Can. J. Math. Vol. **50** (2), 1998 pp. 412–425

ASYMPTOTIC TRANSFORMATIONS OF q-SERIES

RICHARD J. McINTOSH

ABSTRACT. For the q-series $\sum_{n=0}^{\infty} a^n q^{bn^2+cn}/(q)_n$ we construct a companion q-series such that the asymptotic expansions of their logarithms as $q \longrightarrow 1^-$ differ only in the dominant few terms. The asymptotic expansion of their quotient then has a simple closed form; this gives rise to a new q-hypergeometric identity. We give an asymptotic expansion of a general class of q-series containing some of Ramanujan's mock theta functions and Selberg's identities.

In Ramanujan's last letter to Hardy dated January 1920 (see [13, pp. 127–131], [12, pp. 354–355] and [15, pp. 56–61]), he observes that the asymptotic expansions of certain q-series "close" in a striking manner. For instance, when $q = e^{-t}$ and $t \longrightarrow 0^+$,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \sqrt{\frac{2}{5-\sqrt{5}}}\exp\left\{\frac{\pi^2}{15t} - \frac{t}{60}\right\} + o(1).$$

In the same letter Ramanujan notes that it is only in a limited number of cases that the terms in the exponent close as above. He then goes on to explain his discovery of the mock theta functions.

To facilitate printing we employ the notation

$$(a\,;q^k)_0 = 1,$$
$$(a\,;q^k)_n = (1-a)(1-aq^k)(1-aq^{2k})\cdots(1-aq^{k(n-1)})$$

and

$$(a\,;q^k)_\infty = \prod_{m=0}^{\infty}(1-aq^{mk}),$$

where $;q^k$ is usually omitted on the left when $k = 1$. Thus

$$(a\,;q^k)_n = \frac{(a\,;q^k)_\infty}{(aq^{kn}\,;q^k)_\infty}$$

for positive integers n. If n is not a positive integer, we take this as the definition of $(a\,;q^k)_n$. We also need the dilogarithm function $\mathrm{Li}_2(x)$ defined for $x \le 1$ by

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-u)}{u}\,du.$$

Received by the editors October 7, 1996.
AMS subject classification: 11B65, 33D10, 34E05, 41A60.

https://doi.org/10.4153/CJM-1998-022-x Published online by Cambridge University Press

Thus

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

for $|x| \le 1$. This function satisfies the following identities found in [9]:

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(-x) = \frac{1}{2}\,\mathrm{Li}_2(x^2),$$
$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) + \log(x)\log(1-x) = \mathrm{Li}_2(1) = \pi^2/6$$

and

$$\mathrm{Li}_2\left(\frac{x}{x+1}\right) + \frac{1}{2}\log^2(x+1) = -\,\mathrm{Li}_2(-x),$$

where $\log^2 z$ means $(\log_e z)^2$. Moreover,

$$\mathrm{Li}_2\left(\frac{\sqrt{5}-1}{2}\right) + \log^2\left(\frac{\sqrt{5}-1}{2}\right) = \frac{\pi^2}{10}$$

and

$$\mathrm{Li}_2\left(\left(\frac{\sqrt{5}-1}{2}\right)^2\right) + \log^2\left(\frac{\sqrt{5}-1}{2}\right) = \frac{\pi^2}{15}.$$

In all of our asymptotic expansions $q = e^{-t}$ and $t \longrightarrow 0^+$. We say that the asymptotic expansion of a q-series is *closed* when the number of terms in the exponent is finite. For example, the asymptotic expansion (with first few terms)

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^{n(n+c)}}{(q)_n} = \sqrt{\frac{2}{5-\sqrt{5}}}\left(\frac{\sqrt{5}-1}{2}\right)^{c} \exp\Bigg\{ &\frac{\pi^2}{15t} + \left(\frac{15c^2-3c-1}{60} - \frac{c(c-1)}{20}\sqrt{5}\right) t \\
&- c(c-1)\left(\frac{1}{50} + \frac{2c-1}{300}\sqrt{5}\right) t^2 \\
&- c(c-1)\left(\frac{2c-1}{500} + \frac{c^2-c+6}{3000}\sqrt{5}\right) t^3 \\
&- c(c-1)\left(\frac{c^2-c+26}{15000} - \frac{(2c-1)(3c^2-3c-31)}{90000}\sqrt{5}\right) t^4 + O(t^5)\Bigg\},
\end{aligned} \tag{1}$$

proved in [11], closes when $c = 0$ or $c = 1$. To prove these closures we first use the Rogers-Ramanujan identities (see [2, p. 50] or [8, p. 290]) and the Jacobi triple product identity (see for example [2, p. 21]) to obtain

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \frac{(q^2\,;q^5)_\infty (q^3\,;q^5)_\infty (q^5\,;q^5)_\infty}{(q)_\infty} = \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(5n+1)/2}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_n} = \frac{(q\,;q^5)_\infty (q^4\,;q^5)_\infty (q^5\,;q^5)_\infty}{(q)_\infty} = \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(5n+3)/2}.$$

https://doi.org/10.4153/CJM-1998-022-x Published online by Cambridge University Press

Then we use the identities

$$(q)_\infty = \sqrt{\frac{2\pi}{t}} \exp\left\{-\frac{\pi^2}{6t} + \frac{t}{24}\right\} \prod_{m=1}^{\infty} \left(1 - \exp\left\{-\frac{4m\pi^2}{t}\right\}\right), \tag{2}$$

obtained from the transformation formula for the Dedekind η-function (see for example [6, pp. 47–48]), and

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\alpha n^2+\beta n} = \sqrt{\frac{4\pi}{\alpha t}} \exp\left\{\frac{\beta^2 t}{4\alpha}\right\} \sum_{n=1}^{\infty} \cos\left\{\frac{(2n-1)\beta\pi}{2\alpha}\right\} \exp\left\{-\frac{(2n-1)^2\pi^2}{4\alpha t}\right\},$$

obtained from the transformation formula for θ_3 (see for example [7, p. 4]).

