell} dx \\ &= \frac{d}{a} \int_0^\infty x^{d/a-1} e^{-x^{d/a} + \rho x(e^z - 1)} dx \\ &= \psi(\rho(e^z - 1)). \end{aligned}$$

For $d/a \geq 1$, or, equivalently, $a/d \leq 1$, the moment generating function $\psi(z)$ of the Weibull distribution exists, and, therefore, by the result above, the moment generating function $\varphi(z)$ of the discrete distribution Y exists too, and the discrete distribution of Y is characterized by its moments.

3. Results

Next we state our limiting distribution results for $X_{dm,an}$, divided into three cases, namely, $c = 0$, $c \neq 0$ and $a/d \leq 1$, and $c \neq 0$ and $a/d > 1$. We start with the simplest case $c = 0$, which seems to be well known (at least implicitly), and is covered here for the sake of completeness and as another application of our approach.

3.1. Limiting distributions for $c = 0$

Theorem 1. *For the ball replacement matrix*

$$M = \begin{pmatrix} -a & 0 \\ 0 & -d \end{pmatrix},$$

the random variable $X_{dm,an}$, representing the number of white balls when all black balls have been removed, starting with an white and dm black balls, has the following limiting behavior:

1. For fixed m and $n \rightarrow \infty$, the scaled random variable $X_{dm,an}/(an)$ converges in distribution to a Kumaraswamy-distributed random variable:

$$\frac{X_{dm,an}}{an} \xrightarrow{\mathcal{L}} K\left(\frac{d}{a}, m\right).$$

2. For $m, n \rightarrow \infty$ such that $m^{a/d} = o(n)$, the scaled random variable $m^{a/d} X_{dm,an}/(an)$ converges in distribution to a Weibull-distributed random variable:

$$\frac{m^{a/d} X_{dm,an}}{an} \xrightarrow{\mathcal{L}} W\left(\frac{d}{a}, 1\right).$$

3. For $m, n \rightarrow \infty$ such that $n \sim \rho m^{a/d}$, with $\rho \in \mathbb{R}^+$, the random variable $X_{dm,an}/a$ converges to a discrete random variable X with moments

$$\frac{X_{dm,an}}{a} \xrightarrow{\mathcal{L}} X, \quad \text{with} \quad E(X^s) = \sum_{\ell=1}^s \binom{s}{\ell} \rho^\ell \Gamma\left(1 + \frac{a\ell}{d}\right).$$

4. For $m \rightarrow \infty$ and $n = n(m) = o(m^{a/d})$, the random variable $X_{dm,an}$ converges to a limit X , which has all its mass concentrated at 0:

$$X_{dm,an} \xrightarrow{\mathcal{L}} 0.$$

Remark 1. For $a = d = 1$, we obtain the limit laws for the well-known sampling without replacement urn model

$$M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

which seem to be mathematical folklore, although we could not find a proper reference to the literature. In particular, as mentioned in the introduction, we obtain a beta limiting distribution for fixed m and $n \rightarrow \infty$, an exponential distribution for $m, n \rightarrow \infty$ such that $m = o(n)$, and geometric distributions for $m, n \rightarrow \infty$ such that $n \sim \rho m$.

Remark 2. Strictly speaking, for $a/d > 1$, we can only show moment convergence in (2) and (3) by our approach. However, it is possible to improve to general $a, d \in \mathbb{N}$ using different approaches. First, we would like to mention a generating functions approach similar to the

works [6] and [9] to obtain a closed-form expression:

$$P\left\{\frac{X_{dm,an}}{a} = k\right\} = \frac{m}{a} \binom{n}{k} \int_0^1 q^{1/a+k/d-1} (1 - q^{1/a})^{m-1} (1 - q^{1/d})^{n-k} dq, \tag{5}$$

valid for $1 \leq k \leq n$. By using the definition of the beta function, we readily obtain the expressions

$$\begin{aligned} P\left\{\frac{X_{dm,an}}{a} = k\right\} &= \sum_{\ell=1}^m (-1)^{\ell-1} \binom{m}{\ell} \binom{k-1+\ell d/a}{k} / \binom{n+\ell d/a}{n} \\ &= \sum_{\ell=k}^n (-1)^{\ell-k} \binom{n}{\ell} \binom{\ell}{k} / \binom{m+\ell a/d}{m}, \end{aligned}$$

which allow us to extend the results to general $a, d \in \mathbb{N}$. Second, as pointed out to us by the referee, it is possible to use a decoupling approach; see, for example, [11] and [15]: we consider two independent linear death processes, W_t (white) with death rate 1, and B_t (black) with rate $\mu = d/a$, starting at time 0 with n white balls and m black balls. Let $\tau = \inf_{t>0}\{B_t = 0\}$ be the time when the black process dies out. Then $W_\tau = X_{dm,an}/a$. By the independence assumption $P\{\tau < t\} = (1 - e^{-\mu t})^m$, the number of white balls is determined by the binomial distribution $B(n, p)$ with parameter $p = e^{-t}$. Consequently, we can strengthen part 1 of Theorem 1 to almost-sure convergence, and readily (re)obtain the closed-form expression (5) after a simple substitution.

3.2. Limiting distributions for $c \neq 0$ and $a/d \leq 1$

Theorem 2. For the ball replacement matrix

$$M = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix},$$

where $c = pa > 0$ with $p \in \mathbb{N}$, and $a/d \leq 1$, the random variable $X_{dm,an}$, counting the number of white balls when all black balls have been removed, starting with an white and dm black balls, has the following limiting behavior.

1. For fixed m and $n \rightarrow \infty$, the random variable $X_{dm,an}/(an)$ converges in distribution to a Kumaraswamy-distributed random variable $K(d/a, m)$:

$$\frac{X_{dm,an}}{an} \xrightarrow{\mathcal{L}} K\left(\frac{d}{a}, m\right).$$

2. For $m \rightarrow \infty$ and arbitrary $n = n(m)$, possibly constant or a function of m , the random variable $X_{dm,an}/g_{m,n}$ converges in distribution to a Weibull-distributed random variable:

$$\frac{X_{dm,an}}{g_{m,n}} \xrightarrow{\mathcal{L}} W\left(\frac{d}{a}, 1\right).$$

The normalization values $g_{m,n}$ are given as

$$g_{m,n} = g_{m,n}(a, c, d) = \begin{cases} \frac{an + mcd/(d - a)}{m^{a/d}} & \text{for } \frac{a}{d} < 1, \\ \frac{an}{m} + c \log m & \text{for } \frac{a}{d} = 1. \end{cases}$$

Remark 3. The special case $a = c = d = 1$ of Theorem 2 has already been proved by Hwang *et al.* [9]. Furthermore, the special case $a = c = 1, d = 2$ of Theorem 2 reproves the Rayleigh limiting distribution for $\sqrt{m}X_{n,2m}/(n + 2m)$, also stated in Hwang *et al.* [9].

3.3. Limiting distributions for $c \neq 0$ and $a/d > 1$

Theorem 3. *For the ball replacement matrix*

$$M = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix},$$

where $c = pa > 0$ with $p \in \mathbb{N}$, and $a/d > 1$, the random variable $X_{dm,an}$, representing the number of white balls when all black balls have been removed, starting with an white and dm black balls, has the following limiting behavior.

1. For fixed m and $n \rightarrow \infty$, the scaled random variable $X_{dm,an}/(an)$ converges in distribution to a Kumaraswamy-distributed random variable:

$$\frac{X_{dm,an}}{an} \xrightarrow{\mathcal{L}} K\left(\frac{d}{a}, m\right).$$

2. For $m, n \rightarrow \infty$ such that $m^{a/d} = o(n)$, the moments of the random variable $m^{a/d} X_{dm,an}/(an)$ converge to the moments of the Weibull-distributed random variable $W(d/a, 1)$.
3. For $m, n \rightarrow \infty$ such that $n \sim \rho m^{a/d}$, with $\rho \in \mathbb{R}^+$, the moments of the random variable $X_{dm,an}$ converge:

$$E(X_{dm,an}^s) \rightarrow a^s \sum_{\ell=0}^s \rho^\ell \Gamma\left(1 + \frac{a\ell}{d}\right) \sum_{r=\ell}^s \left\{ \begin{matrix} s \\ r \end{matrix} \right\} \vartheta_{r,\ell;0,0}, \quad s \geq 1.$$

The values $\vartheta_{s,\ell;h,g}$ satisfy a system of recurrence relations given in Proposition 1 below.

4. For $m \rightarrow \infty$ and arbitrary $n = n(m)$ satisfying $n = o(m^{a/d})$, the moments of the random variable $X_{dm,an}$ converge:

$$E(X_{dm,an}^s) \rightarrow a^s \sum_{r=0}^s \left\{ \begin{matrix} s \\ r \end{matrix} \right\} \vartheta_{r,0;0,0}, \quad s \geq 1.$$

The values $\vartheta_{s,\ell;h,g}$ satisfy a system of recurrence relations given in Proposition 1 below.

Note that we can prove convergence of the moments, but we are not able to show that the resulting moment sequences define a unique distribution.

4. The structure of the moments

4.1. A recurrence for the moments

By the definition of Pólya–Eggenberger urn models with ball replacement matrix given by (1), the probability generating function $h_{m,n}(v) := \sum_{k \geq 0} P\{X_{dm,an} = k\} v^k$ of $X_{dm,an}$ satisfies the recurrence (recall that $c = pa$, $p \in \mathbb{N}_0$)

$$h_{m,n}(v) = \frac{an}{an + dm} h_{m,n-1}(v) + \frac{dm}{an + dm} h_{m-1,n+p}(v) \quad \text{for } n \geq 0, m \geq 1, \quad (6)$$

with initial values given by $h_{0,n}(v) = v^{an}$ for all $n \geq 0$.

Our aim is to derive limiting distributions of $X_{dm,an}$ for $\max\{m, n\} \rightarrow \infty$. To do this, we will give a precise description of the moments, which enables us to obtain an exact expression for the expected value, and to determine the limiting distributions using the so-called method

of moments, i.e. by applying the moment convergence theorem of Fréchet and Shohat, the second central limit theorem; see, e.g. [19]. Of course, it follows from (6) that the moments $e_{m,n}^{[s]} := E(X_{dm,an}^s)$ satisfy the recurrence

$$e_{m,n}^{[s]} = \frac{an}{an + dm} e_{m,n-1}^{[s]} + \frac{dm}{an + dm} e_{m-1,n+p}^{[s]} \quad \text{for } n \geq 0, m \geq 1, \tag{7}$$

with initial values $e_{0,n}^{[s]} = a^s n^s$ for $n \in \mathbb{N}_0$.

