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NOTES ON CONTACT RICCI SOLITONS

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Abstract A compact contact Ricci soliton (whose potential vector field is the Reeb vector field) is Sasaki–Einstein. A compact contact homogeneous manifold with a Ricci soliton is Sasaki–Einstein.

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1. Introduction

A *contact manifold* (M, η) is a smooth manifold M^{2n+1} together with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. This means that $d\eta$ has maximal rank $2n$ on the contact distribution $D = \ker \eta$. The duality of η defines a unique vector field ξ, the *Reeb vector field*. The *Reeb flow* is a one-parameter group of diffeomorphisms ϕ_t generated by the Reeb vector field ξ.

A *Ricci soliton* is a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M, g) by

$$\tfrac{1}{2}\mathfrak{L}_V g + \mathrm{Ric} - \lambda g = 0, \tag{1.1}$$

where V is a vector field (the potential vector field) and λ is a constant on M. Obviously, a Ricci soliton with V Killing is an Einstein metric. Compact Ricci solitons are the fixed points of the Ricci flow,

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Ric},$$

projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds. The Ricci soliton is said to be shrinking, steady or expanding if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. Hamilton [**11**] and Ivey [**12**] proved that a Ricci soliton on a compact manifold has constant curvature in dimensions 2 and 3, respectively. If the vector field V is the gradient of a potential function, then g is called a gradient Ricci soliton. We refer the reader to [**4**, **10**] for details about Ricci solitons or gradient Ricci solitons.

Then it is interesting to consider a *contact Ricci flow* which evolves by the Reeb flow and a (time-dependent) evolving factor at the same time. We will call a solution of the evolution equation a *contact Ricci soliton*. We have the contact Ricci soliton equation:

$$\tfrac{1}{2}\mathfrak{L}_\xi g + \mathrm{Ric} - \lambda g = 0. \tag{1.2}$$

A contact manifold with ξ a Killing vector field is called a *K-contact manifold*.

One of the main purposes of the present paper is to prove the following theorem.

Theorem 1.1. *A contact Ricci soliton is shrinking and is Einstein K-contact.*

On the other hand, Boyer and Galicki [**3**] proved the following result.

Theorem 1.2. *A compact Einstein K-contact manifold is Sasakian.*

Thus, together with Theorem 1.1 we have the following.

Corollary 1.3. *A compact contact Ricci soliton is Sasaki–Einstein.*

In the second half of § 3, we prove a homogeneous contact metric manifold admitting a gradient Ricci soliton is either Einstein or locally isometric to $E^{n+1} \times S^n(4)$ (Theorem 3.6). Moreover, we show that a compact contact homogeneous manifold with a Ricci soliton is Sasaki–Einstein (Corollary 3.7).

2. Preliminaries

We start by reviewing briefly the fundamental materials about contact Riemannian (CR) geometry and contact pseudo-Hermitian geometry. We refer the reader to [**2**, **17**] for further details. All manifolds in the present paper are assumed to be connected, oriented and of class C^∞.

2.1. Contact Riemannian structures

A $(2n+1)$-dimensional manifold M is a *contact manifold* if it is equipped with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form η, there exists a unique vector field ξ, called the *Reeb vector field*, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X. It is well known that there also exists a Riemannian metric g and a $(1,1)$-tensor field φ such that

$$\eta(X) = g(X,\xi), \qquad d\eta(X,Y) = g(X,\varphi Y), \qquad \varphi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

where X and Y are vector fields on M. From (2.1), it follows that

$$\varphi\xi = 0, \qquad \eta \circ \varphi = 0, \qquad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.2}$$

A Riemannian manifold M equipped with structure tensors (η, g) satisfying (2.1) is said to be a *contact Riemannian manifold* or *contact metric manifold* and it is denoted by $M = (M, \eta, g)$. Given a contact Riemannian manifold M, we define a $(1,1)$-tensor field h by $h = \frac{1}{2}\mathfrak{L}_\xi\varphi$. Then we may observe that h is symmetric and satisfies

$$h\xi = 0, \quad h\varphi = -\varphi h, \tag{2.3}$$

$$\nabla_X\xi = -\varphi X - \varphi hX, \tag{2.4}$$

where ∇ is the Levi-Cività connection. From (2.3) and (2.4) we see that each flow of ξ is a geodesic flow.

