A CLASS OF RIGHT-ORDERABLE GROUPS

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1. Introduction. A group $G$ is called right-orderable (or an RO-group) if there exists an order relation $\preceq$ on $G$ such that $a \preceq b$ implies $ac \preceq bc$ for all $a, b, c$ in $G$. This is equivalent to the existence of a subsemigroup $P$ of $G$ such that $P \cap P^{-1} = \{e\}$ and $P \cup P^{-1} = G$. Given the order relation $\preceq$, $P$ can be taken to be the set of positive elements and conversely, given $P$, define $a \preceq b$ if and only if $ba^{-1} \in P$. A group $G$ together with a given right-order relation on $G$ is called right-ordered. A subgroup $C$ of a right-ordered group $G$ is called convex if for every $g$ in $G$ and $x$ in $C$, $e \preceq g \preceq C$ implies $g \in G$. The set of all convex subgroups of $G$ is ordered by inclusion and closed with respect to unions and intersections. However there is not much more one can say in general regarding this set. We shall call a right-order $P$ on $G$ a $C$-right-order if the set of convex subgroups form a system with torsion-free abelian factors. P. Conrad [2] has looked at a number of equivalent conditions for a group $G$ to be $C$-right-ordered. Our main concern here is to investigate the properties of an RO-group $G$ in which every right-order is a $C$-right-order. We call such a group a $C_1$-group. In Lemma 3.1 we show that a right-order $P$ is a $C$-right-order if and only if it satisfies the property:

(*) For all $x, y$ in $P$ there exist $u, v$ in $\text{sgr}(x, y)$ (the semigroup generated by $x$ and $y$) such that $ux \geq vy$.

Thus in particular an RO-group $G$ is a $C_1$-group if it satisfies the property:

(**) For all $x, y$ in $G$ there exist $u, v$ in $\text{sgr}(x, y)$ such that $ux = vy$.

We call $G$ a $C_2$-group if it satisfies (**). Finally we denote by $C_2$ the largest subgroup closed subclass of $C_1$. Then $RO \supseteq C_1 \supseteq C_2 \supseteq RO \cap C_3$, and all these inclusions are proper (Corollary 3.3, Theorem 3.5).

In Section 2 we note a few properties of $C_3$-groups. In particular we show that locally solvable $C_3$-groups are locally nilpotent-by-finite (Theorem 2.6). This is not true of $C_2$-groups (Theorem 3.5), however orderable locally solvable $C_2$-groups are locally nilpotent and finitely generated orderable solvable $C_1$-groups are nilpotent (Theorem 3.6).

2. $C_3$-groups. We start by observing that the class $C_3$ is subgroup-closed and closed under periodic extensions; moreover a group $G$ is in $C_3$ if every two-generator subgroup of $G$ is in $C_3$. B. H. Neumann has shown that $G$ is in $C_3$ if every two-generator subgroup of $G$ is nilpotent.

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648
Lemma 2.1. Let $H$ be a subgroup in the centre of a group $G$. If $G/H$ is in $C_3$, then $G$ is in $C_3$.

**Proof.** Let $x, y \in G$. Then there exist $u, v \in \text{sgr}(x, y)$ such that $ux = v y$ for some $z \in H$. Thus $vyux = u x v y$, $uv \in \text{sgr}(x, y)$.

Corollary 2.2. If every two-generator subgroup of $G$ is nilpotent-by-periodic, then $G$ is in $C_3$.

Lemma 2.3. A direct product of $C_3$-groups is in $C_3$.

**Proof.** It is clearly enough to show that if $H_1, H_2$ are $C_3$-groups, then so is $G = H_1 \times H_2$. Take any $x = x_1 x_2, y = y_1 y_2$ in $G$ with $x_1, y_1 \in H_1$. Since $H_1 \subset C_3$, there exist $a = a_1 a_2, b = b_1 b_2$ in $\text{sgr}(x, y)$ such that $a_1 x_1 = b_1 y_1$. Also, since $H_2 \subset C_3$, there exist $(a_2 x_2)^m h_2, (a_2 x_2)^n k_2$ in $\text{sgr}(ax, by)$, with $m, n$ positive integers, $h_2, k_2$ in $H_2$, such that $h_2 a_2 x_2 = k_2 b_2 y_2$. Then $(a_1 x_1)^m h_2 a x (a_1 x_1)^n k_2 b y = (a_1 x_1)^m k_2 b y (a_1 x_1)^n h_2 a x$, and of course $(a_1 x_1)^m h_2 a x, (a_1 x_1)^n k_2 b y, a, b$ are all in $\text{sgr}(x, y)$.

