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BOUNDEDNESS OF SINGULAR INTEGRAL OPERATORS OF CALDERON TYPE, III

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§1. Introduction

In this paper we investigate the boundedness of Cauchy kernels. The Cauchy kernel associated with a locally integrable real-valued function $\theta(x)$ is defined by

$$(1)\qquad \mathfrak{C}[\theta](x, y) = (1 + i\theta(y))/\{(x - y) + i(\Theta(x) - \Theta(y))\}\,,$$

where $\Theta(x) = \int_0^x \theta(z)\,dz$. This kernel plays an important role in harmonic analysis on the graph $\{(x, \Theta(x));\, x \in (-\infty, \infty)\}$. For $p > 1$ and a non-negative function $\omega(x)$, let L^p_ω denote the space of functions $f(x)$ with $\|f\|_{p\omega} = \left\{\int_{-\infty}^{\infty} |f(x)|^p \omega(x)\,dx\right\}^{1/p} < \infty$. In the case $\omega(x) \equiv 1$, we write simply L^p and $\|\cdot\|_p$. We say that $\mathfrak{C}[\theta]$ is of type (p, ω) if, for any $f \in L^p_\omega$,

$$(2)\qquad \mathfrak{C}[\theta]f(x) = \lim_{\varepsilon\to 0}\int_{\varepsilon<|x-y|<1/\varepsilon} \mathfrak{C}[\theta](x, y) f(y)\,dy$$

exists almost everywhere (a.e.) and $\|\mathfrak{C}[\theta]\|_{p\omega} = \sup\{\|\mathfrak{C}[\theta]f\|_{p\omega}/\|f\|_{p\omega};\, 0 < \|f\|_{p\omega} < \infty\} < \infty$. We also write $\|\mathfrak{C}[\theta]\|_p$ in the case $\omega(x) \equiv 1$. We say that $\omega(x)$ satisfies the Muckenhoupt (A_p) condition if

$$(\mathrm{A}_p)\qquad \sup_I\,(m_I\omega)(m_I\omega^{-1/(p-1)})^{p-1} < \infty\,,$$

where "$\sup_I$" denotes the supremum over all finite intervals I and $m_I\omega = (1/|I|)\int_I \omega(x)\,dx$ ($|I|$: the measure of I). It is well-known that Calderón-Zygmund kernels are of type (p, ω) if $\omega(x)$ satisfies (A_p) ([2]). We shall show that the analogous property is valid for some Cauchy kernels. We say that a locally integrable function $f(x)$ is of bounded mean oscillation if $\|f\|_{\mathrm{BMO}} = \sup_I m_I|f - m_I f| < \infty$. The space BMO of functions of bounded mean oscillation, modulo constants, is a Banach space with norm $\|\cdot\|_{\mathrm{BMO}}$. We show

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THEOREM 1. *If* $\theta \in BMO$, *then* $\mathfrak{C}[\theta]$ *is of type* (p, ω) *for any* $\omega(x)$ *with* (A_p).

It is necessary to study whether "$\theta \in \mathrm{BMO}$" is a sharp condition for which $\mathfrak{C}[\theta]$ is of type (p, ω). In this paper we work only with $p = 2$ and $\omega(x) \equiv 1$. Let us define the distance $d(\theta)$ between $\theta(x)$ and 0 by the supremum of $\|\mathfrak{C}[\theta + u] - \mathfrak{C}[0]\|_2$ over all real numbers u. As a response to this subject, we show

THEOREM 2. $\lim_{t\to 0} d(t\theta) = 0$ *if and only if* $\theta \in BMO$.

§ 2. Notation and Lemmas

We use C for absolute constants. The value of C differs in general from one occasion to another. Let L^∞ denote the Banach space of functions $f(x)$ with norm $\|f\|_\infty = \text{ess.}\sup_x |f(x)| < \infty$ and L^1_{loc} the totality of locally integrable functions. The maximal function of $f \in L^1_{\text{loc}}$ is defined by $f^*(x) = \sup_{x\in I} m_I|f|$, where "$\sup_{x\in I}$" is the supremum over all finite intervals I containing x. For a measurable set E in $(-\infty, \infty)$, $\chi_E(x)$ denotes the characteristic function of E. Given $0 < \varepsilon < \eta$ and a real number y, we put $\chi^{(y)}_{\varepsilon,\eta}(x) = \chi^{(y)}_\varepsilon(x) - \chi^{(y)}_\eta(x)$, where $\chi^{(y)}_s(x) = \chi_{[y-s,y+s]^c}(x)$ $(s = \varepsilon, \eta)$. We write $\mathfrak{D}[\theta](x, y) = \mathfrak{C}[\theta](x, y) - \mathfrak{C}[0](x, y)$. For $f \in L^1_{\text{loc}}$, we put $\mathfrak{D}^*[\theta]f(x) = \sup\{|\mathfrak{D}[\theta](\chi^{(x)}_{\varepsilon,\eta} f)(x)|;\ 0 < \varepsilon < \eta\}$. The norm $\|\mathfrak{D}^*[\theta]\|_{p\omega}$ is defined analogously as $\|\mathfrak{C}[\theta]\|_{p\omega}$. Here are some lemmas necessary for the proofs of our theorems.