A contact Riemannian manifold for which ξ is Killing is called a *K-contact manifold.* It is easy to see that a contact Riemannian manifold is K-contact if and only if $h = 0$.

2.2. Contact pseudo-Hermitian almost-CR structures

For a contact manifold M, the tangent space T_pM of M at each point $p \in M$ is decomposed as $T_pM = D_p \oplus \{\xi\}_p$ (direct sum), where we denote $D_p = \{v \in T_pM \mid \eta(v) = 0\}$. Then the $2n$-dimensional distribution (or sub-bundle) $D\colon p \to D_p$ is called the *contact distribution* (or *contact sub-bundle*). Its associated almost-CR structure is given by the holomorphic sub-bundle

$$\mathcal{H} = \{X - iJX : X \in \Gamma(D)\}$$

of the complexification $\mathbb{C}TM$ of the tangent bundle TM, where $J = \varphi|_D$, the restriction of φ to D. Then we see that each fibre $\mathcal{H}_p$, $p \in M$, is of complex dimension n and $\mathcal{H} \cap \bar{\mathcal{H}} = \{0\}$. Furthermore, we have $\mathbb{C}D = \mathcal{H} \oplus \bar{\mathcal{H}}$. For the real representation $\{D, J\}$ of $\mathcal{H}$ we define the Levi form by

$$L\colon \Gamma(D) \times \Gamma(D) \to \mathcal{F}(M), \quad L(X, Y) = -d\eta(X, JY),$$

where $\mathcal{F}(M)$ denotes the algebra of differentiable functions on M. Then we see that the Levi form is Hermitian and positive definite. We call the pair (η, L) (or (η, J)) a *contact strictly pseudo-convex, pseudo-Hermitian structure* on M. We say that *the almost-CR structure is integrable* if $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$. Then the pair (η, J) is called a *contact strictly pseudo-convex (integrable) CR structure* and (M, η, J) is called a *contact strictly pseudo-convex CR manifold* or a *contact strictly pseudo-convex integrable pseudo-Hermitian manifold.*

For a given contact strictly pseudo-convex pseudo-Hermitian manifold M, the pseudo-Hermitian structure is integrable if and only if M satisfies the integrability condition $\Omega = 0$, where Ω is a $(1,2)$-tensor field on M defined by

$$\Omega(X, Y) = (\nabla_X\varphi)Y - g(X + hX, Y)\xi + \eta(Y)(X + hX) \tag{2.5}$$

for all vector fields X, Y on M [**17**, Proposition 2.1]. It is well known that for three-dimensional contact Riemannian manifolds their associated CR structures are always integrable. In addition, we define the *pseudo-Hermitian torsion* $A = \varphi h$ (cf. [**5**]).

A *Sasakian manifold* is a contact strictly pseudo-convex CR manifold whose Reeb flow is isometric (or, equivalently, vanishing pseudo-Hermitian torsion). From (2.5) it follows at once that a Sasakian manifold is also determined by the condition

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X \tag{2.6}$$

for all vector fields X and Y on the manifold.

3. Proofs of the results

The following lemma has a crucial role in proving Theorem 1.1.

Lemma 3.1. *If (g,V) is a Ricci soliton of a Riemannian manifold, then we have*

$$\tfrac{1}{2}\|\mathfrak{L}_V g\|^2 = dr(V) + 2\operatorname{div}(\lambda V - SV), \tag{3.1}$$

where r denotes the scalar curvature of g and S denotes the Ricci operator defined by $\operatorname{Ric}(X,Y) = g(SX,Y)$.

