GROUPS WHOSE PROJECTIVE CHARACTER DEGREES ARE POWERS OF A PRIME†

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1. Introduction. Let G be a finite group, and $P: G \to GL(n, \mathbb{C})$ be such that for all $x, y \in G$

(i) $P(x)P(y) = \alpha(x, y)P(xy)$, and

(ii) $P(1) = I_n$,

where $\alpha(x, y) \in \mathbb{C}^*$; then P is a *projective representation* of G with cocycle α and degree n. For other basic definitions concerning projective representations see **[4]**.

DEFINITION 1.1. We shall say that G is a (p, α)*-group* if for some cocycle α of G all the irreducible projective representations of G with cocycle α have degree a power of the prime number p.

The main result of this paper is the following.

THEOREM B. *If G is a (p, α)-group, then G is solvable.*

We now review some of the known facts about $(p, 1)$-groups where 1 denotes the trivial cocycle of a group. In particular, the following theorem is well-known (see (2.3) of **[5]**).

THEOREM A. *G is a $(p, 1)$-group if and only if G has a normal Abelian p-complement.*

Thus we have that Theorem B is true when α is the trivial cocycle of G. However, Theorem A does not hold in general when G is a (p, α)-group and α is not cohomologous to 1, since A_4 provides a counter-example.

Before commencing, we remark that the degrees of projective representations are unaffected under projective equivalence, so that if G is a (p, α)-group it is also a (p, β)-group for $\beta \sim \alpha$, that is for β cohomologous to α. Thus in what follows it is no loss to assume that the cocycle α under consideration is special (see **[4]** for this definition). As a consequence of this we need only consider the set of irreducible projective characters of G with cocycle α, denoted by $\mathrm{Proj}(G, \alpha)$; these in particular are class functions. Also it follows from the use of a covering group and ordinary character theory that both the orthogonality relations and Frobenius' reciprocity hold for $\mathrm{Proj}(G, \alpha)$. Finally we state and use without further reference the well-known fact that $o([\alpha])$ in $H^2(G, \mathbb{C}^*)$ divides $\xi(1)$ for all $\xi \in \mathrm{Proj}(G, \alpha)$.

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2. Preliminary Results.

Definition 2.1. Let G be a group with cocycle α, $N \trianglelefteq G$, and $\zeta \in \text{Proj}(N, \alpha_N)$. Then the g-conjugate ζ^g of ζ, for $g \in G$, is defined by

$$\zeta^g(x) = \alpha(g, xg^{-1})\alpha(x, g^{-1})(\alpha(g, g^{-1}))^{-1}\zeta(gxg^{-1})$$

for all $x \in N$.

It is routine to check that the above definition defines an action of G on $\text{Proj}(N, \alpha_N)$ for which Clifford's theorem as stated for ordinary characters holds. We denote the stabilizer of ζ in G by $I_G(\zeta)$.

We now investigate the way that normal subgroups and certain quotient groups of G effectively inherit the property of being (p, α)-groups from G.

Lemma 2.2. *Let G be a (p, α)-group, $N \trianglelefteq G$, and $\zeta \in \text{Proj}(N, \alpha_N)$. Then*

(i) *N is a (p, α_N)-group;*
(ii) *$I_G(\zeta)/N$ is a (p, β)-group for some cocycle β of $I_G(\zeta)/N$, and $[G:I_G(\zeta)]$ is a power of p.*

Proof. (i) Let $\zeta \in \text{Proj}(N, \alpha_N)$ *and* $\xi \in \text{Proj}(G, \alpha)$ such that $[\xi_N, \zeta] \neq 0$. By Clifford's theorem we have

$$\xi_N = e(\zeta_1 + \ldots + \zeta_t)$$

where $\zeta = \zeta_1, \ldots, \zeta_t$ are the distinct G-conjugates of ζ, and so $\xi(1) = et\,\zeta(1)$. However, since $\xi(1)$ is a power of p we have that $\zeta(1)$ is also a power of p.

(ii) Let $T = I_G(\zeta)$; then from Clifford's theorem there exists a cocycle β of T/N and a bijection from $\text{Proj}(T/N, \beta)$ onto $\{\eta \in \text{Proj}(T, \alpha_T):[\eta_N, \zeta] \neq 0\}$ defined by $\tau \mapsto \tau\kappa$ where $\kappa \in \text{Proj}(T, \alpha_T\beta^{-1})$. Let $\tau \in \text{Proj}(T/N, \beta)$ and $\tau\kappa = \eta$ then again by Clifford's theorem $\eta^G = \xi$ for $\xi \in \text{Proj}(G, \alpha)$. Thus

$$\eta^G(1) = [G:T]\eta(1) = [G:T]\tau(1)\kappa(1) = \xi(1)$$

and so both $[G:T]$ and $\tau(1)$ are powers of p. ■

We have shown that, if G is a (p, α)-group, then we have some restraints on $\text{Proj}(N, \alpha_N)$ where $N \trianglelefteq G$. However, we can achieve very strong control over the projective characters of G between G and $S \in \text{Syl}_p(G)$.