The q-series

$$\sum_{n=0}^{\infty} \frac{a^n q^{bn^2+cn}}{(q)_n}$$

in general does not have a closed asymptotic expansion. We conjecture that it has a companion series

$$\sum_{n=0}^{\infty} \frac{\tilde{a}^n q^{\tilde{b}n^2+\tilde{c}n}}{(q)_n}$$

such that the quotient

$$\left(\sum_{n=0}^{\infty} \frac{a^n q^{bn^2+cn}}{(q)_n}\right) \Big/ \left(\sum_{n=0}^{\infty} \frac{\tilde{a}^n q^{\tilde{b}n^2+\tilde{c}n}}{(q)_n}\right)$$

has a closed asymptotic expansion.

We begin with the main theorem proved in [11].

THEOREM 1. *Let a, b, c, and q be real numbers with $a > 0$, $b > 0$ and $|q| < 1$. Let z denote the positive root of $az^{2b} + z - 1 = 0$. When $q = e^{-t}$ and $t \longrightarrow 0^+$ we obtain for each nonnegative integer p,*

$$\log \sum_{n=0}^{\infty} \frac{a^n q^{bn^2+cn}}{(q)_n} = \left\{\mathrm{Li}_2(1-z) + b\log^2 z\right\}t^{-1} + c\log z - \frac{1}{2}\log\left\{z + 2b(1-z)\right\} + \sum_{k=1}^{p} R_k(b,c,z)t^k + O(t^{p+1}), \tag{3}$$

where $R_1, R_2, \ldots, R_p$ are rational functions of b, c and z.

Equation (3) often holds for negative values of a. When $b = 0$ and $|a| < 1$ we use Euler's first identity (see for example [2, p. 19])

$$\sum_{n=0}^{\infty} \frac{u^n}{(q)_n} = \prod_{m=0}^{\infty} \frac{1}{1 - uq^m} \tag{4}$$

with $u = aq^c$ to express the sum on the left side of (3) as an infinite product. Its asymptotic expansion is then obtained by the Euler-Maclaurin sum formula. When $b = 1/2$ and

$a > -1$ the left side of (3) can also be expressed as an infinite product, this time using Euler's second identity

$$\sum_{n=0}^{\infty} \frac{u^n q^{n(n-1)/2}}{(q)_n} = \prod_{m=0}^{\infty}(1 + uq^m) \tag{5}$$

with $u = aq^{c+1/2}$. When $0 < b < 1/2$, numerical computations suggest that (3) holds for all a, and when $b > 1/2$ numerical computations suggest that (3) holds for $a > -(2b-1)^{2b-1}/(2b)^{2b}$. In the last case the equation $az^{2b} + z - 1 = 0$ has two positive solutions. We must take the smaller one to insure that $z + 2b(1-z) > 0$. Throughout this paper we assume that (3) holds in these cases.

Let z be the smallest positive solution of $az^{2b} + z - 1 = 0$. For $a, b > 0$ define

$$a' = a^{-1/(2b)}, \quad b' = 1/(4b), \quad c' = c/(2b), \quad z' = 1 - z,$$

and observe that

$$a'(z')^{2b'} + z' - 1 = 0.$$

This transformation is an involution (that is, has order 2). We ask how the asymptotic series for

$$\log \sum_{n=0}^{\infty} \frac{a^n q^{bn^2+cn}}{(q)_n} \quad \text{and} \quad \log \sum_{n=0}^{\infty} \frac{(a')^n q^{b'n^2+c'n}}{(q)_n}$$

are related.

Using the symbolic algebra program MAPLE [10] together with [11], we obtain

$$\begin{aligned} \log \sum_{n=0}^{\infty} \frac{a^n q^{bn^2+cn}}{(q)_n} &= \left\{\mathrm{Li}_2(1-z) + b\log^2 z\right\}t^{-1} + c\log z - \frac{1}{2}\log\left\{z + 2b(1-z)\right\} \\ &\quad + \frac{(1-z)P_1(b,c,z)}{\{z+2b(1-z)\}^3}t + \sum_{k=2}^{7} \frac{z(1-z)P_k(b,c,z)}{\{z+2b(1-z)\}^{3k}}t^k + O(t^8), \end{aligned}$$

where $P_1, P_2, \ldots, P_7$ are polynomials in b, c and z. Surprisingly, it turns out that

$$\begin{aligned} \log \sum_{n=0}^{\infty} \frac{(a')^n q^{b'n^2+c'n}}{(q)_n} &= \left\{\mathrm{Li}_2(z) + \frac{\log^2(1-z)}{4b}\right\}t^{-1} + \frac{c}{2b}\log a + \frac{1}{2}\log(2b) + c\log z \\ &\quad - \frac{1}{2}\log\left\{z + 2b(1-z)\right\} + \left\{\frac{c^2}{4b} - \frac{1}{24} - \frac{(1-z)P_1(b,c,z)}{\{z+2b(1-z)\}^3}\right\}t \\ &\quad + \sum_{k=2}^{7}(-1)^k \frac{z(1-z)P_k(b,c,z)}{\{z+2b(1-z)\}^{3k}}t^k + O(t^8). \end{aligned}$$