Proof. We adapt a local coordinate system (x^i). Then (1.1) implies

$$\tfrac{1}{2}\mathfrak{L}_V g^{ij} + R^{ij} - \lambda g^{ij} = 0. \tag{3.2}$$

From the above equation (3.2) we compute

$$\begin{aligned}\tfrac{1}{2}\|\mathfrak{L}_V g\|^2 &= -R^{ij}\mathfrak{L}_V g_{ij} + \lambda g^{ij}\mathfrak{L}_V g_{ij} \\ &= -\mathfrak{L}_V r + g_{ij}\mathfrak{L}_V R^{ij} + \lambda g_{ij}\mathfrak{L}_V g^{ij}.\end{aligned} \tag{3.3}$$

We compute the second term of the last equation:

$$\begin{aligned} g_{ij}\mathfrak{L}_V R^{ij} &= g_{ij}\nabla_V R^{ij} - g_{ij}\nabla_\alpha V^i R^{\alpha j} - g_{ij}\nabla_\alpha V^j R^{i\alpha} \\ &= g_{ij}\nabla_V R^{ij} - 2g_{ij}\nabla_\alpha V^i R^{\alpha j} \\ &= 2dr(V) - 2\operatorname{div} SV, \end{aligned} \tag{3.4}$$

where we have used $dr(V) = 2V^\beta \nabla_\alpha R^\alpha_\beta$. Since $g_{ij}\mathfrak{L}_V g^{ij} = 2\operatorname{div} V$, using (3.3) and (3.4) we obtain (3.1). □

Remark 3.2. By using Green's Theorem it follows from (3.1) that 'a compact Ricci soliton (M,g,V) with constant scalar curvature is a trivial Ricci soliton'.

Now we suppose that a contact manifold (M,η) admits a Ricci soliton (g,ξ). Then from (1.1) we get

$$\tfrac{1}{2}(g(\nabla_X\xi,Y) + g(\nabla_Y\xi,X)) + \operatorname{Ric}(X,Y) - \lambda g(X,Y) = 0 \tag{3.5}$$

Use (2.4) to obtain $\operatorname{Ric}(X,Y) - g(\varphi hX,Y) - \lambda g(X,Y) = 0$, or

$$SX = \lambda X + AX \tag{3.6}$$

for any vector field X on M. Setting $X = \xi$ in (3.6), and since $h\xi = 0$, we have $S\xi = \lambda\xi$. From (2.3) we can see that $\operatorname{tr} A$ vanishes. Hence, from (3.6) we also find that the scalar curvature r $(= (2n+1)\lambda)$ is constant. Now, we assume that $V = \xi$ in (3.1); then we have ξ a Killing vector field. Then (3.6) yields again that M is Einstein. Since $\operatorname{Ric}(\xi,\xi) = 2n$ for a K-contact manifold [**2**], we see that $\lambda > 0$. This completes the proof of Theorem 1.1.

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By (2.5) and (2.6), we have the following corollary.

Corollary 3.3. *Let (M, η) be a contact manifold whose associated pseudo-Hermitian structure (η, J) is CR-integrable. If (M, η) admits a contact Ricci soliton (g, ξ), then M is Sasaki–Einstein.*

A K-contact manifold of constant curvature has constant sectional curvature $+1$ [**16**]. Since the Weyl tensor always vanishes on a three-dimensional Riemannian manifold, we have the following result.

Corollary 3.4. *A complete and simply connected contact 3-manifold (M, η) admitting a contact Ricci soliton (g, ξ) is a unit sphere.*

It is notable that every closed 3-manifold admits a contact structure [**13**].

Corollary 3.5. *A conformally flat contact Ricci soliton (M, η, g, ξ) is of constant curvature $+1$.*

Perelman [**14**] proved that a Ricci soliton on a compact manifold is a gradient Ricci soliton and hence the potential vector field V of a compact Ricci soliton is the sum of the gradient of a function and a Killing vector field. For details we refer the reader to [**10**].

