Chunikhin's existence theorem for subgroups of a finite group

C. D. H. Cooper

We give a simplified proof of a general theorem of Chunikhin on existence of subgroups of a finite group. The proof avoids the technical device of "indexials" which Chunikhin set up for this purpose.

1. Introduction

In [2] (and later in [1], pp. 79-100), Chunikhin proves a very general theorem which asserts, for any finite group and any normal series of that group, the existence of a subgroup having a certain relationship with the terms of the normal series. It includes as special cases the existence of a Hall \( \pi \)-subgroup in a \( \pi \)-soluble group and the existence of subgroups of all possible \( \pi \)-orders in a \( \pi \)-supersoluble group. In this paper we give a much more direct proof than the one in [1], avoiding the elaborate machinery of "indexials" which Chunikhin sets up.

Throughout the paper, all groups are assumed to be finite.

**THEOREM 1** (Chunikhin [1], pp. 79-100). Suppose that the group \( G \) has a series 
\[ 1 = G_0 \leq G_1 \leq \ldots \leq G_{2n} = G \] such that if \( 1 \leq i \leq n-1 \), \( G_{2i} \trianglelefteq G \) and \( G = G_{2i}N_{G_0}(G_{2i-1}) \). If \( 0 \leq i \leq n-1 \), let \( \theta_i \) be the set of primes which divide \( |G_{2j+1}/G_{2j}| \) for some \( j \in \{i, \ldots, n-1\} \). Then there exists a subgroup \( H \) of \( G \) such that if \( H_i = H \cap G_i \) for \( 0 \leq i \leq 2n \),

\[
(1) \quad G_{2i+1} = H_{2i+1}G_{2i} \quad \text{for} \quad 0 \leq i \leq n-1 ,
\]

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(2) \( H_i \triangleleft H \) for \( 0 \leq i \leq 2n \),

(3) \( H_{2i}/H_{2i-1} \) is a nilpotent \( \Theta_i \)-group for \( 1 \leq i \leq n \),

(4) \( H_{2i+1}/H_{2i} \cong G_{2i+1}/G_{2i} \) for \( 0 \leq i \leq n-1 \),

(5) \( |H_{2i}/H_{2i-1}| \) divides \( |G_{2i}:G_{2i-1}| \) for \( 1 \leq i \leq n \),

(6) \( H \) is a \( \Theta_0 \)-group.

2. Some definitions

If \( \pi \) is any set of primes, \( \pi' \) denotes the complement of \( \pi \) in the set of all primes. A \( \pi \)-number is an integer whose only prime divisors are elements of \( \pi \), and a \( \pi \)-group is a group whose order is a \( \pi \)-number. A Hall \( \pi \)-subgroup, \( H \), of a group \( G \) is a \( \pi \)-subgroup of \( G \) whose index in \( G \) is a \( \pi' \)-number.

A group \( G \) is \( \pi \)-supersoluble if each chief factor is either a cyclic group of order \( p \) for some \( p \in \pi \), or a \( \pi' \)-group. \( G \) is \( \pi \)-soluble if each composition factor (chief factor) is either a \( p \)-group for some \( p \in \pi \) or a \( \pi' \)-group. \( G \) is \( \pi \)-separable if each composition factor (chief factor) is a \( \pi \)-group or a \( \pi' \)-group. \( G \) is \( \pi \)-decomposable if the order of each composition factor (chief factor) is divisible by at most one prime from \( \pi \). \( G \) is \( \pi \)-partible if the order of each composition factor (chief factor) is a \( \pi' \)-number, or is divisible by at most one prime from \( \pi \).

We are here following Gorenstein [3] in the use of the term "\( \pi \)-separable". Chunikhin uses "\( \pi \)-separable" to refer to what we call \( \pi \)-decomposable. The fact that the above definitions, with the exception of \( \pi \)-supersolubility, can be stated in terms of composition factors or chief factors follows from the fact that every chief factor is a direct product of isomorphic copies of some composition factor.