Lemma 2.4. A polycyclic $C_3$-group is nilpotent-by-finite.

**Proof.** Let $G$ be a counterexample with $l(G)$ minimum where $l(G)$ is the number of infinite factors in any series of $G$ with cyclic factors. Replacing $G$ with a suitable normal subgroup of finite index if necessary, we may assume that it is nilpotent-by-abelian and torsion-free. Let $N$ be the Fitting subgroup of $G$. By the minimality of $G$, $N$ is abelian (because $G/N'$ nilpotent-by-finite implies $G$ nilpotent-by-finite), $G/N$ is infinite cyclic, and the centre of $G$ is trivial (see Lemma 2.1).

Let $G = \langle N, t \rangle$, write $N$ additively and regard it as a module over the integral group ring $\mathbb{Z} \langle t \rangle$. Let $A$ be an indecomposable submodule of $N$. Then $A$ can be identified with an additive subgroup of the complex numbers on which the action of $t$ is that of multiplication by an algebraic integer $\tau$ whose minimal polynomial over the rationals has degree equal to $l(A)$. If all the roots of this polynomial have absolute value one, then by a theorem of Kronecker, $\tau$ is an $n$th root of unity for some integer $n$. But then $t^n$ centralizes $A$, and $G_1 = \langle N, t^n \rangle$ has a non-trivial centre, so that $G_1$ and hence $G$ is nilpotent-by-finite. Thus $|\tau| \neq 1$, and replacing $t$ with a suitable power of $t$, if necessary, we may assume that $|\tau| < \frac{1}{3}$.

Choose any non-zero $a \in A$. By hypothesis there exist $u, v \in \text{sgr}(at, ta)$ such that $uat = vta$. Then we have:

$$t^{i+1}(ar^i + \ldots + ar^1 + ar) = t^{i+1}(ar^i + \ldots + ar^1 + a),$$

where $ar = t^{-i}at, 1 \leq \alpha_i \leq \ldots \leq \alpha_r, 1 \leq \beta_1 \leq \ldots \leq \beta_i, i \leq a_i \leq i + 1$ and $i \leq \beta_i \leq i + 1$ for all $i$. But

$$|\tau^i + \ldots + \tau^1 + \tau| \leq (|\tau|^a + \ldots + |\tau|)$$

$$+ |\tau| < \sum_{n=1}^{\infty} \left( \frac{1}{4} \right)^n + \frac{1}{4} + \frac{7}{12},$$

where $\ast$.
while
\[ |r^{\alpha_1} + \ldots + r^{\alpha_l} + 1| \geq 1 - (|r|^{\alpha_1} + \ldots + |r|^{\alpha_l}) > 1 - \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \frac{2}{3}. \]
and we reach a contradiction.

**Lemma 2.5.** If \( G = \langle A, t \rangle \) is a \( C_r \)-group and \( A = \langle a_1, \ldots, a_k \rangle^G \) is abelian, then \( A \) is finitely generated and \( G \) is nilpotent-by-finite.

**Proof.** The existence of \( u_t, v_t \) in \( \text{sgr} \langle a_t, ta_t \rangle \) such that \( u_t a_t v_t = v_t a_t \) shows that \( \langle a_t \rangle^G = \langle a_t, a_t^{-1}, \ldots, a_t^{-r} \rangle \) for some integer \( r_t \). The rest follows from Lemma 2.4.