Our first observation for determining the structure of the moments is that $e_{m,n}^{[s]}$ is a polynomial of degree s in n ; in other words, the s th moment is of the form $e_{m,n}^{[s]} = \sum_{\ell=0}^s \lambda_{s,\ell,m} n^\ell$, where the numbers $\lambda_{s,\ell,m}$ are independent of n . In the following we obtain an explicit result for $\lambda_{s,s,m}$, and a recursive description of the quantities $\lambda_{s,\ell,m}$, $1 \leq \ell \leq s - 1$, in terms of $\lambda_{s,i,j}$, with $\ell + 1 \leq i \leq s$ and $1 \leq j \leq m$.

Lemma 1. *The s th moments $e_{m,n}^{[s]} = E(X_{dm,an}^s)$ of the random variable $X_{dm,an}$ satisfy the expansion $e_{m,n}^{[s]} = \sum_{k=0}^s \lambda_{s,k,m} n^k$, where $\lambda_{s,s,m} = a^s / \binom{m+as/d}{m}$. Furthermore, the values $\lambda_{s,\ell,m}$, $1 \leq \ell \leq s - 1$, satisfy the recurrence relations*

$$\lambda_{s,\ell,m} = \sum_{k=0}^{m-1} \frac{\binom{m}{k}}{\binom{m+a\ell/d}{k}} \mu_{s,\ell,m-k},$$

where

$$\mu_{s,\ell,m} := \frac{a}{a\ell + dm} \sum_{k=\ell+1}^s \binom{k}{\ell-1} (-1)^{k-\ell-1} \lambda_{s,k,m} + \frac{dm}{a\ell + dm} \sum_{k=\ell+1}^s \binom{k}{\ell} p^{k-\ell} \lambda_{s,k,m-1}.$$

For $\ell = 0$, we have

$$\lambda_{s,0,m} = \sum_{k=0}^{m-1} \mu_{s,0,k}, \quad \text{with } \mu_{s,0,m} := \sum_{k=1}^s p^k \lambda_{s,k,m}.$$

The initial values are given by $\lambda_{s,s,0} = a^s$ and $\lambda_{s,\ell,0} = 0$ for $0 \leq \ell \leq s - 1$.

Proof. In order to prove the stated expansion of $e_{m,n}^{[s]}$, we start with the ansatz $e_{m,n}^{[s]} = \sum_{\ell=0}^s \lambda_{s,\ell,m} n^\ell$, and obtain from the recurrence relation (7) the equation

$$(an + dm) \sum_{\ell=0}^s \lambda_{s,\ell,m} n^\ell = an \sum_{\ell=0}^s \lambda_{s,\ell,m} (n-1)^\ell + dm \sum_{\ell=0}^s \lambda_{s,\ell,m-1} (n+p)^\ell. \tag{8}$$

By comparing the coefficients of n^ℓ for $0 \leq \ell \leq s + 1$ in (8) we obtain a system of $s + 2$ equations, i.e.

$$\begin{aligned} \lambda_{s,s,m} &= \lambda_{s,s,m}, \\ dm\lambda_{s,\ell,m} + a\lambda_{s,\ell-1,m} &= a \sum_{k=\ell-1}^s (-1)^{k-\ell+1} \lambda_{s,k,m} \binom{k}{\ell-1} \\ &\quad + dm \sum_{k=\ell}^s p^{k-\ell} \lambda_{s,k,m-1} \binom{k}{\ell}, \quad 1 \leq \ell \leq s, \\ \lambda_{s,0,m} &= \sum_{k=0}^s p^k \lambda_{s,k,m-1}, \end{aligned}$$

with initial values $\lambda_{s,s,0} = a^s$ and $\lambda_{s,\ell,0} = 0$ for $0 \leq \ell \leq s - 1$, which are determined by $e_{0,n}^{[s]} = a^s n^s$. The first equation is trivially true, so there remain $s + 1$ equations, which determine the values $\lambda_{s,\ell,m}$, $0 \leq \ell \leq s$. The term $\lambda_{s,\ell-1,m}$ on the left-hand side cancels with the first summand of $\sum_{k=\ell-1}^s (-1)^{k-\ell+1} \lambda_{s,k,m} \binom{k}{\ell-1}$ on the right-hand side, and we obtain

$$dm\lambda_{s,\ell,m} = -a\ell\lambda_{s,\ell,m} + a \sum_{k=\ell+1}^s (-1)^{k-\ell+1} \lambda_{s,k,m} \binom{k}{\ell-1} + dm\lambda_{s,k,m-1} + dm \sum_{k=\ell+1}^s p^{k-\ell} \lambda_{s,k,m-1} \binom{k}{\ell}, \quad 1 \leq \ell \leq s.$$

The key step is to note that, for computing the values $\lambda_{s,\ell,m}$ for $1 \leq \ell \leq s$, only values $\lambda_{s,i,m}$ and $\lambda_{s,i,m-1}$ with $\ell + 1 \leq i \leq s$ are needed, which allows us to recursively describe these values. Hence, we can obtain, for the values $\lambda_{s,\ell,m}$, the recurrence relations

$$\lambda_{s,\ell,m} = \frac{dm}{dm + a\ell} \lambda_{s,\ell,m-1} + \mu_{s,\ell,m} \quad \text{for } 1 \leq \ell \leq s,$$

with

$$\mu_{s,\ell,m} := \frac{a}{dm + a\ell} \sum_{k=\ell+1}^s \binom{k}{\ell-1} (-1)^{k-\ell-1} \lambda_{s,k,m} + \frac{dm}{dm + a\ell} \sum_{k=\ell+1}^s \binom{k}{\ell} p^{k-\ell} \lambda_{s,k,m-1}.$$

In the case $\ell = 0$ we directly obtain

$$\lambda_{s,0,m} = \lambda_{s,0,m-1} + \sum_{k=1}^s p^k \lambda_{s,k,m-1}$$

and, further,

$$\lambda_{s,0,m} = \lambda_{s,0,m-1} + \mu_{s,0,m-1} \quad \text{with} \quad \mu_{s,0,m} := \sum_{k=1}^s p^k \lambda_{s,k,m}.$$

Using induction with respect to m and n , we conclude that recurrence (7) has a unique solution for the given initial values $\lambda_{s,s,0} = a^s$ and $\lambda_{s,\ell,0} = 0$ for $0 \leq \ell \leq s - 1$.

Now we will compute $\lambda_{s,s,m}$. We have

$$dm\lambda_{s,s,m} + a\lambda_{s,s-1,m} = -sa\lambda_{s,s,m} + a\lambda_{s,s-1,m} + dm\lambda_{s,s,m-1},$$

leading to

$$(dm + as)\lambda_{s,s,m} = dm\lambda_{s,s,m-1}.$$

This gives

$$\lambda_{s,s,m} = \frac{dm}{dm + as} \lambda_{s,s,m-1} \quad \text{and, further,} \quad \lambda_{s,s,m} = \frac{m!}{(m + as/d)_m} = \frac{a^s}{\binom{m+as/d}{m}},$$

where $(m + as/d)_m = (m + as/d) \cdots (1 + as/d)$ is written using the falling factorials notation.

By Lemma 1 we can derive arbitrarily high moments of $X_{dm,an}$. In particular, we will derive the expectation of $X_{dm,an}$, and use Lemma 1 to prove our limiting distribution results.

4.2. Factorial moments and an explicit formula for the moments in the case $c = 0$

The computation of several moments using Lemma 1 suggested that an alternative description of the moments can be obtained. Let

$$e_{m,n}^{(s)} = E\left(\left(\frac{X_{dm,an}}{a}\right)_s\right) = E\left(\frac{X_{dm,an}}{a} \left(\frac{X_{dm,an}}{a} - 1\right) \dots \left(\frac{X_{dm,an}}{a} - s + 1\right)\right)$$

denote the s th factorial moment of the normalized random variable $X_{dm,an}/a$, with initial value $e_{0,n}^{(s)} = (n)_s$. In the following we obtain an alternative expansion for $e_{m,n}^{(s)}$ in terms of $(n)_s$, the falling factorial powers of n .

Lemma 2. *The s th factorial moments $e_{m,n}^{(s)} = E((X_{dm,an}/a)_s)$ of the normalized random variable $X_{dm,an}/a$ satisfies the expansion $e_{m,n}^{(s)} = \sum_{\ell=0}^s \Lambda_{s,\ell,m}(n)_\ell$, with $\Lambda_{s,s,m} = 1/\binom{m+as/d}{m}$ and $\Lambda_{s,\ell,m}$, $1 \leq \ell \leq s - 1$, recursively described by*

$$\Lambda_{s,\ell,m} = \frac{1}{\binom{m+a\ell/d}{m}} \sum_{k=0}^{m-1} \binom{k+a\ell/d}{k} \sum_{j=\ell+1}^s \binom{j}{\ell} (p)_{j-\ell} \Lambda_{s,j,k}.$$

The initial values are given by $\Lambda_{s,s,0} = 1$ and $\Lambda_{s,\ell,0} = 0$ for $0 \leq \ell \leq s - 1$.

Moreover, the values $\lambda_{s,\ell,m}$, with $e_{m,n}^{[s]} = E(X_{dm,an}^s) = \sum_{\ell=0}^s n^\ell \lambda_{s,\ell,m}$, arising in the expansion of the ordinary moments of $X_{dm,an}$, are related to $\Lambda_{s,j,m}$ via

$$\lambda_{s,\ell,m} = a^s \sum_{r=\ell}^s \sum_{j=\ell}^r (-1)^{j-\ell} \begin{Bmatrix} s \\ r \end{Bmatrix} \begin{bmatrix} j \\ \ell \end{bmatrix} \Lambda_{r,j,m}, \quad 0 \leq \ell \leq s,$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ denotes the unsigned Stirling numbers of the first kind and $\begin{Bmatrix} n \\ k \end{Bmatrix}$ denotes the Stirling numbers of the second kind.