Thus if we replace t by $-t$ and hence q by $1/q$ in the above equation, then our two asymptotic series appear to agree from the t^2 term onward. With this in mind we define

$$\tilde{a} = -a^{-1/(2b)}, \quad \tilde{b} = 1/2 - 1/(4b), \quad \tilde{c} = 1/2 - c/(2b), \quad \tilde{z} = 1/(1-z).$$

Again we see that

$$\tilde{a}\tilde{z}^{2\tilde{b}} + \tilde{z} - 1 = 0.$$

https://doi.org/10.4153/CJM-1998-022-x Published online by Cambridge University Press

This transformation is a trinvolution (that is, has order 3). For $a > 0$ and $b > 1/2$ we obtain

$$\log \sum_{n=0}^{\infty} \frac{\tilde{a}^n q^{\tilde{b}n^2+\tilde{c}n}}{(q)_n} = \left\{-\mathrm{Li}_2(z) - \frac{\log^2(1-z)}{4b}\right\} t^{-1} + \frac{c}{2b}\log a + \frac{1}{2}\log(2b) + c\log z - \frac{1}{2}\log\{z+2b(1-z)\} + \left\{\frac{1}{24} - \frac{c^2}{4b} + \frac{(1-z)P_1(b,c,z)}{\{z+2b(1-z)\}^3}\right\} t + \sum_{k=2}^{7} \frac{z(1-z)P_k(b,c,z)}{\{z+2b(1-z)\}^{3k}} t^k + O(t^8).$$

Hence

$$\left(\sum_{n=0}^{\infty} \frac{a^n q^{bn^2+cn}}{(q)_n}\right) \Big/ \left(\sum_{n=0}^{\infty} \frac{\tilde{a}^n q^{\tilde{b}n^2+\tilde{c}n}}{(q)_n}\right) = \frac{1}{\sqrt{2ba^{c/b}}} \exp\left\{\left(\frac{\pi^2}{6} + \frac{\log^2 a}{4b}\right) t^{-1} + \left(\frac{c^2}{4b} - \frac{1}{24}\right) t + O(t^8)\right\}.$$

Note the absence of the variable z in the above formula. Numerical computations support the conjecture that

$$\left(\sum_{n=0}^{\infty} \frac{a^n q^{bn^2+cn}}{(q)_n}\right) \Big/ \left(\sum_{n=0}^{\infty} \frac{\tilde{a}^n q^{\tilde{b}n^2+\tilde{c}n}}{(q)_n}\right) = \frac{1}{\sqrt{2ba^{c/b}}} \exp\left\{\left(\frac{\pi^2}{6} + \frac{\log^2 a}{4b}\right) t^{-1} + \left(\frac{c^2}{4b} - \frac{1}{24}\right) t\right\} \left[1 + O\left(\exp\left(-\frac{K}{t}\right)\right)\right], \tag{6}$$

where K is a positive constant which in general depends on a and b but not on c.

The cases $b = 1/2$ and $b = 1$ warrant further analysis. When $b = 1/2$, $a > 0$, $|q| < 1$ and $|a^{-1}q^{-c+1/2}| < 1$ we have $\tilde{a} = -a^{-1}$, $\tilde{b} = 0$, $\tilde{c} = -c + 1/2$ and

$$\left(\sum_{n=0}^{\infty} \frac{a^n q^{n^2/2+cn}}{(q)_n}\right) \Big/ \left(\sum_{n=0}^{\infty} \frac{(-a^{-1})^n q^{(-c+1/2)n}}{(q)_n}\right) = a^{-c} \exp\left\{\left(\frac{\pi^2}{6} + \frac{\log^2 a}{2}\right) t^{-1} + \left(\frac{c^2}{2} - \frac{1}{24}\right) t\right\} \left[\prod_{m=1}^{\infty}\left(1 - \exp\left(-\frac{4m\pi^2}{t}\right)\right)\right]^{-1} \times \left[1 + 2\sum_{n=1}^{\infty} \exp\left(-\frac{2n^2\pi^2}{t}\right) \cos\left(2nc\pi - \frac{2n\pi \log a}{t}\right)\right]. \tag{7}$$

PROOF OF (7). By Euler's second identity (5) with $u = aq^{c+1/2}$ we obtain

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2/2+cn}}{(q)_n} = \prod_{m=0}^{\infty} (1 + aq^{m+c+1/2}),$$

and from Euler's first identity (4) with $u = -a^{-1}q^{-c+1/2}$ we obtain

$$\sum_{n=0}^{\infty} \frac{(-a^{-1})^n q^{(-c+1/2)n}}{(q)_n} = \prod_{m=0}^{\infty} \frac{1}{1 + a^{-1}q^{m-c-1/2}}.$$

https://doi.org/10.4153/CJM-1998-022-x Published online by Cambridge University Press