A \( \pi \)-supersoluble group is clearly \( \pi \)-soluble. A group is \( \pi \)-soluble if and only if it is both \( \pi \)-separable and \( \pi \)-decomposable. \( \pi \)-separability and \( \pi \)-decomposability each imply \( \pi \)-partibility. Subgroups and factor groups of \( \pi \)-supersoluble, \( \pi \)-soluble, \( \pi \)-separable, \( \pi \)-decomposable and \( \pi \)-partible groups are respectively \( \pi \)-supersoluble, \( \pi \)-soluble, \( \pi \)-separable, \( \pi \)-decomposable and \( \pi \)-partible. \( \pi \)-supersoluble and
π-soluble groups are respectively, \( \pi_1 \)-supersoluble and \( \pi_1 \)-soluble for all \( \pi_1 \subseteq \pi \). A \( \pi \)-separable group is \( \pi' \)-separable.

3. Preliminary lemmas

**Lemma 1.** Suppose \( B \leq A \leq G \) and \( C \leq G \) such that \( B \) permutes with \( C \). Then

\[
\begin{align*}
(a) & \quad A \cap BC = B(AnC), \\
(b) & \quad |AnC|/|BnC| \text{ divides } |A|/|B|, \\
(c) & \quad \text{if } A \leq BC \text{ and } B \leq A \text{ then } AnC/BnC \cong A/B.
\end{align*}
\]

**Proof.** (a) is the modularity law ([4], p. 121).

(b) \( |B(AnC)| = |B|.|AnC|/|BnC| \) whence \( |AnC|/|BnC| = |B(AnC)|/|B| \) which divides \( |A|/|B| \).

(c) \( A = A \cap BC = B(AnC) \) by (a) whence \( A/B \cong AnC/(AnC)nB = AnC/BnC \).

**Lemma 2.** (Schur-Zassenhaus [4], p. 224). If \( H \) is a normal Hall \( \pi' \) subgroup of \( G \), \( G \) contains a Hall \( \pi' \) subgroup.

**Lemma 3.** Conclusions (4) to (6) of Theorem 1 are consequences of (1) to (3).

**Proof.** (4) follows from (1) by the second isomorphism theorem.

(5). Suppose \( 1 \leq i \leq n \). Then

\[
G_{2i-1}H = H_{2i-1}G_{2i-2}H
\]

by (1),

\[
= HG_{2i-2},
\]

since \( G_{2i-2} \subseteq G \). Similarly \( HG_{2i-1} = HG_{2i-2} \) and so \( G_{2i-1} \) is permutable with \( H \). (5) now follows from Lemma 1 (b) on putting \( A = G_{2i} \), \( B = G_{2i-1} \), \( C = H \).

(6). If \( 0 \leq i \leq n-1 \), \( H_{2i+1}/H_{2i} \) is a \( \theta_i \) group by (4), and if \( 1 \leq i \leq n \), \( H_{2i}/H_{2i-1} \) is a \( \theta_i \) group by (3).

Since \( \theta_0 \supseteq \theta_1 \supseteq \ldots \supseteq \theta_n \), \( H \) is a \( \theta_0 \) group.

**Lemma 4.** If \( 1 = G_0 \leq G_1 \leq \ldots \leq G_{2n} = G \) is a chain of subgroups of
$G$ such that if $1 \leq i \leq n$, $G_{2i} \neq G$ and $G = G_{2i} N_G(G_{2i-1})$ and if $S$ is a subgroup of $G$ such that $G_1 \leq S \leq N_G(G_1)$ and $G = G_2 S$, then putting $S_i = S \cap G_i$,

(a) $G_i = G_2 S_i$ for $2 \leq i \leq 2n$, and

(b) $S = S_{2i} N_{G_2}(S_{2i-1})$ and $S_{2i} \subseteq S$ for $1 \leq i \leq n$.

Proof. (a). Suppose $i \geq 2$. Then

$$G_i = G_i \cap G_2 S = G_2 [G_i \cap S] \text{ by Lemma 1 (a)},$$

$$= G_2 S_i.$$  

(b). If $i \geq 2$,

$$S = S \cap G = S \cap G_{2i} N_G(G_{2i-1}),$$

$$= S \cap G_{2i} N_G(G_{2i-1}) \text{ by (a)},$$

$$= S \cap S_{2i} N_G(G_{2i-1})$$

since $G_2 \leq G_{2i-1} \leq N_G(G_{2i-1})$,

$$= S_{2i} \left[ S \cap N_G(G_{2i-1}) \right] \text{ by Lemma 1 (a)},$$

$$= S_{2i} N_{G_2}(G_{2i-1}) \leq S_{2i} N_{G_2}(S_{2i-1}).$$

But $S_{2i} N_{G_2}(S_{2i-1}) \subseteq S$ and so $S = S_{2i} N_{G_2}(S_{2i-1})$.