As an immediate consequence of the above result, we obtain explicit results for all moments in the case $c = 0$.

Corollary 1. *In the case $c = 0$ the s th factorial moment $E((X_{dm,an}/a)_s)$ of the normalized random variable $X_{dm,an}/a$ is given by*

$$E\left(\left(\frac{X_{dm,an}}{a}\right)_s\right) = \frac{(n)_s}{\binom{m+as/d}{m}}.$$

Consequently, the ordinary s th moment of $X_{dm,an}$ is given by

$$E(X_{dm,an}^s) = a^s \sum_{\ell=0}^s n^\ell \sum_{j=\ell}^s (-1)^{j-\ell} \begin{Bmatrix} s \\ j \end{Bmatrix} \begin{bmatrix} j \\ \ell \end{bmatrix} / \binom{m+aj/d}{m}.$$

Proof of Lemma 2. First we note that the factorial moments $e_{m,n}^{(s)}$ satisfy the same recurrence relations (7) as their ordinary counterparts $e_{m,n}^{[s]}$, only the initial condition changes to $e_{0,n}^{(s)} = E(X_{0,an}/a)_s = (n)_s$. We proceed as in the proof of Lemma 1 and obtain, by the ansatz $e_{m,n}^{(s)} = \sum_{k=0}^s \Lambda_{s,k,m}(n)_s$, the equation

$$(an + dm) \sum_{\ell=0}^s \Lambda_{s,\ell,m}(n)_\ell = an \sum_{\ell=0}^s \Lambda_{s,\ell,m}(n-1)_\ell + dm \sum_{\ell=0}^s \Lambda_{s,\ell,m-1}(n+p)_\ell. \quad (9)$$

Next we use the facts that

$$(an + dm)(n)_\ell = (a(n - \ell) + a\ell + dm)(n)_\ell = a(n)_{\ell+1} + a\ell + dm(n)_\ell,$$

$$n(n - 1)_\ell = (n)_{\ell+1},$$

and the binomial theorem for falling factorials

$$(a + b)_n = \sum_{k=0}^n \binom{n}{k} (a)_k (b)_{n-k}, \tag{10}$$

in order to write (9) as

$$\sum_{\ell=0}^s \Lambda_{s,\ell,m} (a(n)_{\ell+1} + (a\ell + dm)(n)_\ell)$$

$$= a \sum_{\ell=0}^s \Lambda_{s,\ell,m} (n)_{\ell+1} + dm \sum_{j=0}^s (n)_j \sum_{\ell=j}^s \binom{\ell}{j} (p)_{\ell-j} \Lambda_{s,\ell,m-1}.$$

The equation above simplifies to

$$\sum_{\ell=0}^s \Lambda_{s,\ell,m} (a\ell + dm)(n)_\ell = dm \sum_{j=0}^s (n)_j \sum_{\ell=j}^s \binom{\ell}{j} (p)_{\ell-j} \Lambda_{s,\ell,m-1}.$$

Comparing the coefficients of $(n)_\ell$, the falling factorial powers of n , we obtain the equations

$$\Lambda_{s,s,m}(as + dm) = dm \Lambda_{s,s,m-1}, \quad \ell = s,$$

$$\Lambda_{s,\ell,m}(as + dm) = dm \Lambda_{s,\ell,m-1} + dm \sum_{j=\ell+1}^s \binom{j}{\ell} (p)_{j-\ell} \Lambda_{s,j,m-1}, \quad 0 \leq \ell \leq s - 1. \tag{11}$$

Consequently, we obtain

$$\Lambda_{s,s,m} = \frac{m}{m + as/d} \Lambda_{s,s,m-1} \quad \text{and, further,} \quad \Lambda_{s,s,m} = \frac{1}{\binom{m+as/d}{m}}.$$

Moreover, we also obtain from (4.2) the stated recurrence relation for $\Lambda_{s,\ell,m}$:

$$\Lambda_{s,\ell,m} = \frac{m}{m + a\ell/d} \Lambda_{s,\ell,m-1} + \frac{m}{m + a\ell/d} \sum_{j=\ell+1}^s \binom{j}{\ell} (p)_{j-\ell} \Lambda_{s,j,m-1}.$$

In order to obtain the stated relation between $\lambda_{s,\ell,m}$ and $\Lambda_{s,j,m}$, $0 \leq \ell, j \leq s$, we use the expansion of the ordinary moments into factorial moments using the Stirling numbers of the first and second kinds:

$$\begin{aligned} E(X_{dm,an}^s) &= a^s E\left(\frac{X_{dm,an}}{a}\right)^s \\ &= a^s \sum_{r=0}^s \left\{ \begin{matrix} s \\ r \end{matrix} \right\} E\left(\frac{X_{dm,an}}{a}\right)_r \\ &= a^s \sum_{r=0}^s \left\{ \begin{matrix} s \\ r \end{matrix} \right\} \sum_{j=0}^r (n)_j \Lambda_{r,j,m} \end{aligned}$$

$$\begin{aligned}
 &= a^s \sum_{r=0}^s \binom{s}{r} \sum_{j=0}^r \Lambda_{r,j,m} \sum_{\ell=0}^j \binom{j}{\ell} (-1)^{j-\ell} n^\ell \\
 &= a^s \sum_{\ell=0}^s n^\ell \sum_{r=\ell}^s \binom{s}{r} \sum_{j=\ell}^r \Lambda_{r,j,m} \binom{j}{\ell} (-1)^{j-\ell}.
 \end{aligned}$$

On the other hand, $E(X_{dm,an}^s) = \sum_{\ell=0}^s n^\ell \lambda_{s,\ell,m}$, which proves the stated result.

Proof of Corollary 1. In the case of $c = ap = 0$, or, equivalently, $p = 0$, we obtain from Lemma 2 the result

$$e_{m,n}^{(s)} = \sum_{\ell=0}^s \Lambda_{s,\ell,m} (n)_\ell = (n)_s \Lambda_{s,s,m} = \frac{(n)_s}{\binom{m+as/d}{m}},$$

since all the terms $\Lambda_{s,\ell,m}$, $0 \leq \ell \leq s - 1$, are 0 due to the factor p :

$$\begin{aligned}
 \Lambda_{s,\ell,m} &= \frac{1}{\binom{m+a\ell/d}{m}} \sum_{k=0}^{m-1} \binom{k+a\ell/d}{k} \sum_{j=\ell+1}^s \binom{j}{\ell} (p)_{j-\ell} \Lambda_{s,j,k} \\
 &= \frac{p}{\binom{m+a\ell/d}{m}} \sum_{k=0}^{m-1} \binom{k+a\ell/d}{k} \sum_{j=\ell+1}^s \binom{j}{\ell} (p-1)_{j-\ell-1} \Lambda_{s,j,k}.
 \end{aligned}$$

This implies that the ordinary moments are explicitly given by

$$E(X_{dm,an}^s) = a^s \sum_{\ell=0}^s n^\ell \sum_{j=\ell}^s (-1)^{j-\ell} \binom{s}{j} \binom{j}{\ell} / \binom{m+aj/d}{m}.$$

4.3. The fine structure of the moments for $a/d > 1$

Next we are going to use Lemma 2 to obtain a refinement of the description of the factorial moments.

Proposition 1. For $a/d > 1$ and $c \neq 0$, the values $\Lambda_{s,\ell,m}$, arising in the expansion of the s th factorial moment $e_{m,n}^{(s)} = E((X_{dm,an}/a)_s) = \sum_{\ell=0}^s (n)_\ell \Lambda_{s,\ell,m}$ of the random variable $X_{dm,an}$, satisfy

$$\Lambda_{s,\ell,m} = \sum_{h=\ell}^s \frac{1}{\binom{m+ah/d}{m}} \sum_{g=0}^{h-\ell} \vartheta_{s,\ell;h,g} m^g,$$

where the values $\vartheta_{s,\ell;j,i}$ satisfy $\vartheta_{s,s;s,0} = 1$,

$$\vartheta_{s,\ell;\ell,0} = - \sum_{j=\ell+1}^s \binom{j}{\ell} (p)_{j-\ell} \sum_{h=j}^s \sum_{i=0}^{h-j} \vartheta_{s,j;h,i} q_i \left(\frac{ah}{d}, \frac{a\ell}{d} \right),$$

and, further, for $\ell + 1 \leq h \leq s - 1$ and $0 \leq g \leq h - \ell$,

$$\vartheta_{s,\ell;h,g} = \sum_{j=\ell+1}^{\min\{h,h-g+1\}} \binom{j}{\ell} (p)_{j-\ell} \sum_{i=\max\{0,g-1\}}^{h-j} \vartheta_{s,j;h,i} p_{i,g} \left(\frac{ah}{d}, \frac{a\ell}{d} \right).$$

The quantities $p_{i,\ell}(X, Y)$, $0 \leq \ell \leq i + 1$, and $q_i(X, Y)$ arising here will be defined in Lemma 3 below.

In order to give the proof of the result above, we need the following identity.