Therefore

$$\left(\sum_{n=0}^{\infty}\frac{a^n q^{n^2/2+cn}}{(q)_n}\right)\Big/\left(\sum_{n=0}^{\infty}\frac{(-a^{-1})^n q^{(-c+1/2)n}}{(q)_n}\right)=\prod_{m=1}^{\infty}(1+aq^{m+c-1/2})(1+a^{-1}q^{m-c-1/2}).$$

By Jacobi's triple product identity we have

$$\prod_{m=1}^{\infty}(1+aq^{m+c-1/2})(1+a^{-1}q^{m-c-1/2})=\frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}a^n q^{n^2/2+cn}.$$

The asymptotic expansion of $(q)_\infty$ is given in (2) and the asymptotic expansion of the sum is obtained from the transformation formula for θ_3. Forming their product completes the proof of (7).

By Euler's identities (4) and (5),

$$\left(\sum_{n=0}^{\infty}\frac{(-a^{-1})^n q^{(-c+1/2)n}}{(q)_n}\right)^{-1}=\prod_{m=1}^{\infty}(1+a^{-1}q^{m-c-1/2})=\sum_{n=0}^{\infty}\frac{a^{-n}q^{n^2/2-cn}}{(q)_n},$$

so (7) can be stated as

$$\begin{aligned}&\left(\sum_{n=0}^{\infty}\frac{a^n q^{n^2/2+cn}}{(q)_n}\right)\left(\sum_{n=0}^{\infty}\frac{a^{-n} q^{n^2/2-cn}}{(q)_n}\right)\\ &\quad=a^{-c}\exp\left\{\left(\frac{\pi^2}{6}+\frac{\log^2 a}{2}\right)t^{-1}+\left(\frac{c^2}{2}-\frac{1}{24}\right)t\right\}\left[\prod_{m=1}^{\infty}\left(1-\exp\left(-\frac{4m\pi^2}{t}\right)\right)\right]^{-1}\\ &\qquad\times\left[1+2\sum_{n=1}^{\infty}\exp\left(-\frac{2n^2\pi^2}{t}\right)\cos\left(2nc\pi-\frac{2n\pi\log a}{t}\right)\right]\end{aligned}\tag{8}$$

for $a>0$ and $|q|<1$.

Equation (6) takes on an interesting form when $a=b=1$. Numerical computations suggest the following refinement:

$$\begin{aligned}&\left(\sum_{n=0}^{\infty}\frac{q^{n^2+cn}}{(q)_n}\right)\Big/\left(\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2/4+(1-c)n/2}}{(q)_n}\right)\\ &\quad=\frac{1}{\sqrt{2}}\exp\left\{\frac{\pi^2}{6t}+\left(\frac{c^2}{4}-\frac{1}{24}\right)t\right\}\left[1+2\cos(c\pi)\left(\frac{1+\sqrt{5}}{2}\right)^{2c-1}\right.\\ &\qquad\times\exp\left\{-\frac{4\pi^2}{5t}+\frac{c(c-1)\sqrt{5}}{10}t+\frac{c(c-1)(2c-1)\sqrt{5}}{150}t^2\right.\\ &\qquad+\frac{c(c-1)(c^2-c+6)\sqrt{5}}{1500}t^3\\ &\qquad\left.\left.-\frac{c(c-1)(2c-1)(3c^2-3c-31)\sqrt{5}}{45000}t^4+O(t^5)\right\}\right].\end{aligned}\tag{9}$$

https://doi.org/10.4153/CJM-1998-022-x Published online by Cambridge University Press

One should compare the tails of the asymptotic expansions (1) and (9). In (9) we immediately see that the cosine term vanishes whenever $c \equiv 1/2 \pmod 1$. For these values of c numerical computations suggest the further refinement:

$$\left(\sum_{n=0}^{\infty} \frac{q^{n^2+cn}}{(q)_n}\right) \Big/ \left(\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2/4+(1-c)n/2}}{(q)_n}\right) = \frac{1}{\sqrt{2}} \exp\left\{\frac{\pi^2}{6t} + \left(\frac{c^2}{4} - \frac{1}{24}\right) t\right\} \prod_{m=1}^{\infty} \left(1 - \exp\left(-\frac{4(2m-1)\pi^2}{t}\right)\right)$$

for $|q| < 1$. We recognize the product on the right as that associated with the identity

$$(-q^{1/2}\,;\,q^{1/2})_\infty = \frac{(q)_\infty}{(q^{1/2}\,;\,q^{1/2})_\infty} = \frac{1}{\sqrt{2}} \exp\left\{\frac{\pi^2}{6t} + \frac{t}{48}\right\} \prod_{m=1}^{\infty} \left(1 - \exp\left(-\frac{4(2m-1)\pi^2}{t}\right)\right),$$

obtained from (2). This leads us to conjecture that

$$\sum_{n=0}^{\infty} \frac{q^{(2n+\mu)(2n+\mu+1)/2}}{(q^2\,;\,q^2)_n} = (-q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2-\mu n}}{(q^2\,;\,q^2)_n} \tag{10}$$

for all integers μ. In support of the above conjectures, we now show that (10) is indeed true.