Theorem 3.6. *Let (M, η, g) be a homogeneous contact metric manifold. If g is a gradient Ricci soliton, then M is either Einstein or locally isometric to $E^{n+1} \times S^n(4)$.*

Proof. Suppose that M admits a non-trivial Ricci soliton that is non-Einstein. Then we first recall the recent result of Petersen and Wylie [**15**] that (M, g) splits as $(M_1, g_1) \times (M_2, g_2)$, where M_1 is Einstein ($\mathrm{Ric}_1 = \lambda_1 g_1$) and M_2 is Euclidean E^k. If $\lambda_1 = 0$, then we easily see that M is Ricci-flat. So, let λ_1 be non-zero. We may assume that ξ splits into the factors M_1 and M_2 as $\xi_1 + \xi_2$ orthogonally. We divide our arguments into the following three cases.

(i) Suppose that $\xi_1 \neq 0$ and $\xi_2 \neq 0$. Then, for any tangent vector field E_2 on M_2, we find $S\nabla_{E_2}\xi_2 = 0$, and hence $\nabla_{E_2}\xi_2$ is tangent to M_2. Since $\nabla_{E_2}\xi_1 = 0$, we get $\nabla_{E_2}\xi$ is tangent to M_2. Then using (2.4) we have

$$g(\varphi E_2 + \varphi h E_2, E_1) = 0 \tag{3.7}$$

for any vector field E_1 tangent to M_1. In a similar argument as above, we have that $\nabla_{E_1}\xi$ is tangent to M_1 and using (2.4) we have

$$g(\varphi E_1 + \varphi h E_1, E_2) = 0 \tag{3.8}$$

for any vector field E_2 tangent to M_2. Since φh is symmetric, from (3.7) and (3.8) it follows that φE_1 (respectively, φE_2) is tangent to M_1 (respectively, M_2). Thus, we have $0 = \varphi(\xi_1 + \xi_2) = \varphi\xi_1 + \varphi\xi_2$, which implies $\varphi\xi_1 = \varphi\xi_2 = 0$. This yields $\xi_1 = \eta(\xi_1)\xi$ and $\xi_2 = \eta(\xi_2)\xi$, which is impossible.

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(ii) Examine the case $\xi_2 = 0$. Then $S\xi = \lambda_1\xi$ and ξ is tangent to M_1. Since $\nabla_{E_2}\xi = 0$, we have $\varphi E_2 + \varphi h E_2 = 0$ for any E_2 tangent to M_2. Applying φ, we get $hE_2 = -E_2$ and $h\varphi E_2 = \varphi E_2$, where we have used $h\varphi = -\varphi h$. Hence, we have that φE_2 is tangent to M_1. Differentiating $S\xi = \lambda_1\xi$ covariantly for φE_2, then using (2.4), we have $S(E_2 - hE_2) = \lambda_1(E_2 - hE_2)$. This implies that $\lambda_1 E_2 = 0$, which is impossible.

(iii) Let us consider the case $\xi_1 = 0$ and $\xi_2 \neq 0$. It follows at once that $S\xi = 0$, and then ξ is tangent to M_2. Then we have $R(X, Y)\xi = 0$ for any vector fields X, Y on M, and hence locally flat in dimension 3 and isometric to $S^n(4) \times E^{n+1}$ in higher dimensions [**1**]. This completes the proof. □

Since the homogeneity implies the constancy of its scalar curvature, with the aforementioned Perelman's remark and using (3.1) and the Boyer–Galicki result (Theorem 1.2), we have the following.

Corollary 3.7. *A homogeneous compact contact manifold admitting a Ricci soliton is Sasaki–Einstein.*

We finish the present work by stating the related results recently obtained.

- If a compact real hypersurface M in a complex number space admits a Ricci soliton whose potential vector field is the Reeb vector field, then M is a sphere [**6**, **7**].
- A real hypersurface in a complex projective or hyperbolic space does not admit a Ricci soliton whose potential vector field is the Reeb vector field [**8**].
- If a compact real hypersurface M of contact type in a complex number space admits a gradient Ricci soliton, then M is a sphere [**9**].
- A Hopf-hypersurface in a complex projective or hyperbolic space, in which the Reeb vector field is a principal vector field, does not admit a gradient Ricci soliton [**9**].

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