**Theorem 2.6.** If \( G \) is a locally solvable \( C_2 \)-group, then \( G \) is locally nilpotent-by-finite.

**Proof.** Assume, by way of induction, that the result holds for finitely generated groups of solvability length less than \( r \), and let \( G \) be a finitely generated group of solvability length \( r \). If \( A \) is the last non-trivial term in the derived series of \( G \), then \( A \) is abelian and \( G/A \) is nilpotent-by-finite. Replacing \( G \) by a suitable subgroup of finite index if necessary, we may assume that \( G/A \) is nilpotent. Then \( A = S^G \), where \( S = \langle a_1, \ldots, a_k \rangle \) for some \( a_1, \ldots, a_k \) in \( A \).

Also there exists a series \( A = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_n = G \) such that, for all \( i \), \( G_i \triangleleft G \) and \( G_i = \langle G_{i-1}, t_i \rangle \) for suitable \( t_i \) in \( G \). Repeated application of Lemma 2.5 shows that \( S^G_{G_i} \) is finitely generated for all \( i = 1, \ldots, m \). Thus \( G \) is polycyclic and the result follows from Lemma 2.4.

3. \( C_1 \) and \( C_2 \)-groups.

**Lemma 3.1.** Let \( P \) be a right-order on a group \( G \). Then the following are equivalent.

(i) \( P \) satisfies condition (*)

(ii) For every \( x, y \) in \( P \setminus \{e\} \), \( x^n y > x \) for some \( n > 0 \).

(iii) If \( C \) and \( D \) are convex subgroups of \( G \) under \( P \) and \( D \) covers \( C \), then \( C \) is normal in \( D \) and \( D/C \) is isomorphic to a subgroup of the additive group of the reals.

(iv) For all \( y \) in \( P \setminus \{e\} \) the set \( \{ x \in G \mid x <^y y \} \) is a convex subgroup of \( G \), where \( \mid x \mid = x \) if \( x \in P \) and \( x^{-1} \) otherwise, and \( x <^y y \) means that \( \mid x \mid^n < y \) for all \( n \).

**Proof.** (i) \( \Rightarrow \) (ii). Suppose that \( x^n y \leq x \) for all \( n > 0 \). By hypothesis there exist \( u, v \) in \( \text{sgr} \langle xy, x \rangle \) such that \( u x v \geq vx \). Since \( v > e, vx > x \). On the other hand \( u x y = x^{\alpha_1} x^{\alpha_2} y \ldots x^{\alpha_r} y \), where \( \alpha_i \geq 1 \) for \( i = 1, \ldots, r \) and \( r \geq 1 \), hence \( u x y \leq x^{\alpha_{r+1}} \ldots x^{\alpha_r} y \leq \ldots \leq x \), a contradiction.

That (ii) \( \Rightarrow \) (i) is trivial. The equivalence of (ii) and (iii) was shown in [2] and the equivalence of (iii) and (iv) in [1]. We mentioned (iii) and (iv) because we will need them in the following.
LEMMA 3.2. Let $A$ and $B$ be RO-groups and $G$ a split extension of $A$ by $B$. If there exists a right-order $P_A$ on $A$, invariant under conjugation by elements of $B$, such that not all the jumps in the chain of convex subgroups of $A$ determined by $P_A$ are centralized by $B$, then $G$ is not a $C_1$-group.

Proof. The result is obvious if $P_A$ is not a $C$-order on $A$. Let $P_B$ be a right-order on $B$ and define a right-order $P$ on $G$ by letting $g = ab$ ($a \in A$, $b \in B$) belong to $P$ if either $e \neq a \in P_A$ or $a = e$ and $b \in P_B$. That $P$ is indeed a right-order follows from the fact that $P_A$ is $B$-invariant. We show that it is not a $C$-order. Let $C \triangleleft D$ be a jump of convex subgroups of $A$ under $P_A$ which is not centralized by $B$, and choose $e < a \in D \setminus C$, $b \in B$ such that $[a, b] \notin C$.