PROOF OF (10). Let $F(\mu)$ be the sum on the left of (10) and $G(\mu)$ the sum on the right of (10). Then

$$\begin{aligned} F(\mu-1) - q^{-\mu} F(\mu) &= \sum_{n=1}^{\infty} \frac{q^{2n^2+(2\mu-1)n+(\mu-1)\mu/2}(1-q^{2n})}{(q^2\,;\,q^2)_n} \\ &= \sum_{n=1}^{\infty} \frac{q^{2n^2+(2\mu-1)n+(\mu-1)\mu/2}}{(q^2\,;\,q^2)_{n-1}} \\ &= \sum_{n=0}^{\infty} \frac{q^{2n^2+(2\mu-3)n+(\mu+1)(\mu+2)/2}}{(q^2\,;\,q^2)_n} \\ &= F(\mu+1) \end{aligned}$$

and

$$\begin{aligned} G(\mu+1) - G(\mu-1) &= \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2/2-(\mu+1/2)n}(1-q^{2n})}{(q^2\,;\,q^2)_n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2/2-(\mu+1/2)n}}{(q^2\,;\,q^2)_{n-1}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} q^{n^2/2-(\mu-1/2)n-\mu}}{(q^2\,;\,q^2)_n} \\ &= -q^{-\mu} G(\mu). \end{aligned}$$

https://doi.org/10.4153/CJM-1998-022-x Published online by Cambridge University Press

Since F and G satisfy the same recurrence, it suffices to prove that $F(\mu) = (-q)_\infty G(\mu)$ for two consecutive values of μ. We will prove it for $\mu = 0$ and $\mu = -1$.

The case $\mu = 0$ follows from the identity [1, p. 575, eq. (R1) with q replaced by $q^{1/2}$ and z replaced by $zq^{1/2}$]:

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+n} z^{2n}}{(q^2\,;\,q^2)_n} = (zq\,;\,q)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} z^n}{(q\,;\,q)_n (zq\,;\,q)_n} \tag{11}$$

with $z = -1$. Putting $z = -q^{-1}$ in (11) we get

$$\begin{aligned} F(-1) &= \sum_{n=0}^{\infty} \frac{q^{2n^2-n}}{(q^2\,;\,q^2)_n} \\ &= (-1)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(q)_n(-1)_n} \\ &= (-q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(q)_n(-q)_{n-1}} \\ &= (-q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}(1+q^n)}{(q^2\,;\,q^2)_n} \\ &= (-q)_\infty \big[G(1) + G(0)\big] \\ &= (-q)_\infty G(-1) \end{aligned}$$

by the recurrence for G. This completes the proof of (10).

Other proofs of (10) were supplied by George Andrews, Basil Gordon, Mike Hirschhorn and Tom Koornwinder. The above proof was based on the ideas of George Andrews and Basil Gordon.

The method used in the proof of Theorem 1, in particular, equation (2.11) in [11] can be suitably modified to yield the complete asymptotic expansion of q-series of the type

$$f(a,b,c\,;\,q) := \sum_{n=0}^{\infty} a^n q^{bn^2+cn} \prod_{i=1}^{m} (q^{k_i}\,;\,q^{k_i})_{s_i n}^{r_i},$$

where $k_i, s_i > 0$. We assume that a, b, c and q are real numbers such that $a > 0$, $b \geq 0$ and $|q| < 1$.

THEOREM 2. *Suppose that the terms in the above sum are positive and unimodal, i.e., they increase until $n = N(q)$ and then decrease to 0. Suppose that the z-equation*

$$az^{2b} \prod_{i=1}^{m} (1 - z^{k_i s_i})^{s_i r_i} = 1$$

has a unique root z with $0 < z < 1$. Let $q = e^{-t}$ and $t \to 0^+$. Then $q^{N(q)} \to z$ and for each nonnegative integer p,

https://doi.org/10.4153/CJM-1998-022-x Published online by Cambridge University Press

$$\log f(a,b,c\,;q) = \Big\{ b\log^2 z - \sum_{i=1}^{m} \frac{r_i}{k_i} \operatorname{Li}_2(1-z^{k_i s_i}) \Big\} t^{-1} - \frac{r+1}{2}\log t + \frac{r+1}{2}\log 2\pi$$
$$+ c\log z + \sum_{i=1}^{m} \frac{r_i}{2} \log \frac{1-z^{k_i s_i}}{k_i} - \frac{1}{2}\log\Big(2b - \sum_{i=1}^{m} \frac{k_i s_i^2 r_i z^{k_i s_i}}{1-z^{k_i s_i}}\Big)$$
$$+ \sum_{j=1}^{p} R_j(b,c,z) t^j + O(t^{p+1}),$$

where $r = r_1 + r_2 + \cdots + r_m$ *and* $R_1, R_2, \ldots, R_p$ *are rational functions of* b, c *and* z.

This theorem can be extended to accommodate additional factors in the terms of $f(a,b,c\,;q)$. For example, if

$$g(a,b,c\,;q) := \sum_{n=0}^{\infty} a^n q^{bn^2+cn}(1-q^{2n+1}) \prod_{i=1}^{m} (q^{k_i}\,;q^{k_i})_{s_i n}^{r_i},$$

then we just add the asymptotic expansion

$$\log(1-q^{2n+1}) = \log(1-z^{2t+1/N})$$
$$= \log(1-z^{2t}) - \frac{z^{2t}\log z}{1-z^{2t}} N^{-1} - \frac{z^{2t}\log^2 z}{2(1-z^{2t})^2} N^{-2} - \cdots$$

to the modified form of equation (2.11) in [11].