Lemma 3. *The sum*

$$\frac{1}{\binom{m+Y}{m}} \sum_{k=0}^{m-1} \frac{\binom{k+Y}{k}}{\binom{k+X}{k}} k^i$$

can be expanded as

$$\frac{1}{\binom{m+Y}{m}} \sum_{k=0}^{m-1} \frac{\binom{k+Y}{k}}{\binom{k+X}{k}} k^i = \sum_{\ell=0}^{i+1} \frac{m^\ell p_{i,\ell}(X, Y)}{\binom{m+X}{m}} - \frac{q_i(X, Y)}{\binom{m+Y}{m}},$$

assuming that $Y + h + 1 - X \neq 0$ for $0 \leq h \leq i, i \geq 0$, with

$$p_{i,\ell}(X, Y) = \sum_{j=\ell}^{i+1} \begin{bmatrix} j \\ \ell \end{bmatrix} (-1)^{j-\ell} \times \sum_{h=\max\{0, j-1\}}^i \frac{\binom{h+1}{j}(X)_{h+1-j}}{(Y+h+1-X)} \sum_{f=h}^i (-X)^{i-f} \begin{Bmatrix} i \\ f \end{Bmatrix} \begin{Bmatrix} f \\ h \end{Bmatrix}, \quad 0 \leq \ell \leq i+1,$$

$$q_i(X, Y) = \sum_{h=0}^i \sum_{f=h}^i (-X)^{i-f} \begin{Bmatrix} i \\ f \end{Bmatrix} \begin{Bmatrix} f \\ h \end{Bmatrix} \frac{(X)_{h+1}}{(Y+1+h-X)}.$$

Proof. We use the identity,

$$\sum_{k=0}^{m-1} \frac{\binom{k+Y}{k}}{\binom{k+X}{k}} (k+X)_j = \frac{\binom{m+Y}{m} (m+X)_{j+1}}{(j+1+Y-X) \binom{m+X}{m}} - \frac{(X)_{j+1}}{j+1+Y-X}, \tag{12}$$

with $j+1+X-Y \neq 0$ and $j \geq 0$, which can be proven using induction with respect to m . The identity was observed by the authors for small values of j using a computer algebra system, and then proved in general by induction. In order to apply the above result to the sum

$$\frac{1}{\binom{m+Y}{m}} \sum_{k=0}^{m-1} \frac{\binom{k+Y}{k}}{\binom{k+X}{k}} k^i,$$

we expand k^i in the following way:

$$\begin{aligned} k^i &= (k+X-X)^i \\ &= \sum_{f=0}^i \binom{i}{f} (-X)^{i-f} (k+X)^f \\ &= \sum_{f=0}^i \binom{i}{f} (-X)^{i-f} \sum_{h=0}^f \begin{Bmatrix} f \\ h \end{Bmatrix} (k+X)_h \\ &= \sum_{h=0}^i (k+X)_h \sum_{f=h}^i \binom{i}{f} \begin{Bmatrix} f \\ h \end{Bmatrix} (-X)^{i-f}. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} & \frac{1}{\binom{m+Y}{m}} \sum_{k=0}^{m-1} \frac{\binom{k+Y}{k}}{\binom{k+X}{k}} k^i \\ &= \sum_{h=0}^i \frac{1}{\binom{m+Y}{m}} \sum_{k=0}^{m-1} \frac{\binom{k+Y}{k}}{\binom{k+X}{k}} (k+X)_h \sum_{f=h}^i \binom{i}{f} \left\{ \begin{matrix} f \\ h \end{matrix} \right\} (-X)^{i-f} \\ &= \sum_{h=0}^i \left(\frac{(m+X)_{h+1}}{(h+1+Y-X)\binom{m+X}{m}} - \frac{(X)_{h+1}}{(h+1+Y-X)\binom{m+Y}{m}} \right) \sum_{f=h}^i \binom{i}{f} \left\{ \begin{matrix} f \\ h \end{matrix} \right\} (-X)^{i-f}. \end{aligned}$$

This proves the stated result for $q_i(X, Y)$. In order to obtain the expressions for $p_{i,\ell}(X, Y)$, $0 \leq \ell \leq i + 1$, we have to expand $(m+X-h)\binom{m+X}{h}$ in terms of ordinary powers of m . We use the binomial theorem for falling factorials (10), and also the expansion

$$(m)_j = \sum_{\ell=0}^j \begin{bmatrix} j \\ \ell \end{bmatrix} (-1)^{j-\ell} m^\ell.$$

Consequently, we obtain

$$\begin{aligned} (m+X)_{h+1} &= \sum_{j=0}^{h+1} \binom{h+1}{j} (m)_j (X)_{h+1-j} \\ &= \sum_{j=0}^{h+1} \binom{h+1}{j} \sum_{\ell=0}^j \begin{bmatrix} j \\ \ell \end{bmatrix} (-1)^{j-\ell} m^\ell (X)_{h+1-j} \\ &= \sum_{\ell=0}^{h+1} m^\ell \sum_{j=\ell}^{h+1} \binom{h+1}{j} \begin{bmatrix} j \\ \ell \end{bmatrix} (-1)^{j-\ell} (X)_{h+1-j}. \end{aligned}$$

Interchanging summations then leads directly to the stated results for $p_{i,\ell}(X, Y)$.

Proof of Proposition 1. We proceed by induction with respect to m and ℓ . We readily observe that $\Lambda_{s,s,m}$ satisfies the stated expansion. Assuming that the values $\Lambda_{s,j,k}$ have the stated expansion for all $k < m$ and $\ell + 1 \leq j \leq s$, we obtain, by Lemma 2 and the induction hypothesis,

$$\begin{aligned} \Lambda_{s,\ell,m} &= \frac{1}{\binom{m+a\ell/d}{m}} \sum_{k=0}^{m-1} \binom{k+a\ell/d}{k} \sum_{j=\ell+1}^s \binom{j}{\ell} (p)_{j-\ell} \Lambda_{s,j,k} \\ &= \sum_{j=\ell+1}^s \binom{j}{\ell} (p)_{j-\ell} \sum_{h=j}^s \sum_{i=0}^{h-j} \vartheta_{s,j;h,i} \frac{1}{\binom{m+a\ell/d}{m}} \sum_{k=0}^{m-1} \frac{\binom{k+a\ell/d}{k}}{\binom{k+ah/d}{k}} k^i. \end{aligned}$$

Before we can apply Lemma 3 we need to check that the conditions $Y+1+g-X \neq 0$ are satisfied for $Y = a\ell/d$, $X = ah/d$, and $0 \leq g \leq i$. We have $1 \leq 1+g \leq i+1 \leq h-j+1 \leq h-\ell$, with equality only in the case $j = \ell + 1$. Hence, from our assumption that $a/d > 1$ we obtain $1+g \leq h-\ell < a(h-\ell)/d$, such that $1+g-a(h-\ell)/d \neq 0$, $0 \leq g \leq i$. We obtain, by

Lemma 3,

$$\Lambda_{s,\ell,m} = \sum_{j=\ell+1}^s \binom{j}{\ell} (p)_{j-\ell} \sum_{h=j}^s \sum_{i=0}^{h-j} \vartheta_{s,j;h,i} \left(\sum_{g=0}^{i+1} \frac{m^g p_{i,g}(ah/d, a\ell/d)}{\binom{m+ah/d}{m}} - \frac{q_i(ah/d, a\ell/d)}{\binom{m+a\ell/d}{m}} \right).$$

This implies the stated result for $\vartheta_{s,\ell;\ell,0}$. Furthermore, we obtain

$$\begin{aligned} & \sum_{j=\ell+1}^s \binom{j}{\ell} (p)_{j-\ell} \sum_{h=j}^s \sum_{i=0}^{h-j} \vartheta_{s,j;h,i} \sum_{g=0}^{i+1} \frac{m^g p_{i,g}(ah/d, a\ell/d)}{\binom{m+ah/d}{m}} \\ &= \sum_{h=\ell+1}^s \frac{1}{\binom{m+ah/d}{m}} \sum_{g=0}^{h-\ell} m^g \sum_{j=\ell+1}^{\min\{h,h-g+1\}} \binom{j}{\ell} (p)_{j-\ell} \sum_{i=\max\{0,g-1\}}^{h-j} \vartheta_{s,j;h,i} p_{i,g} \left(\frac{ah}{d}, \frac{a\ell}{d} \right), \end{aligned}$$

which leads to the stated results.

4.4. Derivation of the expected value

Next we will derive the explicit expressions for the expectation of $X_{dm,an}$ using Lemma 1.

Proposition 2. *The expectation of $X_{dm,an}$ is given as*

$$E(X_{dm,an}) = \begin{cases} \frac{an}{m+1} + cH_m & \text{for } \frac{a}{d} = 1, \\ \frac{an}{\binom{m+a/d}{m}} + \frac{c}{d-a} \left(\frac{dm+a}{\binom{m+a/d}{m}} - a \right) & \text{for } \frac{a}{d} \neq 1. \end{cases}$$

Proof. In order to obtain the expected value of $X_{dm,an}$, we use Lemma 1 to obtain

$$E(X_{dm,an}) = \lambda_{1,1,m}n + \lambda_{1,0,m},$$

where the values $\lambda_{1,1,m}$ and $\lambda_{1,0,m}$ are given by

$$\lambda_{1,1,m} = \frac{a}{\binom{m+a/d}{m}}, \quad \lambda_{1,0,m} = \lambda_{1,0,m-1} + p^1 \lambda_{1,1,m}.$$

This implies that $\lambda_{1,0,m}$ can be written as

$$\lambda_{1,0,m} = p \sum_{k=0}^{m-1} \lambda_{1,k,m} = pa \sum_{k=0}^{m-1} \frac{1}{\binom{k+a/d}{k}}.$$

We have to distinguish between the cases $a/d = 1$ and $a/d \neq 1$. First, assume that $a/d = 1$. We obtain

$$\sum_{k=0}^{m-1} \frac{1}{\binom{k+a/d}{k}} = \sum_{k=0}^{m-1} \frac{1}{\binom{k+1}{k}} = \sum_{k=0}^{m-1} \frac{1}{k+1} = H_m,$$

and, further, $E(X_{dm,an}) = an/(m+1) + cH_m$. For the case $a/d \neq 1$, we use the summation formula

$$\sum_{k=0}^{m-1} \frac{1}{\binom{k+X}{X}} = \frac{m+X}{(1-X)\binom{m+X}{X}} - \frac{X}{1-X},$$

which can be easily deduced from another summation formula, namely (see [7, p. 188]),

$$\sum_{\ell=0}^k \binom{k}{\ell} \frac{(-1)^\ell}{X + \ell} = \frac{1}{X \binom{k+X}{k}},$$

using the fact that $\sum_{k=0}^{m-1} \binom{k}{\ell} = \binom{m}{\ell+1}$. We obtain

$$\sum_{k=0}^{m-1} \frac{1}{\binom{k+a/d}{k}} = \frac{dm + a}{(d - a) \binom{m+a/d}{m}} - \frac{a}{d - a},$$

which directly leads to the result

$$E(X_{dm,an}) = \frac{an}{\binom{m+a/d}{m}} + \frac{c(dm + a)}{(d - a) \binom{m+a/d}{m}} - \frac{ca}{d - a}.$$

4.5. Asymptotic expansions of the expected value

Next we derive asymptotic expansions of the expected value $E(X_{dm,an})$ for $\max\{m, n\} \rightarrow \infty$. These expansions serve as an indicator for the normalizations used in Theorems 1–3 for the random variables $X_{dm,an}$.