LEMMA 3 (The Calderón-Zygmund decomposition: Stein [8, p. 17]). *Let* $f \in L^1$ *and* $\lambda > 0$. *Then there exists a sequence* $\{J_k\}_{k=1}^\infty$ *of mutually disjoint finite intervals such that, with* $J = \bigcup_{k=1}^\infty J_k$,

$$(3) \qquad |J| \le C\|f\|_1/\lambda, \quad m_{J_k}|f| \le 2\lambda \ (k \ge 1), \quad |f(x)| \le \lambda \ \text{ a.e. in } J^c.$$

LEMMA 4 (John-Nirenberg [5]). *Let* $r \ge 1$, $f \in BMO$ *and* I *be a finite interval. Then* $m_I|f - m_I f|^r \le C^r \Gamma(r + 1)\|f\|^r_{\text{BMO}}$.

LEMMA 5 (Coifman-Fefferman [2]). *If* $\omega(x)$ *satisfies* (A_p), *then there exist* $1 < q < p$ *and a constant* B_1 *such that, for any* $f \in L^q_\omega$, $\|f^*\|_{q\omega} \le B_1\|f\|_{q\omega}$.

LEMMA 6 (Coifman-Fefferman [2]). *If* $\omega(x)$ *satisfies* (A_p), *then there exist two constants* $0 < \gamma \le 1$ *and* $B_2 \ge 1$ *such that, for any finite interval* I *and a set* E *in* I, $\omega(E)/\omega(I) \le B_2(|E|/|I|)^\gamma$, *where* $\omega(F) = \int_F \omega(x)\,dx$ $(F \subset (-\infty, \infty))$.

LEMMA 7 (Murai [6]). *For a real-valued function $\theta(x)$ in L^∞, we put*

$$\mathfrak{R}[\theta](x, y) = \frac{1}{x-y} \exp\left\{-i\frac{\Theta(x)-\Theta(y)}{x-y}\right\}. \tag{4}$$

Then there exists an absolute constant $N \geq 2$ such that, for any $r > 1$, $\|\mathfrak{R}^*[\theta]\|_r \leq \{Cr/(r-1)\}\rho(\|\theta\|_\infty)$, where $\rho(t) = (1+t)^N$ $(t \geq 0)$.

LEMMA 8. *For a real-valued function $\theta(x)$ in L^∞ and a real number u, we put*

$$\begin{aligned} &\mathfrak{T}_n[\theta, u](x, y) \\ &= \left\{\frac{\Theta(x)-\Theta(y)}{x-y}\right\}^n \Big/ \{(x-y) + i(\Theta(x)-\Theta(y) + u(x-y))\} \\ &\qquad (n = 0, 1). \end{aligned} \tag{5}$$

Then, for any $r > 1$, $\|\mathfrak{T}_n^*[\theta, u]\|_r \leq \{Cr/(r-1)\}\|\theta\|_\infty^n \rho(\|\theta\|_\infty)$ $(n = 0, 1)$.

Proof. We choose an infinitely differentiable function $\psi_n(x)$ so that $\psi_n(x) = x^n$ $(|x| \leq 1)$ and $\int_{-\infty}^\infty |\hat\psi_n(t)|\rho(t)\,dt < \infty$, where $\hat\psi_n(t) = \int_{-\infty}^\infty e^{-itx}\psi_n(x)\,dx$. We have

$$1/\{(x-y) + i(\Theta(x)-\Theta(y) + u(x-y))\} = \int_0^\infty \mathfrak{R}[s\theta](x, y)e^{-s(1+iu)}\,ds$$

and

$$\psi_n\left\{\frac{\Theta(x)-\Theta(y)}{(x-y)\|\theta\|_\infty}\right\} = C\int_{-\infty}^\infty \hat\psi_n(t)\exp\left\{it\frac{\Theta(x)-\Theta(y)}{(x-y)\|\theta\|_\infty}\right\}dt.$$

Hence

$$\begin{aligned} &\mathfrak{T}_n[\theta, u](x, y) \\ &\quad = \|\theta\|_\infty^n \psi_n\left\{\frac{\Theta(x)-\Theta(y)}{(x-y)\|\theta\|_\infty}\right\} \Big/ \{(x-y) + i(\Theta(x)-\Theta(y) + u(x-y))\} \\ &\quad = C\|\theta\|_\infty^n \int_{-\infty}^\infty \hat\psi_n(t)\,dt \int_0^\infty \mathfrak{R}[s\theta - t\theta/\|\theta\|_\infty](x, y)e^{-s(1+iu)}\,ds. \end{aligned}$$