Before defining another transformation, we give an asymptotic formula for a product similar to (8), but with odd powers of q in the denominators:

$$\Big(\sum_{n=1}^{\infty} \frac{a^n q^{n^2+(2c-1)n}}{(q\,;q^2)_n}\Big)\Big(\sum_{n=1}^{\infty} \frac{a^{-n} q^{n^2-(2c+1)n}}{(q\,;q^2)_n}\Big) = \frac{\pi}{2a^c t}\exp\Big\{\Big(\frac{\pi^2}{12} + \frac{\log^2 a}{4}\Big) t^{-1}$$
$$+ \Big(c^2 + \frac{2}{3}\Big) t + O(t^8)\Big\}.$$

The sum

$$\sum_{n=0}^{\infty} a^n q^{bn^2+cn}(-q)_n$$

satisfies the conditions in Theorem 2 if we take $a > 1/2$ and $b \geq 0$ (in the case $b = 0$ we must take $1/2 < a < 1$). Note that when $|a| < 1/2$ this sum tends to $1/(1-2a)$ and when $a = 1/2$ it is asymptotic to $\sqrt{\pi/(4b+1)t}$. The z-equation for the sum is $az^{2b}(1+z) = 1$. Consider the transformation

$$\tilde a = a^{1/(2b+1)}, \quad \tilde b = b/(2b+1), \quad \tilde c = (c-b)/(2b+1), \quad \tilde z = z/(z+1),$$

and observe that

$$\tilde a \tilde z^{2\tilde b} + \tilde z = 1,$$

which is the $\tilde z$-equation for the sum

$$\sum_{n=0}^{\infty} \frac{\tilde a^n q^{\tilde b n^2 + \tilde c n}}{(q)_n}.$$

https://doi.org/10.4153/CJM-1998-022-x Published online by Cambridge University Press

It turns out that

$$\left(\sum_{n=0}^{\infty} a^n q^{bn^2+cn}(-q)_n\right) \Big/ \left(\sum_{n=0}^{\infty} \frac{\tilde{a}^n q^{\tilde{b}n^2+\tilde{c}n}}{(q)_n}\right)$$
$$= \sqrt{\frac{\pi}{(2b+1)t}} \exp\left\{\left(\frac{\log^2 a}{4b+2} - \frac{\pi^2}{12}\right) t^{-1} - \frac{2c+1}{4b+2}\log a + \left(\frac{6c^2+6c-b+1}{24b+12}\right) t + O(t^6)\right\}.$$

Numerical computations suggest that the error term is in fact $O(e^{-K/t})$ for some positive number K which in general depends on a and b but not on c. A consequence of this is the known fact that Ramanujan's fifth order mock theta functions

$$\psi_0(q) = \sum_{n=0}^{\infty} q^{(n+1)(n+2)/2}(-q)_n \quad \text{and} \quad \psi_1(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}(-q)_n$$

are closed. For a list of Ramanujan's fifth order mock theta functions see [16, pp. 277–278].

The Transformation $n \longrightarrow n - 1/2$**.** Let

$$F(r) = \sum_{n=0}^{\infty} \frac{a^{n+r-1} q^{b(n+r-1)^2+c(n+r-1)}}{(q^r)_n}$$

for real $r \neq 0, -1, -2, \ldots$. We will prove that $F(1/2)$ and $F(1)$ are *asymptotic associates*, that is, their asymptotic expansions agree from the t^2 term onward. The method used in the proof of Theorem 1, in particular, equation (2.11) in [11] can be suitably modified to show that

$$F(r) = \frac{\Gamma(r) z^c t^{r-1}}{\sqrt{z+2b(1-z)}} \exp\left\{\left(b\log^2 z + \mathrm{Li}_2(1-z)\right)t^{-1} + \sum_{k=1}^{p} R_k(b,c,z,r)t^k + O(t^{p+1})\right\},$$

where $az^{2b} + z - 1 = 0$ and $R_1, R_2, \ldots, R_p$ are rational functions of b, c, z and r. Since

$$(q^r)_n = \frac{(q^r)_\infty}{(q^{n+r})_\infty} = \frac{(q^r)_\infty}{(q)_\infty}(q)_{n+r-1},$$

we have

$$F(r) = \frac{1}{(q^r)_\infty} \sum_{n=0}^{\infty} a^{n+r-1} q^{b(n+r-1)^2+c(n+r-1)} (q^{n+r})_\infty$$
$$= \frac{(q)_\infty}{(q^r)_\infty} \sum_{n=0}^{\infty} \frac{a^{n+r-1} q^{b(n+r-1)^2+c(n+r-1)}}{(q)_{n+r-1}}.$$