Lemma 4. *For $m \in \mathbb{N}$ fixed, $n \rightarrow \infty$, and arbitrary a, c , and d , the expected value of $X_{dm,an}$, as given in Theorem 2, is asymptotically given by*

$$E(X_{dm,an}) = \frac{an}{\binom{m+a/d}{m}} + \mathcal{O}(1).$$

For $m \rightarrow \infty$, $a/d \leq 1$, $c \neq 0$, and arbitrary $n = n(m)$, the expected value of $X_{dm,an}$ always tends to ∞ :

$$E(X_{dm,an}) \sim \begin{cases} \frac{an}{m} + c \log m & \text{for } \frac{a}{d} = 1, \\ \Gamma\left(1 + \frac{a}{d}\right) \frac{an + cdm/(d - a)}{m^{a/d}} & \text{for } \frac{a}{d} < 1. \end{cases}$$

For $m \rightarrow \infty$, $a/d > 1$, or $a/d \leq 1$ together with $c = 0$, we have the following three regions in the asymptotic behavior of the expected value of $X_{dm,an}$.

- For $m^{a/d} = o(n)$, we have

$$E(X_{dm,an}) = \Gamma\left(1 + \frac{a}{d}\right) \frac{an}{m^{a/d}} + \mathcal{O}\left(\frac{n}{m^{a/d+1}} + \frac{c}{m^{a/d-1}} + 1\right).$$

- For $n \sim \rho m^{a/d}$, with $\rho \in \mathbb{R}^+$, we have

$$E(X_{dm,an}) = a\Gamma\left(1 + \frac{a}{d}\right)\rho + \frac{ca}{a - d} + \mathcal{O}\left(\frac{1}{m} + \frac{c}{m^{a/d-1}} + \frac{1}{m^{a/d}}\right).$$

- For $n = n(m)$ such that $n = o(m^{a/d})$, we have

$$E(X_{dm,an}) = \frac{ca}{a - d} + \mathcal{O}\left(\frac{n}{m^{a/d}} + \frac{c}{m^{a/d-1}}\right). \tag{13}$$

Remark 4. Our results above say in principle that the asymptotic behavior of $X_{dm,an}$ is governed by the quotient a/d , together with the (non)positivity of c . A similar situation occurs in tenable triangular urn schemes; compare with [11]. A simple explanation for the above results is as follows. For fixed m and n tending to ∞ , the actual values of a , d , and c are irrelevant. For m tending to ∞ such that $a/d < 1$ and $c \neq 0$, the positivity of c ensures that the random variable always tends to ∞ . In the remaining cases with m tending to ∞ such that $a/d > 1$, or $c = 0$ and arbitrary a, d , the random variable $X_{dm,an}$ can be rather small, depending on the growth of $n = n(m)$ compared to m .

Proof of Lemma 4. We use the explicit results stated in Theorem 2. For fixed m and $n \rightarrow \infty$, we use Stirling’s formula for the gamma function, i.e.

$$\Gamma(z) = \left(\frac{z}{e}\right)^z \frac{\sqrt{2\pi}}{\sqrt{z}} \left(1 + \frac{1}{12z} + \frac{1}{288z^2} + \mathcal{O}\left(\frac{1}{z^3}\right)\right), \tag{14}$$

and obtain, for arbitrary a, c , and d , the expansion

$$\begin{aligned} E(X_{dm,an}) &= \frac{an}{\binom{m+a/d}{m}} + \mathcal{O}(1) \\ &= an \frac{\Gamma(m+1)\Gamma(1+a/d)}{\Gamma(m+1+a/d)} + \mathcal{O}(1) \\ &= \Gamma\left(1 + \frac{a}{d}\right) \frac{an}{m^{a/d}} + \mathcal{O}(1). \end{aligned}$$

Next assume that $m \rightarrow \infty, a/d \leq 1$, and $c \neq 0$, and that we have arbitrary $n = n(m)$. We use the asymptotic expansion of the harmonic numbers

$$H_m = \log m + \gamma - \frac{1}{2m} + \mathcal{O}\left(\frac{1}{m^2}\right),$$

where $\gamma = 0.577\,215\,664\,9\dots$ denotes the Euler–Mascheroni constant, to obtain, for $a/d = 1$, the result

$$E(X_{dm,an}) = \frac{an}{m+1} + cH_m \sim \frac{an}{m} + c \log m.$$

For $a/d < 1$, we obtain

$$\begin{aligned} E(X_{dm,an}) &= \frac{an}{\binom{m+a/d}{m}} + \frac{c}{d-a} \left(\frac{dm+a}{\binom{m+a/d}{m}} - a\right) \\ &\sim \Gamma\left(1 + \frac{a}{d}\right) \frac{an}{m^{a/d}} + \frac{cd}{d-a} \Gamma\left(1 + \frac{a}{d}\right) m^{1-a/d}, \end{aligned}$$

where we have again used Stirling’s formula (14). For $m \rightarrow \infty, a/d > 1$, or $a/d \leq 1$ together with $c = 0$, we proceed similarly to the previous cases. For example, assuming that $m^{a/d} = o(n)$, we obtain, by Stirling’s formula (14), the result

$$\begin{aligned} E(X_{dm,an}) &= \frac{an}{\binom{m+a/d}{m}} + \frac{c}{d-a} \left(\frac{dm+a}{\binom{m+a/d}{m}} - a\right) \\ &= \Gamma\left(1 + \frac{a}{d}\right) \frac{an}{m^{a/d}} \left(1 + \mathcal{O}\left(\frac{1}{m}\right)\right) + \mathcal{O}\left(\frac{c}{m^{a/d-1}} + 1\right). \end{aligned}$$

5. Derivation of the limiting distributions

In the following we will present our proofs of Theorems 1–3. First we prove simultaneously the limit laws of Theorems 1–3 in the case of fixed m and n tending to ∞ . Then, we separately provide the remaining proofs of Theorems 1–3 for m tending to ∞ and $n = n(m)$ in Subsections 5.2, 5.3, and 5.4.

5.1. The case of fixed m

We assume that m is an arbitrary but fixed natural number, and derive the limit of $X_{dm,an}$ for n tending to ∞ . Using Lemma 1, we can expand the s th moment of $X_{dm,an}$ for arbitrary values of $a, d, c \in \mathbb{N}$ in powers of n as

$$\begin{aligned} E(X_{dm,an}^s) &= e_{m,n}^{[s]} \\ &= \sum_{k=0}^s \lambda_{s,k,m} n^k \\ &= \lambda_{s,s,m} n^s + \sum_{k=0}^{s-1} \lambda_{s,k,m} n^k \\ &= \lambda_{s,s,m} n^s + \mathcal{O}(n^{s-1}) \\ &= \frac{a^s n^s}{\binom{m+as/d}{m}} + \mathcal{O}(n^{s-1}), \end{aligned}$$

since we assumed that m is an arbitrary but fixed natural number. Consequently, the moments of the normalized random variable $X_{dm,an}/(an)$ satisfy the following asymptotic expansion:

$$E\left(\frac{X_{dm,an}^s}{a^s n^s}\right) = \frac{e_{m,n}^{[s]}}{a^s n^s} = \frac{1}{\binom{m+as/d}{m}} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) = \frac{m! \Gamma(1 + as/d)}{\Gamma(1 + m + as/d)} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right).$$

Hence, the s th moment of the scaled random variable $X_{dm,an}/(an)$ tends to the s th moment of a Kuramaswamy-distributed random variable $K = K(d/a, m)$ with parameters $\alpha = d/a$ and $\beta = m$ for any $s \geq 1$, i.e. $E(X_{dm,an}^s/(a^s n^s)) \rightarrow E(K^s)$. The theorem of Fréchet and Shohat, see [19], states that the moment convergence implies the convergence in distribution, if the moments sequence determines a unique distribution. Hence, we obtain the convergence in distribution of $X_{dm,an}/(an)$ to K .

5.2. Proof of Theorem 1

We use the explicit results for the moments of $X_{dm,an}$ in the case $c = 0$ and arbitrary a, d stated in Corollary 1:

$$E(X_{dm,an}^s) = a^s \sum_{\ell=0}^s n^\ell \sum_{j=\ell}^s (-1)^{j-\ell} \left\{ \begin{matrix} s \\ j \end{matrix} \right\} \left[\begin{matrix} j \\ \ell \end{matrix} \right] / \binom{m + aj/d}{m}.$$

Hence, using Stirling’s formula (14) we obtain, for $m \rightarrow \infty$, the asymptotic expansions

$$\frac{1}{\binom{m+aj/d}{m}} \sim \frac{\Gamma(1 + aj/d)}{m^{aj/d}}, \quad E(X_{dm,an}^s) \sim a^s \sum_{\ell=0}^s n^\ell \sum_{j=\ell}^s (-1)^{j-\ell} \frac{\Gamma(1 + aj/d) \left\{ \begin{matrix} s \\ j \end{matrix} \right\} \left[\begin{matrix} j \\ \ell \end{matrix} \right]}{m^{aj/d}}. \tag{15}$$

Assume first that $m, n \rightarrow \infty$ such that $m^{a/d} = o(n)$. The dominant term in the expansion above is given by $a^s n^s \Gamma(1 + as/d)/m^{as/d}$, and we obtain

$$E\left(\frac{X_{dm,an}^s m^{as/d}}{a^s n^s}\right) \sim \Gamma\left(1 + \frac{as}{d}\right).$$

Hence, in the region $m, n \rightarrow \infty$ such that $m^{a/d} = o(n)$, we can use the moment convergence theorem of Fréchet and Shohat, and obtain the convergence in distribution of $X_{dm,an} m^{a/d}/(an)$ to $W(d/a, 1)$ for $a/d \leq 1$.