If $i = 1$, $S_{2i-1} = S_1 = G_1 \subseteq S$ whence $N_S(S_{2i-1}) = S$.

4. Proof of Theorem 1

We suppose that the theorem is false and throughout this section $G$ is assumed to be a minimal counter-example. We suppose further that the theorem fails for $G$ in respect of the chain $1 = G_0 \leq G_1 \leq \ldots \leq G_{2n} = G$, $(n \geq 1)$ but holds for every shorter chain. For $0 \leq i \leq n-1$, $\theta_i$ denotes the set of primes which divide some $|G_{2j+1}/G_{2j}|$ for $j \geq i$.

**Lemma 5.** If $G_1 \leq S \leq N_G(G_1)$ and $G = G_2 S$ then $S = G$.

Proof. Suppose $S < G$. It follows from Lemma 4 (b) and the fact that $G$ is a minimal counter-example that there exist $H \leq S$ such that if
for $0 \leq i \leq n-1$, $\phi_i$ denotes the set of primes which divide some $|S_{2j+1}/S_{2j}|$ for $j \geq i$, then

(i) $S_{2i+1} = H_{2i+1}S_{2i}$ for $0 \leq i \leq n-1$, 

(ii) $H_i \triangleleft H$ for $0 \leq i \leq 2n$, 

(iii) $H_{2i}/H_{2i-1}$ is a nilpotent $\phi_i$-group for $1 \leq i \leq n$, 

where $H_i$ is defined to be $H \cap S_i$ for $0 \leq i \leq 2n$.

Now if $i \geq 1$, 

$$G_{2i+1} = G_2S_{2i+1} \quad \text{by Lemma 4 (a)}$$

$$= G_2H_{2i+1}S_{2i} \quad \text{by (i)}$$

$$= G_2S_{2i}H_{2i+1}$$

$$= G_{2i}H_{2i+1} \quad \text{by Lemma 4 (a)}.$$

If $i = 0$, 

$$G_{2i+1} = G_1 = S_1 = H_1S_0 \quad \text{by (i)},$$

$$= H_1.$$

Thus (1) holds for $G$.

If $0 \leq j \leq n-1$ it follows from Lemma 1 (b) that $|S_{2j+1}/S_{2j}|$ divides $|G_{2j+1}/G_{2j}|$ and so $\phi_j \subseteq \phi_i$ for $0 \leq i \leq n-1$. Finally, $H \cap S_i = H \cap G_i$ for $0 \leq i \leq 2n$ and so from (ii), (iii) it follows that (2) and (3) hold for $G$. Hence by Lemma 3, the theorem holds for $G$, a contradiction. Hence $S = G$.

**Lemma 6.** If $G_1 \supsetneq G$ then $G_1 = 1$.

**Proof.** Suppose that $G_1 \neq 1$. Using the symbol $\overline{\cdot}$ to denote images of subgroups of $G$ in $G/G_1$, we have by the assumptions on the $G_i$, $\overline{G_i} \triangleleft \overline{G}$ and $\overline{G} = \overline{G_2} \overline{N_G(G_2G_{2i-1})} = \overline{G_2} \overline{N_G(G_2)} \overline{G_{2i-1}}$. Hence by the minimality of $G$, there is a subgroup $H$ of $G$ such that $G_1 \leq H$ and such that if for $0 \leq i \leq n-1$, $\alpha_i$ denotes the set of primes which divide
some $|\overline{G_{2j+1}}/\overline{G_{2j}}|$ for $j \geq i$,

(i) $\overline{G_{2i+1}} = \overline{H_{2i+1}} \overline{G_{2i}}$ for $0 \leq i \leq n-1$,

(ii) $\overline{H_{i}} \trianglelefteq \overline{H}$ for $0 \leq i \leq 2n$,

(iii) $\overline{H_{2i}}/\overline{H_{2i-1}}$ is a nilpotent $\alpha_{i}$-group for $1 \leq i \leq n$,

where $H_{i}$ is defined to be $H \cap G_{i}$ for $0 \leq i \leq 2n$.