Using Lemma 7, $\rho(s+t) \leq \rho(s)\rho(t)$ and $\rho(st) \leq \rho(s)\rho(t)$, we have

$$\begin{aligned} \|\mathfrak{T}_n^*[\theta, u]\|_r &\leq C\|\theta\|_\infty^n \int_{-\infty}^\infty |\hat\psi_n(t)|\,dt \int_0^\infty \|\mathfrak{R}^*[s\theta - t\theta/\|\theta\|_\infty]\|_r e^{-s}\,ds \\ &\leq \{Cr/(r-1)\}\|\theta\|_\infty^n \int_{-\infty}^\infty |\hat\psi_n(t)|\,dt \int_0^\infty \rho(s\|\theta\|_\infty + |t|)e^{-s}\,ds \\ &\leq \left\{[Cr/(r-1)]\int_{-\infty}^\infty |\hat\psi_n(t)|\rho(t)\,dt \int_0^\infty \rho(s)e^{-s}\,ds\right\}\|\theta\|_\infty^n \rho(\|\theta\|_\infty). \end{aligned} \tag{6}$$

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§3. Proof of Theorem 1

Let $\theta \in \mathrm{BMO}$ and let $\omega(x)$ satisfy (A_p). Since $\mathfrak{C}[0]$ is of type (p, ω) ([2]), it is sufficient to show that $\mathfrak{D}[\theta]$ is of type (p, ω). To do this we show that $\mathfrak{D}^*[\theta]$ is of type (p, ω); once this is known, a standard argument easily shows that (2) exists a.e. for any $f \in L^p_\omega$, and hence the (p, ω)-ness of $\mathfrak{D}[\theta]$ follows. For $s \geq 1$, we define

$$(7) \qquad \Lambda_s(f)(x) = (f^s)^*(x)^{1/s}, \qquad \Gamma_s(f)(x) = \sup_{x \in I} \{m_I(|\theta - m_I\theta||f|)^s\}^{1/s}.$$

Note that $f^*(x) \leq \Lambda_s(f)(x)$ and $\Gamma_1(f)(x) \leq \Gamma_s(f)(x)$ $(s \geq 1)$. We choose r so that $1 < r < p/q$, where q is the number associated with p in Lemma 5.

Given $f \in L^p_\omega$ with compact support, we now prove the following good λ inequality:

$$(8) \qquad \omega(x; \mathfrak{D}^*[\theta]f(x) > 2\lambda,\ \Xi(x) \leq \delta\lambda) \leq 4^{-p}\omega(x; \mathfrak{D}^*[\theta]f(x) > \lambda) \quad (\lambda > 0),$$

where $\Xi(x) = \rho(\|\theta\|_{\mathrm{BMO}})\{\Gamma_r(f)(x) + \|\theta\|_{\mathrm{BMO}}\Lambda_r(f)(x)\}$ and a constant δ is determined later. Given $\lambda > 0$, we put

$$(9) \qquad U(\lambda) = \{x; \mathfrak{D}^*[\theta]f(x) > \lambda\}, \qquad \sigma(\lambda) = \omega(U(\lambda)).$$

Then we can write $U(\lambda) = \bigcup_{k=1}^\infty I_k$ with a sequence $\mathfrak{M}(\lambda) = \{I_k\}_{k=1}^\infty$ of mutually disjoint finite open intervals. To prove (8), it is sufficient to show that, for any $I \in \mathfrak{M}(\lambda)$,

$$(10) \qquad |x \in I;\ \mathfrak{D}^*[\theta]f(x) > 2\lambda,\ \Xi(x) \leq \delta\lambda| \leq B_3|I|,$$

where $B_3 = (4^pB_2)^{-1/\gamma}$ (γ, B_2: the constants in Lemma 6); once this is known, Lemma 6 gives $\omega(x \in I; \mathfrak{D}^*[\theta]f(x) > 2\lambda, \Xi(x) \leq \delta\gamma) \leq 4^{-p}\omega(I)$ $(I \in \mathfrak{M}(\lambda))$, and hence adding in $I \in \mathfrak{M}(\lambda)$, we obtain (8).

Let $I = (a, b) \in \mathfrak{M}(\lambda)$. If $\Xi(x) > \delta\lambda$ in I, nothing is to be proved. Assuming that $\Xi(\xi) \leq \delta\lambda$ for some $\xi \in I$, we prove

$$(11) \qquad |x \in I;\ \mathfrak{D}^*[\theta]f(x) > 2\lambda| \leq B_3|I|.$$

We have, with $\chi(x) = \chi_{\tilde{I}}(x)$ $(\tilde{I} = (a - 3|I|, a + 3|I|))$,

$$(12) \qquad \begin{aligned} &|x \in I;\ \mathfrak{D}^*[\theta]f(x) > 2\lambda| \\ &\leq |x \in I;\ \mathfrak{D}^*[\theta](\chi f)(x) > \lambda/2| + |x \in I;\ \mathfrak{D}^*[\theta](\chi^c f)(x) > 3\lambda/2| \\ &(= P_1 + P_2, \text{ say}). \end{aligned}$$

First we estimate P_1. Let $I^* = (a - 4|I|, a + 4|I|)$. We have, with

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$g(x) = (\theta(x) - m_{I*}\theta)\chi_{I*}(x)$ and $G(x) = \int_0^x g(z)\,dz$.