For $m, n \rightarrow \infty$ such that $n \sim \rho m^{a/d}$, with $\rho \in \mathbb{R}^+$, we obtain from (15) the expansion

$$E(X_{dm,an}^s) \sim a^s \sum_{\ell=0}^s (\rho m^{a/d})^\ell \frac{\Gamma(1 + a\ell/d) \{s\}_\ell}{m^{a\ell/d}} = a^s \sum_{\ell=0}^s \left\{ \begin{matrix} s \\ \ell \end{matrix} \right\} \rho^\ell \Gamma\left(1 + \frac{a\ell}{d}\right).$$

Consequently,

$$\lim_{m,n \rightarrow \infty} E\left(\frac{X_{dm,an}^s}{a^s}\right) = \eta_s, \quad \text{where } \eta_s := \sum_{\ell=0}^s \left\{ \begin{matrix} s \\ \ell \end{matrix} \right\} \rho^\ell \Gamma\left(1 + \frac{a\ell}{d}\right).$$

Hence, by (4), there exists a discrete distribution with the moment sequence as stated above. Moreover, for $a/d \leq 1$, we know that the moment sequence determines a unique distribution, with probability mass function as given in (4).

In the remaining case $m \rightarrow \infty$, with arbitrary $n = n(m)$ satisfying that $n = o(m^{a/d})$, we obtain $E(X_{dm,an}^s) \rightarrow 0$ for all $s \geq 1$, which proves the stated result.

5.3. Proof of Theorem 2

The limiting distributions of $X_{dm,an}$ for $a/d \leq 1$, $c \neq 0$, and $m \rightarrow \infty$, will be obtained by giving precise estimates for the s th moments $e_{m,n}^{[s]}$. Lemma 4 suggests the right scaling factors $g_{m,n}$ chosen according to the ratio a/d . We will provide the following estimates for the moments of $X_{dm,an}$:

$$e_{m,n}^{[s]} = \begin{cases} \Gamma\left(1 + \frac{as}{d}\right) \left(\frac{an + mcd/(d-a)}{m^{a/d}}\right)^s \left(1 + \mathcal{O}\left(\frac{1}{m^{a/d}}\right)\right) & \text{for } \frac{a}{d} < 1, \\ s! \left(\frac{an}{m} + c \log m\right)^s \left(1 + \mathcal{O}\left(\frac{1}{\log m}\right)\right) & \text{for } \frac{a}{d} = 1. \end{cases}$$

Note that the above expansions will imply the limiting distribution results by applying the method of moments, i.e.

$$E\left(\frac{X_{dm,an}^s}{g_{m,n}^s}\right) = \frac{e_{m,n}^{[s]}}{g_{m,n}^s} = \begin{cases} \Gamma\left(1 + \frac{as}{d}\right) \left(1 + \mathcal{O}\left(\frac{1}{m^{a/d}}\right)\right) & \text{for } \frac{a}{d} < 1, \\ s! \left(1 + \mathcal{O}\left(\frac{1}{\log m}\right)\right) & \text{for } \frac{a}{d} = 1, \end{cases}$$

since the s th moment of $X_{dm,an}/g_{m,n}$ converges to the s th moment of a Weibull distribution with suitably chosen parameters.

First we consider the case $a/d < 1$. Since we want to prove the asymptotic expansion

$$e_{m,n}^{[s]} = \Gamma\left(1 + \frac{as}{d}\right) \left(\frac{an + mcd/(d-a)}{m^{a/d}}\right)^s \left(1 + \mathcal{O}\left(\frac{1}{m^{a/d}}\right)\right),$$

we have to determine the asymptotic growth of the coefficients $\lambda_{s,\ell,m}$ appearing in the recursive description of the moments $e_{m,n}^{[s]}$ in Lemma 1. The shape of the s th moment $e_{m,n}^{[s]} = \sum_{\ell=0}^s \lambda_{s,\ell,m} n^\ell$ implies that we have to show the following asymptotic expansion of the numbers $\lambda_{s,\ell,m}$:

$$\lambda_{s,\ell,m} = a^\ell \Gamma\left(1 + \frac{as}{d}\right) \binom{s}{\ell} m^{s-\ell-as/d} \frac{(cd)^{s-\ell}}{(d-a)^{s-\ell}} \left(1 + \mathcal{O}\left(\frac{1}{m^{a/d}}\right)\right).$$

To show this, we will use induction with respect to ℓ and apply Euler’s summation formula. The statement is true for $\ell = s$, since we know by Lemma 1 that

$$\lambda_{s,s,m} = \frac{a^s}{(m+as/d)} = a^s \Gamma\left(1 + \frac{as}{d}\right) \binom{s}{s} m^{as/d} \left(1 + \mathcal{O}\left(\frac{1}{m^{a/d}}\right)\right).$$

Using the induction hypothesis for $\ell + 1$ up to $s - 1$, we see that the dominant contribution to $\mu_{s,\ell,m}$ is stemming from the term $\lambda_{s,\ell+1,m-1}$, and we obtain

$$\mu_{s,\ell,m} \sim \binom{\ell+1}{\ell} \frac{dm}{dm+al} p \lambda_{s,\ell+1,m-1}. \tag{16}$$

Owing to Lemma 1 we also have

$$\lambda_{s,\ell,m} = \sum_{k=0}^{m-1} \frac{\binom{m}{k}}{\binom{m+al/d}{k}} \mu_{s,\ell,m-k} = \frac{m!}{\Gamma(m+al/d+1)} \sum_{k=1}^m \frac{\Gamma(k+al/d+1)}{k!} \mu_{s,\ell,k}. \tag{17}$$

Using the induction hypothesis, we obtain the approximation

$$\begin{aligned} \sum_{k=1}^m \frac{\Gamma(k+al/d+1)}{k!} \mu_{s,\ell,k} &= a^{\ell+1} \Gamma\left(1 + \frac{as}{d}\right) \binom{\ell+1}{\ell} \binom{s}{\ell+1} p \frac{(cd)^{s-\ell}}{(d-a)^{s-\ell}} \\ &\times \sum_{k=\lceil \log m \rceil}^m \frac{\Gamma(k+al/d) k^{s-\ell-as/d}}{k!} \left(1 + \mathcal{O}\left(\frac{1}{m^{a/d}}\right)\right). \end{aligned}$$

Now an application of Euler’s summation formula (see, e.g. [7, p. 469]) leads to

$$\begin{aligned} \lambda_{s,\ell,m} &= a^\ell c \Gamma\left(1 + \frac{as}{d}\right) \binom{\ell+1}{\ell} \binom{s}{\ell+1} m^{s-\ell-as/d} \frac{(cd)^{s-\ell-1}}{(d-a)^{s-\ell-1}} \\ &\times \int_0^1 t^{s-\ell-1-a(s-\ell)/d} dt \left(1 + \mathcal{O}\left(\frac{1}{m^{a/d}}\right)\right) \\ &= a^\ell c \Gamma\left(1 + \frac{as}{d}\right) \binom{s}{\ell+1} \binom{\ell+1}{\ell} m^{s-\ell-as/d} c \frac{(cd)^{s-\ell-1}}{(d-1)^{s-\ell-1}(s-\ell-a(s-\ell)/d)} \\ &\times \left(1 + \mathcal{O}\left(\frac{1}{m^{a/d}}\right)\right) \\ &= a^\ell \Gamma\left(1 + \frac{as}{d}\right) \binom{s}{\ell} m^{s-\ell-as/d} \frac{(cd)^{s-\ell}}{(d-a)^{s-\ell}} \left(1 + \mathcal{O}\left(\frac{1}{m^{a/d}}\right)\right), \end{aligned}$$

which proves the stated result for $0 < \ell \leq s$. For $\ell = 0$, we have, owing to Lemma 1,

$$\lambda_{s,0,\ell} = \sum_{k=0}^{m-1} \sum_{i=1}^s p^i \lambda_{s,i,k} = p \sum_{k=0}^{m-1} \lambda_{s,1,k} \left(1 + \mathcal{O}\left(\frac{1}{m^{a/d}}\right)\right).$$

Using Euler’s summation formula and the fact that $c = ap$, we obtain

$$\begin{aligned} \lambda_{s,0,\ell} &= pa\Gamma\left(1 + \frac{as}{d}\right) \binom{s}{1} \frac{(cd)^{s-1}}{(d-a)^{s-1}} \int_0^m t^{s-1-as/d} dt \left(1 + \mathcal{O}\left(\frac{1}{m^{a/d}}\right)\right) \\ &= s\Gamma\left(1 + \frac{as}{d}\right) m^{s-as/d} \frac{c^s d^{s-1}}{(d-a)^{s-1}(s-sa/d)} \left(1 + \mathcal{O}\left(\frac{1}{m^{a/d}}\right)\right) \\ &= \Gamma\left(1 + \frac{as}{d}\right) m^{s-as/d} \frac{(cd)^s}{(d-a)^s} \left(1 + \mathcal{O}\left(\frac{1}{m^{a/d}}\right)\right). \end{aligned}$$

This completes the proof of Theorem 2 for $a/d < 1$.

Now we consider the remaining case $a/d = 1$. We have to prove that

$$e_{m,n}^{[s]} = s! \left(\frac{an}{m} + c \log m\right)^s \left(1 + \mathcal{O}\left(\frac{1}{\log m}\right)\right).$$

This implies that we have to show the following asymptotic expansion of the numbers $\lambda_{s,\ell,m}$:

$$\lambda_{s,\ell,m} = s! \binom{s}{\ell} \frac{a^\ell c^{s-\ell} (\log m)^{s-\ell}}{m^\ell} \left(1 + \mathcal{O}\left(\frac{1}{\log m}\right)\right).$$

We proceed exactly as in the previous case $a/d < 1$. Using (16) and (17), we finally obtain

$$\begin{aligned} \lambda_{s,\ell,m} &= a^{\ell+1} p c^{s-\ell-1} s! \binom{\ell+1}{\ell} \binom{s}{\ell+1} \frac{m!}{(m+\ell)!} \int_1^m \frac{(\log t)^{s-\ell-1}}{t} dt \left(1 + \mathcal{O}\left(\frac{1}{\log m}\right)\right) \\ &= a^\ell c^{s-\ell} s! \binom{\ell+1}{\ell} \binom{s}{\ell+1} \frac{1}{m^\ell} \frac{(\log(m))^{s-\ell}}{s-\ell} \left(1 + \mathcal{O}\left(\frac{1}{\log m}\right)\right) \\ &= s! \binom{s}{\ell} \frac{a^\ell c^{s-\ell} (\log m)^{s-\ell}}{m^\ell} \left(1 + \mathcal{O}\left(\frac{1}{\log m}\right)\right), \end{aligned}$$

which proves the stated result for $0 < \ell \leq s$. The remaining case $\ell = 0$ is treated in a fully analogous manner. Hence, for $m \rightarrow \infty$, the limiting distribution is given by an exponential distribution with parameter 1, which also proves the part $a/d = 1$ of Theorem 2.