If $i \geq 1$, it follows from (i) that $G_{2i+1} = H_{2i+1}G_{2i}$. Moreover $G_{1} \trianglelefteq H$ and so $G_{1} = H_{1} = H_{1}G_{0}$. Thus (1) holds for $G$. From (ii), $H_{i} \trianglelefteq H$ for $i \geq 1$, and clearly $H_{0} \leq H$. Hence (2) holds for $G$. If $j \geq 1$, $\overline{G_{2j+1}}/\overline{G_{2j}} \cong G_{2j+1}/G_{2j}$. Moreover $\overline{G_{1}}/\overline{G_{0}}$ is trivial. Hence if $0 \leq i \leq n-1$, $\alpha_{i} \subseteq \Theta_{i}$. If $i \geq 1$, $\overline{H_{2i}}/\overline{H_{2i}} \cong H_{2i}/H_{2i-1}$ and so by (iii), $\overline{H_{2i}}/\overline{H_{2i-1}}$ is a nilpotent $\Theta_{i}$-group and so (3) holds for $G$.

Thus, by Lemma 3, the theorem holds for $G$, a contradiction. Hence $G_{1} = 1$.

**Lemma 7.** If $G_{1} = 1$ and $G_{2}$ is nilpotent then $G_{2} = 1$.

**Proof.** Suppose that $G_{2} \neq 1$. Using the symbol $\overline{\cdot}$ to denote images of subgroups of $G$ in $G/G_{2}$, then by the assumptions on the $G_{i}$,

$\overline{G_{2i}} \trianglelefteq \overline{G}$ and $\overline{G} = \overline{G_{2i}N_{G}G_{2i-1}} = \overline{G_{2i}}\overline{H_{2i}}\overline{G_{2i-1}}$. Hence by the minimality of $G$, there is a subgroup $K$ of $G$ such that $G_{2} \leq K$ and such that if for $0 \leq i \leq n-1$, $\alpha_{i}$ denotes the set of primes which divide some $|\overline{G_{2j+1}}/\overline{G_{2j}}|$ for $j \geq i$,

(i) $\overline{G_{2i+1}} = \overline{K_{2i+1}} \overline{G_{2i}}$ for $0 \leq i \leq n-1$,

(ii) $\overline{K_{i}} \trianglelefteq \overline{K}$ for $0 \leq i \leq 2n$,

(iii) $\overline{K_{2i}}/\overline{K_{2i-1}}$ is a nilpotent $\alpha_{i}$-group for $1 \leq i \leq n$,

(iv) $\overline{K}$ is an $\alpha_{0}$-group,

where $K_{i}$ is defined to be $K \cap G_{i}$ for $0 \leq i \leq 2n$. 


If \( j \geq 1 \), \( \bar{G}_{2j+1}/\bar{G}_{2j} \equiv G_{2j+1}/G_{2j} \), and so \( \alpha_j = \theta_j \). Since \( \bar{G}_1/\bar{G}_0 \) is trivial, \( \alpha_0 = \alpha_1 = \theta_1 \). Thus by (iv), \( K/G_2 \) is a \( \theta_1 \)-group.

Since \( G_2 \) is nilpotent it contains a unique Hall \( \theta_1 \)-subgroup, \( M \). \( M \) is characteristic in \( G_2 \) and hence normal in \( G \). \( G_2/M \) is a \( \theta_1 \)-group and so \( K/M \) is a \( \theta_1 \)-group. By Lemma 2, there exists a Hall \( \theta_1 \)-subgroup \( H \) of \( K \). Thus \( K = MH \) and \( M \cap H = 1 \).

If \( i \geq 2 \), then since \( M \leq G_2 \), we have by Lemma 1 (a) that
\[
K_i = K \cap G_i = MH \cap G_i = M \langle H \cap G_i \rangle = MH_i \quad \text{where } H_i \text{ is defined to be } H \cap G_i
\]
for \( 0 \leq i \leq 2n \). If \( i \geq 1 \) we have from (i) that
\[
G_{2i+1} = K_{2i+1} \subseteq G_i = MH_{2i+1} G_{2i} = H_{2i+1} G_{2i}.
\]
Moreover \( G_1 \) and \( H_1 \) are trivial, so \( G_1 = H_1 G_0 \). Thus (1) holds for \( G \).