The asymptotic expansion of the last sum is, up to a quantity exponentially small in comparison to the dominant term, independent of r. This is because the primary contribution

https://doi.org/10.4153/CJM-1998-022-x Published online by Cambridge University Press

to the sum arises from those n which are close to the index of the largest term in the sum. More precisely, we have for all $\epsilon > 0$,

$$\sum_{n=0}^{\infty} \frac{a^{n+r-1} q^{b(n+r-1)^2+c(n+r-1)}}{(q)_{n+r-1}} = \sum_{n\in S} \frac{a^{n+r-1} q^{b(n+r-1)^2+c(n+r-1)}}{(q)_{n+r-1}} \left\{1 + o(e^{-\delta/t})\right\},$$

where $S = \{n : 1-\epsilon \le nt/(-\log z) \le 1+\epsilon\}$ and δ is a positive number depending upon ϵ (for a proof see [11, pp. 126–127]). Thus

$$\begin{aligned}\frac{F(s)}{F(r)} &= \frac{(q^r)_\infty}{(q^s)_\infty}\left\{1 + o(e^{-\delta/t})\right\} \\ &= \frac{\Gamma(s)}{\Gamma(r)} t^{s-r} \exp\left\{\sum_{k=1}^{p} \frac{B_k \cdot \{B_{k+1}(s) - B_{k+1}(r)\}}{k(k+1)!} t^k + O(t^{p+1})\right\},\end{aligned}$$

since

$$\log(q^r)_\infty = -\frac{\pi^2}{6} t^{-1} + \left(\frac{1}{2} - r\right)\log t + \frac{1}{2}\log(2\pi) - \log\Gamma(r) - \sum_{k=1}^{p} \frac{B_k B_{k+1}(r)}{k(k+1)!} t^k + O(t^{p+1})$$

by the Euler-Maclaurin sum formula. In particular,

$$\frac{F(1/2)}{F(1)} = \sqrt{\frac{\pi}{t}}\, e^{t/16+o(t^p)}$$

for all positive integers p, which implies that $F(1/2)$ and $F(1)$ are asymptotic associates. We say that $F(1/2)$ is obtained from $F(1)$ by the transformation $n \longrightarrow n - 1/2$ of the q-series

$$\sum_{n=0}^{\infty} \frac{a^n q^{bn^2+cn}}{(q)_n}.$$

(This is tantamount to summing $a^n q^{bn^2+cn}/(q)_n$ over the positive half-integers $1/2, 3/2, 5/2, \ldots$ instead of the nonnegative integers.)

By Euler's second identity and equation (2) we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q)_n} &= 2(-q)_\infty = \sqrt{2}\exp\left\{\frac{\pi^2}{12t} + \frac{t}{24}\right\} \prod_{m=1}^{\infty}\left(1 - \exp\left\{-\frac{2(2m-1)\pi^2}{t}\right\}\right), \\ \sum_{n=0}^{\infty} \frac{q^{n^2/2}}{(q)_n} &= (-q^{1/2})_\infty = \exp\left\{\frac{\pi^2}{12t} - \frac{t}{48}\right\} \prod_{m=1}^{\infty}\left(1 + \exp\left\{-\frac{2(2m-1)\pi^2}{t}\right\}\right), \\ \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q)_n} &= (-q)_\infty = \frac{1}{\sqrt{2}}\exp\left\{\frac{\pi^2}{12t} + \frac{t}{24}\right\} \prod_{m=1}^{\infty}\left(1 - \exp\left\{-\frac{2(2m-1)\pi^2}{t}\right\}\right)\end{aligned}$$

are closed. Applying the transformations $n \longrightarrow n - 1/2$ and $q \longrightarrow q^2$ to the above sums it follows that

$$\sum_{n=1}^{\infty} \frac{q^{n(n-2)}}{(q\,;q^2)_n} = \frac{1 + 2\psi(q)}{q}, \quad \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(q\,;q^2)_n} = \upsilon(-q), \quad \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q\,;q^2)_n} = \psi(q)$$

https://doi.org/10.4153/CJM-1998-022-x Published online by Cambridge University Press

are also closed. (Here $\psi(q)$ and $\upsilon(q)$ are two of Ramanujan's third order mock theta functions. For a list of these see [15, p. 62].)

The closures of Ramanujan's fifth order mock theta functions

$$F_1(q) := \sum_{n=1}^{\infty} \frac{q^{2n(n-1)}}{(q\,;q^2)_n} \quad \text{and} \quad F_0(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q\,;q^2)_n}$$

can be established by applying the transformations $n \longrightarrow n - 1/2$ and $q \longrightarrow q^2$ to the sums

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_n},$$

whose closures we proved in the argument following equation (1). Rogers [3, pp. 36, 58] proved that

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4\,;q^4)_n} = \frac{1}{(-q^2\,;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q^4\,;q^4)_n} = \frac{1}{(-q^2\,;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_n},$$

which implies that the sums on the left are closed. Applying the transformations $q \longrightarrow q^{1/4}$, $n \longrightarrow n - 1/2$ and $q \longrightarrow q^2$ to these sums it follows that the q-series

$$\sum_{n=1}^{\infty} \frac{q^{n(n-1)/2}}{(q\,;q^2)_n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{(q\,;q^2)_n},$$

appearing in the Lost Notebook [13, p. 9], are also closed.