Case 1. $b$ normalizes $D$. In this case $b$ normalizes $C$ as well since $P_A$ is $C$-invariant. Moreover $D/C$ may be identified with a subgroup of the additive group of the reals since $P_A$ is a $C$-order, and the action of $b$ on $D/C$ is that of multiplication by some real number $\beta > 1$ (replacing $b$ by $b^{-1}$ if necessary). Let $\tilde{a} = Ca$ and choose $\tilde{d} = Cd \in D/C$ such that

$$\tilde{d} \geq \tilde{a}/(\beta - 1) > 0.$$ 

For instance $\tilde{d}$ can be a suitable multiple of $\tilde{a}$. We show that the set

$$S = \{x \in G; |x| \ll \tilde{d}\}$$

is not a subgroup and thus $P$ does not satisfy Condition (iv) of Lemma 3.1. The element $ab^{-1}$ belongs to $S$, for

$$(ab^{-1})^n\tilde{d}^{-1} = aa^b \ldots a^{b^{-1}}d^{-b}\tilde{d}^{-n}$$

and

$$C(aa^b \ldots a^{b^{-1}}d^{-b^n}) = \tilde{a} \left(\sum_{t=0}^{n-1} \beta^t\right) - \tilde{d} \beta^n < 0.$$ 

The element $b$ also belongs to $S$, but $a = (ab^{-1})b$ clearly does not.

Case 2. $b$ does not normalize $D$. Since $P_A$ is $B$-invariant, either $D^b \supset D$ or $D \supset D^b$. Replacing $b$ by $b^{-1}$ if necessary, assume that $D^b \supset D$. We show that the set

$$T = \{x \in G; |x| \ll a\}$$

is not a subgroup. The element $ab^{-1}$ is in $T$ since

$$(ab^{-1})^n\tilde{a}^{-1} = aa^b \ldots a^{b^{-1}}a^{-b^nb^{-n}} \in P^{-1}.$$ 

The element $b$ also belong to $T$; but $a = (ab^{-1})b$ does not. This completes the proof.

COROLLARY 3.3. Subgroups and direct products of $C_1$-groups need not be in $C_1$.

Proof. Let $Q$ denote the additive group of the rationals and let $t$ be the automorphism of $Q$ corresponding to multiplication by $-2$. Then $G = \langle Q, t\rangle$ is in $C_1$ but not in $C_2$. That $G$ is not in $C_2$ can be seen by applying Lemma 3.2 to the subgroup $\langle Q, t^2\rangle$. To see that $G \in C_1$ let $P$ be any right-order on $G$. 

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Without loss of generality we may assume \( t \in P \). For any \( x \in Q \cap P \), \( x^{-1} t \in P^{-1} \), hence \( x < t \) and \( Q \) is convex under \( P \). This shows that \( P \) is a \( C \)-order.

Next consider the direct product of \( G \) with an infinite cyclic group: \( H = G \times \langle z \rangle \). Every element of \( H \) can be written uniquely in the form \((t^s z)^r \times t^s\), where \( x \in Q \) and \( r \) and \( s \) are integers. Let
\[
R = \{ (t^s z)^r \times t^s \}; \text{ either } s > 0, \text{ or } s = 0 \text{ and } x > 0, \\
\quad \text{or } s = x = 0 \text{ and } r \geq 0 \}.
\]
It is easy to check that \( R \) is a right-order on \( H \) and that
\[
\langle e \rangle \ll \langle (t^s z) \rangle \ll \langle (t^s z), Q \rangle \ll H
\]
is its convex series. But \( \langle (t^s z) \rangle \) is not normal in \( \langle (t^s z), Q \rangle \), hence by Lemma 3.1, \( R \) is not a \( C \)-order.

Remark. There exist also polycyclic groups which are in \( C_1 \) but not in \( C_2 \).

**Corollary 3.4.** Let \( G \) be a finitely generated, orderable \( C_1 \)-group. Then the system of convex subgroups under any order on \( G \), is central.

**Proof.** Let \( P \) be any order on \( G \). Since \( G \) is finitely generated, there exists \( J < G \) such that \( J \ll G \) is a convex jump under \( P \). Thus there exists \( A \supseteq J \) such that \( G = \langle A, x \rangle \) and \( G/A \) is infinite cyclic. By Lemma 3.2, \( x \) centralizes every convex jump in \( A \) determined by the restriction of \( P \) to \( A \), and hence every convex jump in \( G \). For any \( a \) in \( A \), \( G = \{ A, xa \} \) so that \( xa \) also centralizes every jump in \( G \) and hence so does \( a \).