5.4. Proof of Theorem 3

We use the results of Lemma 2, and (12), to study the moments

$$E(X_{dm,an}^s) = a^s \sum_{\ell=0}^s n^\ell \sum_{r=\ell}^s \sum_{j=\ell}^r (-1)^{j-\ell} \left\{ \begin{matrix} s \\ r \end{matrix} \right\} \left[\begin{matrix} j \\ \ell \end{matrix} \right] \sum_{h=j}^r \frac{1}{\binom{m+ah/d}{m}} \sum_{g=0}^{h-j} \vartheta_{r,j;h,g} m^g$$

for $m \rightarrow \infty$. Interchanging the summations gives

$$E(X_{dm,an}^s) = a^s \sum_{\ell=0}^s n^\ell \sum_{h=\ell}^s \frac{1}{\binom{m+ah/d}{m}} \sum_{g=0}^{h-\ell} m^g \sum_{r=h}^s \sum_{j=g+\ell}^h (-1)^{j-\ell} \left\{ \begin{matrix} s \\ r \end{matrix} \right\} \left[\begin{matrix} j \\ \ell \end{matrix} \right] \vartheta_{r,j;h,g}.$$

Proceeding as in Subsection 5.2 we use the expansions $1/\binom{m+ah/d}{m} \sim \Gamma(1+ah/d)/m^{ah/d}$, and obtain

$$E(X_{dm,an}^s) \sim a^s \sum_{\ell=0}^s n^\ell \sum_{h=\ell}^s \frac{\Gamma(1+ah/d)}{m^{ah/d}} \sum_{g=0}^{h-\ell} m^g \sum_{r=h}^s \sum_{j=g+\ell}^h (-1)^{j-\ell} \left\{ \begin{matrix} s \\ r \end{matrix} \right\} \left[\begin{matrix} j \\ \ell \end{matrix} \right] \vartheta_{r,j;h,g}.$$

First we consider the case $m, n \rightarrow \infty$ such that $m^{a/d} = o(n)$, and directly obtain

$$E(X_{dm,an}^s) \sim a^s \frac{n^s}{m^{as/d}} \Gamma\left(1 + \frac{as}{d}\right), \quad E\left(\frac{m^{as/d} X_{dm,an}^s}{a^s n^s}\right) \sim \Gamma\left(1 + \frac{as}{d}\right).$$

We again use the moment convergence theorem of Fréchet and Shohat and obtain the convergence in distribution of $X_{dm,an} m^{a/d}/(an)$ to the Weibull-distributed random variable $W(d/a, 1)$.

Next assume that $m, n \rightarrow \infty$ such that $n \sim \rho m^{a/d}$, with $\rho \in \mathbb{R}^+$. We have

$$\begin{aligned} E(X_{dm,an}^s) &\sim a^s \sum_{\ell=0}^s (\rho m^{a/d})^\ell \sum_{h=\ell}^s \frac{\Gamma(1 + ah/d)}{m^{ah/d}} \sum_{g=0}^{h-\ell} m^g \sum_{r=h}^s \sum_{j=g+\ell}^h (-1)^{j-\ell} \begin{Bmatrix} s \\ r \end{Bmatrix} \begin{Bmatrix} j \\ \ell \end{Bmatrix} \vartheta_{r,j;h,g} \\ &\sim a^s \sum_{\ell=0}^s \rho^\ell \Gamma\left(1 + \frac{a\ell}{d}\right) \sum_{r=\ell}^s \begin{Bmatrix} s \\ r \end{Bmatrix} \vartheta_{r,\ell;\ell,0}. \end{aligned}$$

It seems difficult to obtain suitable bounds on $\vartheta_{r,\ell;\ell,0}$ in order to prove that the moment sequence determines a unique distribution, which is necessary to apply the theorem of Fréchet and Shohat.

In the remaining case $n = o(m^{a/d})$, only the constant term being independent of n and m , case $\ell = h = g = 0$, in the expansion of $E(X_{dm,an}^s)$ is relevant, and we obtain

$$E(X_{dm,an}^s) \sim a^s \sum_{r=0}^s \begin{Bmatrix} s \\ r \end{Bmatrix} \vartheta_{r,0;0,0}.$$

Note that this expansion is consistent with $\rho = 0$ in the case considered before. Unfortunately, again we are not able to show that the moment sequence determines a unique distribution.

6. Generalization: a biased Pólya–Eggenberger urn model

In the ordinary Pólya–Eggenberger urn model at every step a ball is chosen at random from the urn. For example, if the urn contains n white and m black balls, the probability of choosing a white ball is given by $n/(m + n)$, whereas the probability of choosing a black ball is given by $m/(m + n)$. We now consider a biased Pólya–Eggenberger urn model defined as follows. Starting with an urn with ball replacement matrix

$$M = \begin{pmatrix} -1 & 0 \\ c & -1 \end{pmatrix},$$

we associate with the states of the urn a sequence P of positive real numbers $P = (p_m)_{m \in \mathbb{N}_0}$, with $p_0 = 0$ and $p_m \in \mathbb{R}^+$, where P is independent of n . For the sake of simplicity, we have chosen $a = d = 1$ in M . The cases $d > 1$ or $a > 1$ (with $c = pa$) can be reobtained by properly choosing the sequence $P = (p_m)_{m \in \mathbb{N}_0} = (dm/a)_{m \in \mathbb{N}_0}$. Assuming that the urn contains n white and m black balls, for this class of biased diminishing urns, the probability of choosing a white ball is given by $n/(n + p_m)$, whereas the probability of choosing a black ball is given by $p_m/(n + p_m)$. Let $X_{m,n}$ denote the random variable that represents the number of white balls remaining in the urn when all the black balls have been removed. By definition we have the following recurrence for $P\{X_{m,n} = k\}$:

$$P\{X_{m,n} = k\} = \frac{n}{n + p_m} P\{X_{m,n-1} = k\} + \frac{p_m}{n + p_m} P\{X_{m-1,n+c} = k\},$$

with initial values $P\{X_{0,n} = n\} = 1$ for $n \in \mathbb{N}_0$. We also have the following recurrence for the moments $e_{m,n}^{[s]} = E(X_{m,n}^s)$:

$$e_{m,n}^{[s]} = \frac{n}{n + p_m} e_{m,n-1,m}^{[s]} + \frac{p_m}{n + p_m} e_{m-1,n+c}^{[s]}$$

with initial values $e_{0,n}^{[s]} = n^s$ for $n \in \mathbb{N}_0$. Obviously, the recurrence for the moment sequence is almost identical to the previous recurrence (7). This suggests that, as before, the s th moment is again a polynomial of degree s in n , with coefficients depending only on m . The next result makes this precise—we recursively determine the moments of $X_{m,n}$ for a given sequence $P = (p_m)_{m \in \mathbb{N}_0}$, and also obtain an alternative description for the factorial moments of $X_{m,n}$, similar to Lemmas 1–2.

Proposition 3. *The s th moment $e_{m,n}^{[s]} = E(X_{m,n}^s)$ of the random variable $X_{m,n}$ satisfies the expansion $e_{m,n}^{[s]} = \sum_{\ell=0}^s \lambda_{s,\ell,m} n^\ell$. The values $\lambda_{s,\ell,m}$ are recursively given by*

$$\lambda_{s,s,m} = \prod_{k=1}^m \frac{p_k}{p_k + s} \quad \text{and} \quad \lambda_{s,\ell,m} = \sum_{k=0}^{m-1} \mu_{s,\ell,m-k} \prod_{j=m+1-k}^m \frac{p_j}{p_j + k}$$

where

$$\mu_{s,\ell,m} := \frac{1}{p_m + \ell} \sum_{k=\ell+1}^s \binom{k}{\ell-1} (-1)^{k-\ell-1} \lambda_{s,k,m} + \frac{p_m}{p_m + \ell} \sum_{k=\ell+1}^s \binom{k}{\ell} c^{k-\ell} \lambda_{s,k,m-1}$$

For $\ell = 0$, we have

$$\lambda_{s,0,m} = \sum_{k=0}^{m-1} \mu_{s,0,k}, \quad \text{with} \quad \mu_{s,0,m} := \sum_{k=1}^s \lambda_{s,k,m} c^k$$

The initial values are given by $\lambda_{s,s,0} = 1$ and $\lambda_{s,\ell,0} = 0$ for $0 \leq \ell \leq s - 1$.

Furthermore, the s th factorial moment

$$e_{m,n}^{(s)} = E((X_{m,n})_s) = E(X_{m,n}(X_{m,n} - 1) \cdots (X_{m,n} - s + 1))$$

of the random variable $X_{m,n}$ satisfies the expansion $e_{m,n}^{(s)} = \sum_{k=0}^s \Lambda_{s,k,m}(n)_k$. The values $\Lambda_{s,k,m}$ are recursively given by $\Lambda_{s,s,m} = \prod_{k=1}^m p_k / (p_k + s)$ and

$$\Lambda_{s,\ell,m} = \left(\prod_{h=1}^m \frac{p_h}{p_h + \ell} \right) \sum_{k=0}^{m-1} \left(\prod_{h=1}^k \frac{p_h}{p_h + \ell} \right) \sum_{j=\ell+1}^s \binom{j}{\ell} (c)_{j-\ell} \Lambda_{s,j,k-1}$$

with initial values given by $\Lambda_{s,s,0} = 1$ and $\Lambda_{s,\ell,0} = 0$ for $0 \leq \ell \leq s - 1$.

The proof of the above result is fully analogous to the proofs of Lemmas 1 and 2, and is therefore omitted. We refrain from studying this new generalized urn problem in full generality, and only state the following immediate consequences.