The closures of many q-series can be established by applying the transformation $n \longrightarrow n - 1/2$ to other closed q-series. For example, this method can be used to show that Ramanujan's seventh order mock theta functions [12, p. 355] (also see [4, pp. 132–133] and [5, p. 286])

$$F_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1})_n}, \quad F_1(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q^{n+1})_{n+1}}, \quad F_2(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^{n+1})_{n+1}}$$

are closed. For this we need the Selberg identities [14, p. 5] (where the right sides of the last two identities are incorrectly interchanged in [14]):

$$\begin{aligned}
A(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2\,;q^2)_n(-q)_{2n}} &= \frac{(q^3\,;q^7)_\infty(q^4\,;q^7)_\infty(q^7\,;q^7)_\infty}{(q^2\,;q^2)_\infty} \\
&= \frac{1}{(q^2\,;q^2)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(7n+1)/2}, \\
B(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2\,;q^2)_n(-q)_{2n}} &= \frac{(q^2\,;q^7)_\infty(q^5\,;q^7)_\infty(q^7\,;q^7)_\infty}{(q^2\,;q^2)_\infty} \\
&= \frac{1}{(q^2\,;q^2)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(7n+3)/2}, \\
C(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2\,;q^2)_n(-q)_{2n+1}} &= \frac{(q\,;q^7)_\infty(q^6\,;q^7)_\infty(q^7\,;q^7)_\infty}{(q^2\,;q^2)_\infty} \\
&= \frac{1}{(q^2\,;q^2)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(7n+5)/2}
\end{aligned}$$

https://doi.org/10.4153/CJM-1998-022-x Published online by Cambridge University Press

and the related functions

$$A_r(q) := \sum_{n=0}^{\infty} \frac{q^{2(n+r-1)^2}}{(q^{2r}\,;q^2)_n(-q^{2r-1})_{2n}},$$
$$B_r(q) := \sum_{n=0}^{\infty} \frac{q^{2(n+r-1)(n+r)}}{(q^{2r}\,;q^2)_n(-q^{2r-1})_{2n}},$$
$$C_r(q) := \sum_{n=0}^{\infty} \frac{q^{2(n+r-1)(n+r)}}{(q^{2r}\,;q^2)_n(-q^{2r-1})_{2n+1}}.$$

Since these functions are closed when $r = 1$ it is not difficult to show that they are also closed when $r = 1/2$. The closures of $F_0(q)$, $F_1(q)$ and $F_2(q)$ are proved by manipulating the series $C_{1/2}(q)$, $B_{1/2}(q)$ and $A_{1/2}(q)$ respectively, and then replacing q by $q^{1/2}$. Note that

$$\begin{aligned}
\frac{1}{(q\,;q^2)_n(-1)_{2n}} &= \frac{1}{(q\,;q^2)_n(-q\,;q^2)_n(-1\,;q^2)_n} \\
&= \frac{1}{(q^2\,;q^4)_n(-1\,;q^2)_n} \\
&= \frac{1}{2(q^2\,;q^4)_n(-q^2\,;q^2)_{n-1}} \\
&= \frac{(q^2\,;q^2)_{n-1}}{2(q^2\,;q^4)_n(q^4\,;q^4)_{n-1}} \\
&= \frac{(q^2\,;q^2)_{n-1}}{2(q^2\,;q^2)_{2n-1}} \\
&= \frac{1}{2(q^{2n}\,;q^2)_n}.
\end{aligned}$$

ACKNOWLEDGEMENTS. The author is grateful to Basil Gordon for several helpful suggestions made during the preparation of this paper. Support by the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged.

REFERENCES

1. G. E. Andrews, *q-identities of Auluck, Carlitz, and Rogers*. Duke Math. J. **33**(1966), 575–581.
2. ______, *The Theory of Partitions*. Addison-Wesley, Reading, 1976.
3. ______, *q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra*. CBMS conference series **66**, Amer. Math. Soc., Providence, Rhode Island, 1986.
4. ______, *The fifth and seventh order mock theta functions*. Trans. Amer. Math. Soc. **293**(1986), 113–134.
5. ______, *Mock theta functions*. Proc. Sympos. Pure Math. **49**(1989), 283–298.
6. T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*. Second edition, Springer-Verlag, New York, 1990.
7. R. Bellman, *A Brief Introduction to Theta Functions*. Holt, Rinehart and Winston, New York, 1961.
8. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*. Fourth edition, Oxford University Press, London, 1960.
9. L. Lewin, *Polylogarithms and Associated Functions*. North-Holland, New York, 1981.
10. MAPLE, A large symbolic manipulation program developed at the University of Waterloo. MAPLE is a registered trademark of Waterloo Maple Soltware.

11. R. J. McIntosh, *Some asymptotic formulae for q-hypergeometric series*. J. London Math. Soc. (2) **51**(1995), 120–136.
12. S. Ramanujan, *Collected Papers*. Cambridge University Press, 1927; reprinted by Chelsea, New York, 1962.
13. ______, *The Lost Notebook and Other Unpublished Papers*. Narosa, New Delhi, 1988.
14. A. Selberg, *Über die Mock-Thetafunktionen siebenter 0rdnung*. Arch. Math. og Naturvidenskab **41**(1938), 3–15.
15. G. N. Watson, *The final problem: an account of the mock theta functions*. J. London Math. Soc. **11**(1936), 55–80.
16. ______, *The mock theta functions (2)*. Proc. London Math. Soc. (2) **42**(1937), 274–304.

Department of Mathematics and Statistics
University of Regina
Regina, SK
S4S 0A2
email: mcintosh@math.uregina.ca