If \( i \geq 2 \), it follows from (ii) that \( K_i \equiv K \) and so \( MH_i \equiv MH \).

Hence
\[
H_i \leq MH_i \cap H = H_i (M \cap H) \quad \text{by Lemma 1 (a),}
\]
\[
= H_i.
\]
Thus \( H_i \equiv H \). Moreover \( H_2 = G_2 \equiv H \) and \( H_1 = G_1 = 1 \equiv H \). Thus (2) holds for \( G \).

If \( i \geq 2 \), \( \bar{K}_{2i}/\bar{K}_{2i-1} \equiv K_{2i}/K_{2i-1} = MH_{2i}/MH_{2i-1} \equiv H_{2i}/H_{2i-1} \) and so by (iii), \( H_{2i}/H_{2i-1} \) is a nilpotent \( \alpha_i \)-group and so a nilpotent \( \theta_1 \)-group. Since \( H_1 = 1 \), \( H_2/H_1 \equiv H_2 \) and is nilpotent since \( G_2 \) is nilpotent. Since \( K/M \) is a \( \theta_1 \)-group, so is \( H_2 \). Hence (3) holds for \( G \) and so by Lemma 3, the theorem holds for \( G \), a contradiction. Hence \( G_2 = 1 \).

Proof of Theorem 1. We obtain our ultimate contradiction through an interplay of Lemmas 5, 6 and 7. Taking \( S = H_G(G_1) \) in Lemma 5 we conclude that \( G_1 \equiv G \). Hence by Lemma 6, \( G_1 = 1 \). Thus if \( S \) is any subgroup of \( G \) such that \( G = G_2S \), then by Lemma 5, \( S = G \). Hence \( G_2 \)
is contained in the Frattini subgroup of \( G \), whence it is nilpotent. By Lemma 7, \( G_2 = 1 \). Since the theorem holds for \( G \) in respect of the shorter chain \( 1 = G_2 \leq G_3 \leq \ldots \leq G_{2n} = G \), it must hold in respect of the original chain, a contradiction.

5. Consequences of Theorem 1

THEOREM 2. Suppose that

\[ 1 = G_0 < G_2 < G_4 < \ldots < G_{2n} = G \]

is a normal series for \( G \). If for \( 1 \leq i \leq n-1 \), the factor \( G_{2i}/G_{2i-2} \) contains a single conjugacy class of Hall \( \pi \)-subgroups and if \( G/G_{2n-2} \) contains a Hall \( \pi \)-subgroup then \( G \) contains a Hall \( \pi \)-subgroup. If these Hall \( \pi \)-subgroups are soluble, \( G \) contains a soluble Hall \( \pi \)-subgroup.

Proof. For \( 1 \leq i \leq n \), choose \( G_{2i-1} \) so that \( G_{2i-1}/G_{2i-2} \) is a Hall \( \pi \)-subgroup of \( G_{2i}/G_{2i-2} \). If \( 1 \leq i \leq n-1 \), all Hall \( \pi \)-subgroups of \( G_{2i}/G_{2i-2} \) are conjugate whence \( G/G_{2i-2} = G_{2i}/G_{2i-2} \), \( N_{G_{2i-2}}(G_{2i-1}/G_{2i-2}) \) and so \( G = G_{2i}N_G(G_{2i-1}) \). By Theorem 1 there exists a subgroup \( H \) of \( G \) having properties (1) to (6).

\[ |G : H| = \prod_{i=1}^{n} \frac{|G_{2i-1}:G_{2i-2}|}{|H_{2i-1}:H_{2i-2}|} \times \prod_{i=1}^{n} \frac{|G_{2i}:G_{2i-1}|}{|H_{2i}:H_{2i-1}|} = \prod_{i=1}^{n} \frac{|G_{2i}:G_{2i-1}|}{|H_{2i}:H_{2i-1}|} \text{ by (4)}, \]

which divides \( \prod_{i=1}^{n} |G_{2i}:G_{2i-1}| \) and so is a \( \pi' \)-number. Since for \( 0 \leq i \leq n-1 \), \( G_{2i+1}/G_{2i} \) is a \( \pi \)-group, \( \theta_i \subseteq \pi \) for all \( i \). In particular \( \theta_0 \subseteq \pi \). \( H \) is a \( \theta_0 \)-group by (6) and so a \( \pi \)-group. Hence it is a Hall \( \pi \)-subgroup of \( G \).