**Theorem 3.5.** There exist finitely generated metabelian \( C_2 \)-groups which are not nilpotent-by-finite, and therefore the class \( C_2 \)-contains the class \( R \cap C_3 \) properly.

**Proof.** Let \( G = \langle a, t; a^t a^{-4} a^5 = e, [a, a^t] = e \rangle \). Then \( A = \langle a \rangle^2 \) is an abelian group of rank 2 which can be identified with the subgroup of the additive group of the complex numbers generated by the numbers \((2 + i)^n\), \( n \in \mathbb{Z} \) on which \( t \) acts as multiplication by \( 2 + i \). Our reason for choosing \( 2 + i \) is that none of its powers is real.

Let \( H \) be any subgroup of \( G \) and \( P \) any order on \( H \). If \( H \subseteq A \) or if \( H \cap A = \langle e \rangle \), then \( H \) is abelian and \( P \) is a \( C \)-order. Otherwise \( H = \langle A \cap H, u \rangle \), where \( u = b t^m \) for some \( b \in A, n \geq 1 \), and \( u \) acts on \( A \cap H \) as multiplication by the non-real gaussian integer \( \xi = (2 + i)^n \). Notice that a gaussian integer \( h + ki \) satisfies the equation \( x^2 - 2hx + h^2 + k^2 = 0 \), so that by choosing \( m > 0 \) such that the real part of \( \xi^m \) is negative, we find a power of \( \xi \) which satisfies an equation \( x^2 + rx + s = 0 \) with \( r > 0 \) and \( s > 0 \). Thus for all \( c \in A \cap H, c^u v^m c^{-u} v^{-m} = e \) as well as \( c v^m c^{-u} v^{-2m} = e \), and therefore if \( c \) is in \( P \), either \( v^m \) or \( v^{-2m} \) is in \( P^{-1} \).

We now show that \( A \cap H \) is convex. By changing \( P \) to \( P^{-1} \) if necessary, we may assume that \( u \in P \). Suppose that \( b > u d > e \) for some \( b, d \in A \cap H, j \in \mathbb{Z} \). If \( j \geq 0 \) then \( d^{w-j} > u^{-j} > e \). If \( j \geq 0 \) then \( b d^{-1} > u^j \geq e \). Thus as-
sume that \( c > u^j \) for some \( c \in A \cap H, \ j \geq 0 \). Notice that \( c > u^j \) implies \( cu^j > u^{2j} \) and \( c \ u^{ij} = c \ u^j > u^{2j} \), thus if \( j \neq 0 \), replacing \( c \) by another suitable element of \( A \cap H \), we may assume \( j \geq 2m \). Thus we have \( c > u^i > e \) and hence \( cu^{-i} > e \) and \( u^i c u^{-i} > e \) for \( i = 0, 1, \ldots, 2m \). In particular \( c, c^{-m} \) and \( c^u c^{-2m} \) are all in \( P \). This is not possible, therefore \( j = 0 \) and \( A \cap H \) is convex. This implies that \( P \) is a \( C \)-order and hence that \( G \) is a \( C_3 \)-group.

It is easy to check that \( G \) is not nilpotent-by-finite and therefore by Theorem 2.6 it is not a \( C_3 \)-group.

**Theorem 3.6.** Let \( G \) be a finitely generated solvable order able \( C_1 \)-group. Then \( G \) is nilpotent.

**Proof.** Let \( G \) be a counterexample of smallest solvability length, and \( P \) any order on \( G \). By Corollary 3.4, the system of convex subgroups of \( G \) is central. Moreover, as \( G \) is finitely generated, it has a descending central series

\[
G = G_0 > G_1 > \ldots G_n > G_{n+1} > \ldots
\]

from \( G \) to \( G_n = \cap_{i=0}^{\infty} G_n \), where \( G_n > G_{n+1} \) is a convex jump under \( P \). If \( G_n = G_n \) for some \( n \), then \( G \) is nilpotent and we have the required contradiction. If \( G_n \neq \langle e \rangle \), observe that \( G/G_n \) satisfies the hypotheses of the theorem since any quotient of a \( C_1 \)-group is in \( C_1 \) if it is an \( RO \)-group. Thus we may replace \( G \) by \( G/G_n \) and assume \( G_n = \langle e \rangle \), so that \( G \) becomes a residually finitely generated torsion-free nilpotent group and hence residually \( F_p \) for all primes \( p \), where \( F_p \) is the class of finite \( p \)-groups.