$$\mathfrak{D}[\theta](x, y)$$
$$(13) \qquad = i\left\{g(y) - \frac{G(x) - G(y)}{x - y}\right\} \Big/ \{(x-y) + i(G(x) - G(y) + m_{I*}\theta(x-y))\}$$
$$(x \in I,\ y \in \tilde{I}) .$$

Since $\|g\|_1 \leq C\|\theta\|_{\mathrm{BMO}}|I|$, the Calderón-Zygmund decomposition shows that there exists a sequence $\{J_k\}_{k=1}^{\infty}$ of mutually disjoint finite intervals such that, with $J = \bigcup_{k=1}^{\infty} J_k$,

$$(14) \qquad \begin{array}{l} |J| \leq (B_3/4)|I|, \quad m_{J_k}|g| \leq C\|\theta\|_{\mathrm{BMO}} \quad (k \geq 1) \\ |g(x)| \leq C\|\theta\|_{\mathrm{BMO}} \quad \text{a.e. in } J^c . \end{array}$$

We put $h(x) = g(x)$ $(x \in I_* - (\bigcup_{k \in \Lambda} J_k))$, $h(x) = m_{J_k} g$ $(x \in J_k,\ k \in \Lambda)$ and $h(x) = 0$ $(x \in I_*^c)$, where $\Lambda = \{k;\ J_k \subset I^*\}$ and I_* is the smallest interval containing $\bigcup_{k \in \Lambda} J_k$. We may assume $I_* \supset \tilde{I}$, adding small intervals if necessary. Put $H(x) = \int_d^x h(z)\,dz$ (d: a point in $J - I$). Then we have, for any $x \in I$, $y \in \tilde{I}$,

$$(15) \qquad \begin{aligned} \mathfrak{D}[\theta](x, y) &= i\mathfrak{T}_0[h, m_{I*}\theta](x, y)g(y) - i\mathfrak{T}_1[h, m_{I*}\theta](x, y) \\ &\quad + i\{\mathfrak{T}_0[g, m_{I*}\theta](x, y) - \mathfrak{T}_0[h, m_{I*}\theta](x, y)\}g(y) \\ &\quad - i\left\{\frac{G(x) - G(y)}{x - y} - \frac{H(x) - H(y)}{x - y}\right\}\mathfrak{T}_0[g, m_{I*}\theta](x, y) \\ &\quad - i\left\{\frac{H(x) - H(y)}{x - y}\right\}\{\mathfrak{T}_0[g, m_{I*}\theta](x, y) - \mathfrak{T}_0[h, m_{I*}\theta](x, y)\} \\ &\quad (= \mathfrak{D}_1(x, y) + \mathfrak{D}_2(x, y) + \cdots + \mathfrak{D}_5(x, y), \text{ say}) . \end{aligned}$$

Since $\|h\|_\infty \leq C\|\theta\|_{\mathrm{BMO}}$, Lemma 8 shows that

$$(16) \qquad \begin{aligned} &|x \in I;\ \mathfrak{D}_1^*(\chi f)(x) > \lambda/10| \leq |x;\ \mathfrak{T}_0^*[h, m_{I*}\theta](g\chi f)(x) > \lambda/10| \\ &\quad \leq \{C\|\mathfrak{T}_0^*[h, m_{I*}\theta]\|_r \|g\chi f\|_r/\lambda\}^r \\ &\quad \leq \{[Cr/(r-1)]\rho(\|\theta\|_{\mathrm{BMO}})\Gamma_r(f)(\xi)/\lambda\}^r|I| \leq \{C\delta r/(r-1)\}^r|I| . \end{aligned}$$

We have analogously

$$(17) \qquad \begin{aligned} &|x \in I;\ \mathfrak{D}_2^*(\chi f)(x) > \lambda/10| \leq \{C\|\mathfrak{T}_1^*[h, m_{I*}\theta]\|_r \|\chi f\|_r/\lambda\}^r \\ &\quad \leq \{[Cr/(r-1)]\|\theta\|_{\mathrm{BMO}}\rho(\|\theta\|_{\mathrm{BMO}})\Lambda_r(f)(\xi)/\lambda\}^r|I| \leq \{C\delta r/(r-1)\}^r|I| . \end{aligned}$$

Let $J^* = \bigcup_{k \in \Lambda} J_k^*$, where J_k^* is the open interval with the same midpoint as J_k and length $2|J_k|$. Then $|J^*| \leq (B_3/2)|I|$. We have, for any $x \in I - J^*$,

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$$\mathfrak{D}_3^*(\chi f)(x) \leq \int_{-\infty}^{\infty} |\mathfrak{T}_0[g, m_{I^*}\theta](x, y) - \mathfrak{T}_0[h, m_{I^*}\theta](x,y)||(g\chi f)(y)|\, dy$$

$$\leq \int_{-\infty}^{\infty} \left\{\left|\int_y^x (g(z) - h(z))\, dz\right| \Big/ (x-y)^2\right\} |(g\chi f)(y)|\, dy$$

$$\leq \sum_{k\in\Lambda} \int_{J_k} \left\{\left(\int_{J_k} |g(z) - h(z)|\, dz\right) \Big/ (x-y)^2\right\} |(g\chi f)(y)|\, dy$$

$$\leq \|\theta\|_{\mathrm{BMO}} \sum_{k\in\Lambda} |J_k| \int_{J_k} |(g\chi f)(y)|/(x-y)^2\, dy \ (= \mathfrak{E}(x), \text{ say}) .$$