If the Hall \( \pi \)-subgroups of the factors of \( G \) are soluble, \( G_{2i-1}/G_{2i-2} \), and hence by (4) \( H_{2i-1}/H_{2i-2} \) is soluble for \( 1 \leq i \leq n \).
Since, by (3), \( H_{2i}/H_{2i-1} \) is nilpotent for \( 1 \leq i \leq n \), \( H \) is soluble.\

**COROLLARY.** The theorem holds if \( 1 = G_0 < G_2 < \ldots < G_{2n} = G \) is a composition series.

**Proof.** If \( A \) is a direct product of isomorphic copies of \( B \), then \( A \) has a single conjugacy class of Hall \( \pi \)-subgroups if and only if \( B \) has. Since each chief factor of \( G \) is a direct product of isomorphic copies of some composition factor, the assumptions on the composition factors carry over to the chief factors.

**THEOREM 3** ([1], Theorem 3.9.1). If \( G \) is \( \pi \)-partible then it contains a Hall \( \pi \)-subgroup. If for some \( \pi_1 \subseteq \pi \), \( G \) is \( \pi_1 \)-decomposable and (\( \pi-\pi_1 \))-separable then it contains a \( \pi_1 \)-soluble Hall \( \pi \)-subgroup.

**Proof.** Let \( 1 = G_0 < G_2 < G_4 < \ldots < G_{2n} = G \) be a chief series for \( G \). If \( \rho \) is the set of prime divisors for some chief factor then by the \( \pi \)-partibility of \( G \),

(i) \( \rho \subseteq \pi \), or

(ii) \( \rho \subseteq \pi' \), or

(iii) \( \rho \cap \pi = \{p\} \) for some prime \( p \).

In case (i) the factor is a \( \pi \)-group and so has a unique Hall \( \pi \)-subgroup (namely itself). In case (ii) the factor is a \( \pi' \)-group and so has a unique Hall \( \pi \)-subgroup (namely the trivial subgroup). In case (iii) the factor has a single conjugacy class of Hall \( \pi \)-subgroups (namely the Sylow \( p \)-groups). Hence by Theorem 2, there exists a Hall \( \pi \)-subgroup \( H \) of \( G \) satisfying (1) to (6) of Theorem 1.

Suppose that \( G \) is \( \pi_1 \)-decomposable and (\( \pi-\pi_1 \))-separable. If \( \rho \) is the set of prime divisors of \( |G_{2i}/G_{2i-2}| \), \( \rho \subseteq \pi'_1 \) or \( \rho \cap \pi_1 = \{p\} \) by \( \pi_1 \)-decomposability. If \( \rho \cap \pi_1 = \{p\} \) then by (\( \pi-\pi_1 \))-separability, \( \rho \cap (\pi-\pi_1) = \emptyset \) that is \( \rho \cap \pi = \rho \cap \pi_1 \). Let \( \tau \) be the set of prime divisors of \( |H_{2i-1}/H_{2i-2}| \). Then by (4), \( \tau \subseteq \rho \) and by (6), \( \tau \subseteq \theta_0 = \pi \). Hence \( \tau \subseteq \rho \cap \pi \). Thus either \( \tau \subseteq \rho \subseteq \pi' \) or

* In fact by (5) and (6), each \( H_{2i}/H_{2i-1} \) is trivial.
Finally for $1 \leq i \leq n$, $H_{2i-1}/H_{2i-1}$ is, by (5), a $\pi'$-group.* Hence $H$ is $\pi_1$-soluble.

References


School of Mathematics,
Macquarie University,
North Ryde, NSW.

* In fact, since $H$ is a $\pi$-group, $H_{2i}/H_{2i-1}$ is trivial.