Let \( N \) be a maximal normal abelian subgroup of \( G \) containing the last non-trivial subgroup of the derived series of \( G \). By a result of Learner (see [5, Lemma 6.25]), \( G/N \) is also residually \( F_p \) for all primes \( p \), and hence orderable (see [3]). Also \( G/N \in C_1 \), and thus it is nilpotent by our choice of \( G \). We now use the following result to complete the proof.

**Lemma 3.7.** Let \( G \) be an orderable \( C_1 \)-group. If there exists \( \langle e \rangle \neq A < G, A \) abelian and \( G/A \) finitely generated torsion-free nilpotent, then \( Z(G) \cap A \neq \langle e \rangle \), where \( Z(G) \) is the centre of \( G \).

The above lemma applies with \( A = N \). Thus \( Z(G) \cap N = Z_1 \neq \langle e \rangle \) and \( G/Z_1 \) is again orderable since \( Z_1 \) is an isolated subgroup in the centre of \( G \). Since \( G \) satisfies the maximal condition on normal subgroups, repeated application of Lemma 3.7 shows that \( N \leq Z_k(G) \), the \( k \)-th centre of \( G \), for some finite \( k \). Thus \( G \) is nilpotent.

**Proof of Lemma 3.7.** Use induction on \( l(G/A) \), the number of factors in any infinite cyclic series of \( G/A \). Suppose \( l(G/A) = 1 \). Then \( G = \langle A, c \rangle \). Take any \( e \neq a \) in \( A \) and let \( A_1 = \langle a \rangle^G \). Let \( P_1 \) be any \( G \)-order on \( A_1 \). Then \( P_1 \) can be extended to a \( G \)-order \( P \) on \( A \) since \( G \) is a metabelian orderable group. By Lemma 3.2, \( c \) centralizes every jump in \( A \) determined by \( P \) and hence every jump in \( A_1 \) determined by \( P_1 \). Thus if \( A_1 \) has finite rank then \( A_1 \cap Z(G) \neq \langle e \rangle \).
\langle e \rangle$, as required. If $A_1$ has infinite rank, then it is freely generated by the elements $a^i$, $i \in \mathbb{Z}$. In this case let $\xi$ be any positive transcendental number and let $P_1$ consist of those elements $(a^{x_1})^{e_1} \ldots (a^{x_n})^{e_n}$ such that $\sum_{i=1}^n r_i \xi^{n_i} \geq 0$. This is an archimedean $G$-order on $A_1$ and so $A_1 \leq Z(G)$.

Now suppose that $l(G/A) = n > 1$. Then there exists $H \subset G$ such that $A \leq H$, $G = \langle H, d \rangle$, and $l(G/H) = 1$. Any right-order on $H$ can be extended to a right-order on $G$. Thus $H \in C_1$ and by the induction hypothesis, $Z(H) \cap A = B \neq \langle e \rangle$. Now $D = \langle A, d \rangle$ is isolated in $G$ and any right-order on $D$ can be extended to a right-order on $G$ since there exists a series from $D$ to $G$ with torsion-free abelian factors. Thus $D \in C_1$ and by the first part of the proof, for any $e \neq b \in B$, $Z(D) \cap \langle b \rangle^D \neq \langle e \rangle$. Thus $Z(G) \cap B \neq \langle e \rangle$ and hence $Z(G) \cap A \neq \langle e \rangle$.

**Remark.** It follows from Corollary 3.4 and Theorem 3.6 that if $G$ is an orderable $C_\omega$-group, then the system of convex subgroups under any order on $G$ is central and $G$ is locally nilpotent if it is locally solvable. In the latter case every partial right-order can be extended to a total right-order (see [4]). In general a solvable $C_\omega$-group does not have this property as can easily be seen by considering the group $\langle a, b; b^{-1}ab = a^{-1} \rangle$.

**References**


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