Since

$$\int_{I-J^*} \mathfrak{E}(x)\, dx \leq C\|\theta\|_{\mathrm{BMO}} \|g\chi f\|_1 \leq C\rho(\|\theta\|_{\mathrm{BMO}}) \Gamma_1(f)(\xi)\, |I| \leq C\delta\lambda |I| ,$$

we have

(18) $$|x \in I - J^*;\ \mathfrak{D}_3^*(\chi f)(x) > \lambda/10| \leq C\delta |I| .$$

Since

$$\mathfrak{D}_4^*(\chi f)(x) \leq \|\theta\|_{\mathrm{BMO}} \sum_{k\in\Lambda} |J_k| \int_{J_k} |(\chi f)(x)|/(x-y)^2\, dy \qquad (x \in I - J^*) ,$$

we have

(19) $$|x \in I - J^*;\ \mathfrak{D}_4^*(\chi f)(x) > \lambda/10| \leq (10/\lambda) C \|\theta\|_{\mathrm{BMO}} f^*(\xi)\, |I| \leq C\delta |I| .$$

In the same manner as $\mathfrak{D}_3^*$, we have

(20) $$|x \in I - J^*;\ \mathfrak{D}_5^*(\chi f)(x) > \lambda/10| \leq C\delta |I| .$$

Consequently, we have, by (16), $\cdots$, (20) and $|J^*| \leq (B_3/2)|I|$,

(21) $$P_1 \leq \{C\delta r/(r-1) + C\delta + B_3/2\} |I| .$$

Next we estimate P_2. Let $x \in I$. Since $\mathfrak{D}^*[\theta]f(a) \leq \lambda$, we have, for any $0 < \varepsilon < \eta$,

(22)
$$\begin{aligned} |\mathfrak{D}[\theta](\chi_{\varepsilon,\eta}^{(x)}\chi^c f)(x)| &\leq |\mathfrak{D}[\theta](\chi_{\varepsilon,\eta}^{(x)}\chi^c f)(x) - \mathfrak{D}[\theta](\chi_{\varepsilon,\eta}^{(a)}\chi^c f)(a)| \\ &\quad + |\mathfrak{D}[\theta](\chi_{\varepsilon,\eta}^{(a)}\chi^c f)(a)| \\ &\leq |\mathfrak{D}[\theta](\chi_{\varepsilon}^{(x)}\chi^c f)(x) - \mathfrak{D}[\theta](\chi_{\varepsilon}^{(a)}\chi^c f)(a)| \\ &\quad + |\mathfrak{D}[\theta](\chi_{\eta}^{(x)}\chi^c f)(x) - \mathfrak{D}[\theta](\chi_{\eta}^{(a)}\chi^c f)(a)| + \lambda \\ &\quad (= Q_\varepsilon + Q_\eta + \lambda, \text{ say}) . \end{aligned}$$

Let $V_1 = (x-\varepsilon, x+\varepsilon)^c \cap (a-\varepsilon, a+\varepsilon)^c$, $V_2 = (x-\varepsilon, x+\varepsilon)^c \backslash (a-\varepsilon, a+\varepsilon)^c$ and $V_3 = (a-\varepsilon, a+\varepsilon)^c \backslash (x-\varepsilon, x+\varepsilon)^c$. Then

$$Q_\varepsilon \le \int_{V_1} |\mathfrak{D}[\theta](x, y) - \mathfrak{D}[\theta](a, y)||(\chi^c f)(y)|\, dy$$

(23) $$+ \int_{V_2} |\mathfrak{D}[\theta](x, y)||(\chi^c f)(y)|\, dy + \int_{V_3} |\mathfrak{D}[\theta](a, y)||(\chi^c f)(y)|\, dy$$

$$(= Q_{\varepsilon 1} + Q_{\varepsilon 2} + Q_{\varepsilon 3}, \text{ say}) .$$

We have

(24) $$\begin{aligned} Q_{\varepsilon 1} &\le \int_{-\infty}^{\infty} |\mathfrak{D}[\theta](x, y) - \mathfrak{D}[\theta](a, y)||(\chi^c f)(y)|\, dy \\ &\le \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} |\mathfrak{D}[\theta](x, y) - \mathfrak{D}[\theta](a, y)||(\partial\chi_k f)(y)|\, dy , \end{aligned}$$

where $\partial\chi_k(y)$ denotes the characteristic function of $\tilde{I}_k - \tilde{I}_{k-1}$ and $\tilde{I}_\ell$ is the open interval with midpoint a and length $2^\ell|\tilde{I}|$ ($\ell \ge 0$). Let $\tilde{\theta}_k(y) = (\theta(y) - m_{\tilde{I}_k}\theta)\chi_{\tilde{I}_k}(y)$ ($k \ge 1$). Then we have, for any $x \in I$, $y \in \tilde{I}_k$,