Lemma 2.3. *Let G be a (p, α)-group, let $L \leq G$ be such that $p \nmid [G:L]$ and let $\zeta \in \text{Proj}(L, \alpha_L)$. Then ζ extends to G.*

Proof. Let $A = \{\xi \in \text{Proj}(G, \alpha):[\xi_L, \zeta] \neq 0\}$, and ξ_s denote an element of A of smallest degree; then $\zeta(1) \leq \xi_s(1)$. By Frobenius' reciprocity let $\zeta^G = \sum_{\xi_i \in A} b_i\xi_i$ where b_i is a positive integer for all i. Then

$$\zeta^G(1) = [G:L]\zeta(1) = \sum_{\xi_i \in A} b_i\xi_i(1)$$

so since $p \nmid [G:L]$ we have

$$(\zeta(1))_p = \xi_s(1)\Big(\sum_{\xi_i \in A} b_i \xi_i(1)(\xi_s(1))^{-1}\Big)_p.$$

Thus $\xi_s(1)/(\zeta(1))_p$. We have shown that $\zeta(1) = \xi_s(1)$, and so $\zeta = (\xi_s)_L$. ■

We remark that 2.3 shows that L is a (p, α_L)-group. Finally in this section we note the following important result.

LEMMA 2.4. *If G is a solvable (p, α)-group, then the Hall p'-subgroups of G are Abelian.*

Proof. This is immediate from [**2**, p. 245]. ■

3. The solvability of (p, α)-groups. In order to minimize repetition we fix the following notation for the rest of this section. Let G be a non-solvable group of minimal order such that G is a (p, α)-group for some cocycle α of G, and let $S \in \mathrm{Syl}_p(G)$. Also if $N \trianglelefteq G$ and $N \leq L \leq G$ we shall denote the group L/N by $\bar{L}$. We begin by applying the results of §2 to G.

LEMMA 3.1. (i) $G' = G$. (ii) *If $S \leq L < G$, then L is solvable and L/L' is a p-group.*

Proof. (i) Suppose $G' < G$. Then, by 2.2(i) and the minimality of G, we have that G' is solvable. However it then follows that G is solvable, a contradiction.

(ii) L is solvable by the minimality of G and 2.3. Suppose L/L' is not a p-group. Then L has a non-trivial linear character, μ, such that $L'S \leq \ker \mu$, and we let $x \in L - \ker \mu$. Now let $\zeta \in \mathrm{Proj}(L, \alpha_L)$. Then by 2.3 we have that ζ and $(\mu\zeta)$ extend respectively to ξ and $\xi' \in \mathrm{Proj}(G, \alpha)$. However, since α is special, $\det(\xi)$ is a linear character of G and so by (i) we must have

$$(\det(\zeta))(x) = (\det(\xi))(x) = 1;$$

but then similarly

$$(\det(\xi'))(x) = (\det(\mu\zeta))(x) = (\mu(x))^{\xi(1)} \neq 1,$$

contrary to (i). ■

We now show that G must in fact be simple.

LEMMA 3.2. *G is simple.*

Proof. Suppose M is a non-trivial maximal normal subgroup of G, so that $\bar{G} = G/M$ is simple by 3.1(i). Let $\zeta \in \mathrm{Proj}(M, \alpha_M)$ and $T = I_G(\zeta)$. Then by 2.2 $\bar{T}$ is a solvable subgroup of $\bar{G}$ of p-power index, and it follows from 2.4 that $\bar{G}$ has an Abelian Hall p'-subgroup $\bar{H}$. Now if $\bar{x} \in \bar{H}$ with $\bar{x} \neq \bar{1}$ we have that $[\bar{G} : C_{\bar{G}}(\bar{x})]$ is a power of p, which is impossible in a non-Abelian simple group. Thus no such M exists. ■

Now in the case $p = 2$, we remark without proof (for the sake of brevity) that we have derived enough information to show that G is not a simple group via the Classification theorem, the main reduction being the facts that $2 \nmid |H^2(G, \mathbb{C}^*)|$ and that the centralizer in G of a 2-central involution is solvable. Thus a contradiction to 3.2 can be obtained.

However, for p odd we can avoid the use of the Classification theorem as follows.

THEOREM 3.3. *p is not odd.*

Proof. Suppose p is odd; then by [**1**, p. 221] we have

$$S \cap G' = \langle S \cap (N_G(K))' : 1 < K \text{ char } S \rangle.$$

Let $Z = Z(J(S))$ and $T = N_G(Z)$, where $J(S)$ is the Thompson subgroup of S. Now let $1 < K$ char S and $L = N_G(K)$, so that $S \leq L < G$. Then L is solvable by 3.1(ii) and hence is p-constrained by [**3**, (6.3.3)]. Also L is p-stable by 2.4 and [**3**, (8.1.2)]. Thus we obtain from [**3**, (8.2.11)] that $L = 0_{p'}(L)N_L(Z)$, and so $S \cap L' = S \cap (N_L(Z))' \leq S \cap T'$; consequently $S \cap G' = S \cap T'$ since Z char S. Finally from 3.1(ii) we have that T/T' is a non-trivial p-group so that $S \not\leq T'$, and we conclude that $S \not\leq G'$ contrary to 3.1(i). ■

We have thus established Theorem B.

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