Corollary 2. *In the biased urn model with $c = 0$ the factorial moments $E((X_{m,n})_s) = E(X_{m,n}(X_{m,n} - 1) \cdots (X_{m,n} - s + 1))$ of the random variable $X_{m,n}$ are given by*

$$E((X_{m,n})_s) = (n)_s \prod_{k=1}^m \frac{p_k}{p_k + s}$$

Consequently, for $c = 0$ and any given sequence $P = (p_m)_{m \in \mathbb{N}_0}$ satisfying $\sum_{m \geq 1} 1/p_m < \infty$, we obtain the following limiting distribution results.

- For m fixed and $n \rightarrow \infty$, the normalized random variable $X_{m,n}/n$ converges in distribution to a random variable V_m , with convergence of all moments:

$$\frac{X_{m,n}}{n} \xrightarrow{\mathcal{L}} V_m, \quad \mathbb{E}\left(\frac{X_{m,n}^s}{n^s}\right) \rightarrow \mathbb{E}(V_m^s) = \prod_{k=1}^m \frac{p_k}{p_k + s}.$$

Moreover, V_m can be written as the exponential of a weighted sum of m independent exponential random variables $\varepsilon_k \stackrel{\mathcal{L}}{=} \text{Exp}(1)$:

$$V_m = \exp\left(-\sum_{k=1}^m \frac{\varepsilon_k}{p_k}\right).$$

- For n fixed and $m \rightarrow \infty$, the random variable $X_{m,n}$ converges in distribution to a random variable Z_n , with convergence of all moments:

$$X_{m,n} \xrightarrow{\mathcal{L}} Z_n, \quad \mathbb{E}(X_{m,n}^s) \rightarrow \mathbb{E}(Z_n^s) = \sum_{j=0}^s (n)_j \left\{ \begin{matrix} s \\ j \end{matrix} \right\} \prod_{k=1}^{\infty} \frac{p_k}{p_k + j}.$$

- For $\min\{m, n\} \rightarrow \infty$, the normalized random variable $X_{m,n}/n$ converges in distribution to a random variable W , with convergence of all moments:

$$\frac{X_{m,n}}{n} \xrightarrow{\mathcal{L}} W, \quad \mathbb{E}\left(\frac{X_{m,n}^s}{n^s}\right) \rightarrow \mathbb{E}(W^s) = \prod_{k=1}^{\infty} \frac{p_k}{p_k + s}.$$

Moreover, W can be written as the exponential of a series of independent exponential random variables $\varepsilon_k \stackrel{\mathcal{L}}{=} \text{Exp}(1)$:

$$W = \exp\left(-\sum_{k=1}^{\infty} \frac{\varepsilon_k}{p_k}\right).$$

Moreover, the random variables Z_n/n and V_m both converge in distribution to W , with convergence of all moments, for n and m tending to ∞ .

Note that related questions (and random variables) have been studied in the literature, for example, in the context of random walks [4], or probability laws related to the Jacobi theta and Riemann zeta functions [1], but questions directly related to the distribution of $X_{m,n}$ were not considered before, to the best of the authors' knowledge.

Proof of Corollary 2. In the case $c = 0$ the factorial moments $e_{m,n}^{(s)}$ of $X_{m,n}$ are given by

$$e_{m,n}^{(s)} = \sum_{k=0}^s \Lambda_{s,k,m}(n)_k = (n)_s \Lambda_{s,s,m} = (n)_s \prod_{\ell=1}^m \frac{p_\ell}{p_\ell + s},$$

since the values $\Lambda_{s,k,m}$, $0 \leq k \leq s - 1$, all have a factor c . The assumption $\sum_{m \geq 1} 1/p_m < \infty$ on the sequence P ensures that the product $\prod_{\ell=1}^m p_\ell/(p_\ell + s)$ converges for $s \geq 1$ and m tending to ∞ . Consequently, the limiting distributions are obtained in a straightforward way

using Fréchet–Shohat’s moment convergence theorem; see, e.g. [19]. In order to decompose the random variables V_m and W into sums of exponential random variables, we proceed as follows. Let $\varepsilon \stackrel{\mathcal{L}}{=} \text{Exp}(\lambda)$ denote an exponentially distributed random variable with parameter λ . The Laplace transform $E(e^{-t\varepsilon})$ of ε is given by $E(e^{-t\varepsilon}) = 1/(1+t/\lambda)$. Let $(\varepsilon_\ell)_{\ell \in \mathbb{N}}$ be independent, identically $\text{Exp}(1)$ -distributed random variables. Using the fact that $(1/\lambda)\varepsilon \stackrel{\mathcal{L}}{=} \text{Exp}(\lambda)$, we obtain

$$E\left(\exp\left(-t \sum_{\ell=1}^m \frac{\varepsilon_\ell}{p_\ell}\right)\right) = \prod_{\ell=1}^m E(e^{-t\varepsilon_\ell/p_\ell}) = \prod_{\ell=1}^m \frac{1}{1+t/p_\ell}.$$

The moments of the random variables V_m and W are given by

$$E(V_m^s) = \prod_{\ell=1}^m \frac{p_\ell}{p_\ell + s} = \prod_{\ell=1}^m \frac{1}{1+s/p_\ell}, \quad E(W^s) = \prod_{\ell=1}^\infty \frac{p_\ell}{p_\ell + s} = \prod_{\ell=1}^\infty \frac{1}{1+s/p_\ell}.$$

Hence, we obtain the stated decompositions

$$V_m \stackrel{\mathcal{L}}{=} \exp\left(-\sum_{k=1}^m \frac{\varepsilon_k}{p_k}\right) \quad \text{and} \quad W \stackrel{\mathcal{L}}{=} \exp\left(-\sum_{k=1}^\infty \frac{\varepsilon_k}{p_k}\right).$$

Remark 5. A particularly interesting case is the biased sampling without replacement urn

$$M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

with sequence $P = (p_m)_{m \in \mathbb{N}_0} = (m^2)_{m \in \mathbb{N}_0}$. The factorial moments

$$E((X_{m,n})_s) = E(X_{m,n}(X_{m,n} - 1) \cdots (X_{m,n} - s + 1))$$

and the ordinary moments of $X_{m,n}$ are given by

$$E((X_{m,n})_s) = (n)_s \prod_{k=1}^m \frac{1}{1+s/k^2}, \quad E(X_{m,n}^s) = \sum_{j=0}^s \left\{ \begin{matrix} s \\ j \end{matrix} \right\} (n)_j \prod_{k=1}^m \frac{1}{1+j/k^2}.$$

A closed-form expression for the product $\prod_{k=1}^m 1/(1+s/k^2)$ is readily obtained using the Euler product form of the sine function, together with the product form of the gamma function:

$$\sin(\pi z) = \pi z \prod_{k=1}^\infty \left(1 - \frac{z^2}{k^2}\right), \quad \Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^\infty \frac{e^{z/k}}{1+z/k}. \tag{18}$$

Since $\sinh(z) = -i \sin(iz)$, where i denotes the imaginary unit, we obtain $\sinh(\pi z) = \pi z \prod_{k=1}^\infty (1+z^2/k^2)$. Moreover, by (18) we have

$$\begin{aligned} \frac{\Gamma(m+1)^2}{\Gamma(m+1+iz)\Gamma(m+1-iz)} &= \frac{(m+1)^2+z^2}{(m+1)^2} \prod_{k=1}^\infty \frac{(1+(m+1)/k)^2+z^2/k^2}{(1+(m+1)/k)^2} \\ &= \frac{(m+1)^2+z^2}{(m+1)^2} \prod_{k=1}^\infty \frac{(k+m+1)^2+z^2}{(k+m+1)^2} \\ &= \prod_{k=m+1}^\infty \left(1 + \frac{z^2}{k^2}\right). \end{aligned}$$

Consequently, we obtain, for the product $\prod_{k=1}^m 1/(1 + s/k^2)$ appearing in $E((X_{m,n})_s)$, the expression

$$\prod_{k=1}^m \frac{1}{1 + s/k^2} = \frac{\pi \sqrt{s} \Gamma(m + 1)^2}{\sinh(\pi \sqrt{s}) \Gamma(m + 1 + i\sqrt{s}) \Gamma(m + 1 - i\sqrt{s})};$$

moreover,

$$\prod_{k=1}^{\infty} \frac{1}{1 + s/k^2} = \frac{\pi \sqrt{s}}{\sinh(\pi \sqrt{s})}.$$

Note that the random variable $W = \exp(-\sum_{k=1}^{\infty} \varepsilon_k/k^2)$ arising in the limit $\min\{m, n\} \rightarrow \infty$ is closely related to distributions considered by Biane *et al.* [1] in the context of Brownian excursions and theta functions. For example, we can further show that the random variable W has support $[0, 1]$, and its distribution function can be expressed in terms of the Jacobi theta function $\Theta(q) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2}$,

$$P\{W \leq q\} = 1 - \Theta(q), \quad 0 \leq q \leq 1.$$

We also refer the reader to Crane *et al.* [3], who studied somewhat related urn models.

7. Conclusion

By applying the method of moments we were able to describe in a quite precise manner the asymptotic behavior of a class of 2×2 urn models with replacement matrix

$$M = \begin{pmatrix} -a & 0 \\ c & -d \end{pmatrix}, \quad a, d, p \in \mathbb{N} \text{ and } c = ap.$$

In Table 1 we give a short summary of our findings, using the asymptotic small-o and equivalence notation.

It is an interesting question to ask whether the approach used for a study of 2×2 urn models can be generalized to an analysis of certain diminishing urn models with more types of balls. Moreover, the biased variant of the considered urn models has interesting connections to distributions considered by Biane *et al.* [1]. Furthermore, as mentioned in Remark 2, it seems that alternative approaches, e.g. generating functions and birth and death processes, offer new perspectives and insights into the problems discussed in this work. The authors are currently investigating these matters.

TABLE 1.

	$n \rightarrow \infty$	$m \rightarrow \infty$	$m \rightarrow \infty$	$m \rightarrow \infty$
	Fixed m	$m = o(n^{a/d})$	$m \sim \rho n^{a/d}, \rho \in \mathbb{R}^+$	$n = o(m^{a/d})$
$a/d \leq 1, c \in \mathbb{N}$	Kuramaswamy	Weibull	Weibull	Weibull
$a/d > 1, c \in \mathbb{N}$	Kuramaswamy	Moment convergence	Moment convergence	Moment convergence
$a, d \in \mathbb{N}, c = 0$	Kuramaswamy	Weibull	Discrete distribution	Degenerate

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