$$\begin{aligned} &\mathfrak{D}[\theta](x, y) - \mathfrak{D}[\theta](a, y) \\ &= i\Big[\Big\{\tilde{\theta}_k(y) - \frac{1}{x-y}\int_y^x \tilde{\theta}_k(z)\, dz\Big\}\Big/\{(x-y) + i(\Theta(x) - \Theta(y))\} \\ &\quad - \Big\{\tilde{\theta}_k(y) - \frac{1}{a-y}\int_y^a \tilde{\theta}_k(z)\, dz\Big\}\Big/\{(a-y) + i(\Theta(a) - \Theta(y))\}\Big] \\ &= i\tilde{\theta}_k(y)\Big\{\frac{1}{(x-y) + i(\Theta(x) - \Theta(y))} - \frac{1}{(a-y) + i(\Theta(a) - \Theta(y))}\Big\} \\ &\quad - i\Big\{\frac{1}{x-y}\int_y^x \tilde{\theta}_k(z)\, dz\Big\}\Big\{\frac{1}{(x-y) + i(\Theta(x) - \Theta(y))} \\ &\qquad - \frac{1}{(a-y) + i(\Theta(a) - \Theta(y))}\Big\} \\ &\quad - i\Big\{\frac{1}{x-y}\int_y^x \tilde{\theta}_k(z)\, dz - \frac{1}{a-y}\int_y^a \tilde{\theta}_k(z)\, dz\Big\}\Big/ \\ &\qquad \{(a-y) + i(\Theta(a) - \Theta(y))\} \end{aligned}$$

(25) $$\begin{aligned} &= \tilde{\theta}_k(y)\Big\{\frac{\Theta(x) - \Theta(y)}{x-y} - \frac{\Theta(a) - \Theta(y)}{a-y}\Big\}\Big/ \\ &\qquad \Big\{(x-y)\Big(1 + i\frac{\Theta(x) - \Theta(y)}{x-y}\Big)\Big(1 + i\frac{\Theta(a) - \Theta(y)}{a-y}\Big)\Big\} \\ &\quad - i\tilde{\theta}_k(y)(x-a)\Big/\Big\{(x-y)(a-y)\Big(1 + i\frac{\Theta(a) - \Theta(y)}{a-y}\Big)\Big\} \\ &\quad - \Big\{\frac{1}{x-y}\int_y^x \tilde{\theta}_k(z)\, dz\Big\}\Big\{\frac{\Theta(x) - \Theta(y)}{x-y} - \frac{\Theta(a) - \Theta(y)}{a-y}\Big\}\Big/ \\ &\qquad \Big\{(x-y)\Big(1 + i\frac{\Theta(x) - \Theta(y)}{x-y}\Big)\Big(1 + i\frac{\Theta(a) - \Theta(y)}{a-y}\Big)\Big\} \end{aligned}$$

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$$+ i\left\{\frac{1}{x-y}\int_y^x \tilde{\theta}_k(z)\,dz\right\}(x-a)\Big/ \left\{(x-y)(a-y)\left(1+i\frac{\Theta(a)-\Theta(y)}{a-y}\right)\right\}$$
$$- i\left\{\frac{1}{x-y}\int_a^x \tilde{\theta}_k(z)\,dz\right\}\Big/\{(a-y)+i(\Theta(a)-\Theta(y))\}$$
$$+ i\left\{\frac{(x-a)}{(x-y)(a-y)}\int_y^a \tilde{\theta}_k(z)\,dz\right\}\Big/\{(a-y)+i(\Theta(a)-\Theta(y))\}\,.$$

Note that $|m_{\tilde{I}_\ell}\theta - m_{\tilde{I}_{\ell-1}}\theta| \le C\|\theta\|_{\mathrm{BMO}}$ ($\ell \ge 1$). Since

$$\int_{\tilde{I}} |\tilde{\theta}_k(z)|\,dz \le \int_{\tilde{I}} |\theta(z) - m_{\tilde{I}_0}\theta|\,dz + |m_{\tilde{I}_0}\theta - m_{\tilde{I}_k}\theta||\tilde{I}| \le Ck\|\theta\|_{\mathrm{BMO}}|\tilde{I}|\,,$$

we have, for any $x \in I$, $y \in \tilde{I}_k - \tilde{I}_{k-1}$,

$$\left|\frac{\Theta(x)-\Theta(y)}{x-y} - \frac{\Theta(a)-\Theta(y)}{a-y}\right|$$
$$(26)\qquad = \left|\frac{1}{x-y}\int_y^x \tilde{\theta}_k(z)\,dz - \frac{1}{a-y}\int_y^a \tilde{\theta}_k(z)\,dz\right|$$
$$= \left|\frac{1}{x-y}\int_a^x \tilde{\theta}_k(z)\,dz - \frac{(x-a)}{(x-y)(a-y)}\int_y^a \tilde{\theta}_k(z)\,dz\right|$$
$$\le (C/|\tilde{I}_k|)\int_{\tilde{I}} |\tilde{\theta}_k(z)|\,dz + (C2^{-k}/|\tilde{I}_k|)\int_{\tilde{I}_k} |\tilde{\theta}_k(z)|\,dz \le C\|\theta\|_{\mathrm{BMO}}k2^{-k}\,.$$

By (25) and (26), we have

$$|\mathfrak{D}[\theta](x,y) - \mathfrak{D}[\theta](a,y)|$$
$$\le C\rho(\|\theta\|_{\mathrm{BMO}})\{|\tilde{\theta}_k(y)| + \|\theta\|_{\mathrm{BMO}}\}k2^{-k}/|\tilde{I}_k| \qquad (x\in I,\, y \in \tilde{I}_k - \tilde{I}_{k-1})\,,$$

and hence

$$(27)\qquad Q_{\varepsilon 1} \le C\rho(\|\theta\|_{\mathrm{BMO}})\sum_{k=1}^{\infty}(k2^{-k}/|\tilde{I}_k|)\int_{-\infty}^{\infty}\{|\tilde{\theta}_k(y)| + \|\theta\|_{\mathrm{BMO}}\}|(\partial\chi_k f)(y)|\,dy$$
$$\le C\rho(\|\theta\|_{\mathrm{BMO}})\left\{\sum_{k=1}^{\infty}k2^{-k}\right\}\{\Gamma_1(f)(\xi) + \|\theta\|_{\mathrm{BMO}}f^*(\xi)\} \le C\Xi(\xi) \le C\delta\lambda\,.$$

If $\varepsilon < 2|I|$, then $Q_{\varepsilon 2} = Q_{\varepsilon 3} = 0$. If $\varepsilon \ge 2|I|$, then we have, with $\tilde{\theta}_V(y) = (\theta(y) - m_V\theta)\chi_V(y)$ $(V = (a-\varepsilon, b))$,

$$Q_{\varepsilon 2} = \int_{V_2}\left|\left\{\tilde{\theta}_V(y) - \frac{1}{x-y}\int_y^x \tilde{\theta}_V(z)\,dz\right\}\Big/\right.$$
$$(28)\qquad \left.\{(x-y)+i(\Theta(x)-\Theta(y))\}\right||(\chi^c f)(y)|\,dy$$

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$$\le (C/|V|)\int_{V_2}\{|\tilde{\theta}_V(y)| + \|\theta\|_{\mathrm{BMO}}\}|(\chi^c f)(y)|\,dy$$
$$\le C\{\Gamma_1(f)(\xi) + \|\theta\|_{\mathrm{BMO}} f^*(\xi)\} \le C\Xi(\xi) \le C\delta\lambda .$$

We have analogously $Q_{\varepsilon 3} \le C\delta\lambda$. Consequently, $Q_\varepsilon \le C\delta\lambda$. In the same manner, we have $Q_\eta \le C\delta\lambda$. Since $0 < \varepsilon < \eta$ are abritrary, we have, by (22),

$$\mathfrak{D}^*[\theta](\chi^c f)(x) \le (1 + C\delta)\lambda \qquad (x \in I) . \tag{29}$$

Using (21) and (29), we choose δ so small that $P_1 \le B_3|I|$ and $P_2 = 0$. Then we have (11) according to (12). Hence we obtain (10). This completes the proof of (8).

Now we deduce the (p, ω)-ness of $\mathfrak{D}^*[\theta]$ from (8). By Lemma 4, we have, with a constant $D_1 \ge 1$,

$$\Gamma_r(f)(x) = \sup_{x\in I}\left\{(1/|I|)\int_I (|\theta(y) - m_I\theta||f(y)|)^r\,dy\right\}^{1/r}$$
$$\le \sup_{x\in I}\left\{(1/|I|)\int_I |\theta(y) - m_I\theta|^{pr/(p-qr)}\,dy\right\}^{(p-qr)/pr}$$
$$\times \left\{(1/|I|)\int_I |f(y)|^{p/q}\,dy\right\}^{q/p} \le D_1\|\theta\|_{\mathrm{BMO}}\Lambda_{p/q}(f)(x) ,$$

and hence $\Xi(x) \le CD_1\|\theta\|_{\mathrm{BMO}}\rho(\|\theta\|_{\mathrm{BMO}})\Lambda_{p/q}(f)(x)$. We have, by Lemma 5,

$$\begin{aligned}\|\Xi\|_{p\omega} &\le CD_1\|\theta\|_{\mathrm{BMO}}\rho(\|\theta\|_{\mathrm{BMO}})\|(f^{p/q})^*\|_{q\omega}^{q/p} \\ &\le CD_1B_1\|\theta\|_{\mathrm{BMO}}\rho(\|\theta\|_{\mathrm{BMO}})\|(f^{p/q})\|_{q\omega}^{q/p} = CD_1B_1\|\theta\|_{\mathrm{BMO}}\rho(\|\theta\|_{\mathrm{BMO}})\|f\|_{p\omega} .\end{aligned} \tag{30}$$

Let $\kappa(\lambda) = \omega(x;\, \Xi(x) > \lambda)$ $(\lambda > 0)$. Then (8) shows that

$$\sigma(2\lambda) \le \kappa(\delta\lambda) + 4^{-p}\sigma(\lambda) \qquad (\lambda > 0) . \tag{31}$$

As in the proof of Lemma 12 in [7], we can easily verify that the following formal calculus holds true.

Integrating each quantity in (31) by $\lambda^{p-1}d\lambda$ from 0 to infinity, we have, with a constant D_2,

$$\int_0^\infty \lambda^{p-1}\sigma(\lambda)\,d\lambda \le (C/\delta)^p\int_0^\infty \lambda^{p-1}\kappa(\lambda)\,d\lambda \le \{(D_2/\delta)\|\theta\|_{\mathrm{BMO}}\rho(\|\theta\|_{\mathrm{BMO}})\|f\|_{p\omega}\}^p .$$

This shows that $\|\mathfrak{D}^*[\theta]f\|_{p\omega}/\|f\|_{p\omega} \le (D_2/\delta)\|\theta\|_{\mathrm{BMO}}\,\rho(\|\theta\|_{\mathrm{BMO}})$. Taking the supremum over all $f \in L^p_\omega$ with compact support, we have

$$\|\mathfrak{D}^*[\theta]\|_{p\omega} \le (D_2/\delta)\,\|\theta\|_{\mathrm{BMO}}\,\rho(\|\theta\|_{\mathrm{BMO}}) . \tag{32}$$

This completes the proof of Theorem 1.

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§ 4. Proof of Theorem 2

Let $\theta \in \mathrm{BMO}$. Then we have, by (32), $d(t\theta) = O(t)$ $(t \to 0)$, and hence $\lim_{t\to 0} d(t\theta) = 0$. Suppose that $\lim_{t\to 0} d(t\theta) = 0$. We choose $s > 0$ so that $d(s\theta) \leq \tau$, where τ is determined later. Given a finite interval I, we denote by w its midpoint. Put $\bar{\theta}(x) = s\theta(x)$, $\tilde{\theta}(x) = \bar{\theta}(x) - m_I\bar{\theta}$ and

$$\mathfrak{G}(x, y) = \{\tilde{\theta}(x) - \tilde{\theta}(y)\} \Big/ \left\{(x - y) + i \int_y^x \tilde{\theta}(z)\, dz\right\}. \tag{33}$$

Since $\mathfrak{G}(x, y)/i$ is the sum of $\mathfrak{D}[\tilde{\theta}](x, y)$ and the dual kernel of $\overline{\mathfrak{D}[\tilde{\theta}](x, y)}$, we have $\|\mathfrak{G}\|_2 \leq 2\tau$. We have, for almost all x in I,

$$\begin{aligned}
|I|\{\bar{\theta}(x) - m_I\bar{\theta}\} &= \int_I \{\tilde{\theta}(x) - \tilde{\theta}(y)\}\, dy \\
&= \int_I \mathfrak{G}(x, y)\left\{(x - y) + i \int_y^x \tilde{\theta}(z)\, dz\right\} dy \\
&= (x - w)\mathfrak{G}\chi_I(x) - \mathfrak{G}\{(\cdot - w)\chi_I\}(x) \\
&\quad + i\left\{\int_w^x \tilde{\theta}(z)\, dz\right\}\mathfrak{G}\chi_I(x) - i\mathfrak{G}\left\{\left(\int_w^{\cdot} \tilde{\theta}(z)\, dz\right)\chi_I\right\}(x).
\end{aligned}$$

Thus

$$\begin{aligned}
&|I|\int_I |\bar{\theta}(x) - m_I\bar{\theta}|\, dx \\
&\leq \left\{\int_I (x - w)^2\, dx\right\}^{1/2} \|\mathfrak{G}\chi_I\|_2 + \sqrt{|I|}\,\|\mathfrak{G}\{(\cdot - w)\chi_I\}\|_2 \\
&\quad + \left[\int_I \left\{\int_w^x \tilde{\theta}(z)\, dz\right\}^2 dx\right]^{1/2} \|\mathfrak{G}\chi_I\|_2 + \sqrt{|I|}\left\|\mathfrak{G}\left\{\left(\int_w^{\cdot} \tilde{\theta}(z)\, dz\right)\chi_I\right\}\right\|_2 \\
&\leq C\left\{1 + (1/|I|)\int_I |\tilde{\theta}(z)|\, dz\right\}\|\mathfrak{G}\|_2 |I|^2 \\
&\leq C\tau\left\{1 + (1/|I|)\int_I |\bar{\theta}(z) - m_I\bar{\theta}|\, dz\right\}|I|^2.
\end{aligned} \tag{34}$$

Now we choose τ so that $C\tau \leq 1/2$. Then $(1/|I|)\int_I |\bar{\theta}(x) - m_I\bar{\theta}|\, dx \leq 1/2$. Taking the supremum over all finite intervals I, we obtain $\|\bar{\theta}\|_{\mathrm{BMO}} \leq 1/2$, which shows $\theta \in \mathrm{BMO}$. This completes the proof of Theorem 2.

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