3

(B,N)-Pairs; Parabolic, Levi, and Reductive Subgroups; Centralisers of Semi-Simple Elements

3.1 (B,N)-Pairs

We review properties of reductive groups related to existence of a (B,N)-pair. For an abstract group, having a (B,N)-pair is a very strong condition; many of the theorems we will give for reductive groups follow from this single property.

Definition 3.1.1 We say that two subgroups B and N of a group G form a **(B,N)-pair** (also called a **Tits system**) for G if:

(i) B and N generate G and $T := B \cap N$ is normal in N.
(ii) The group $W := N/T$ is generated by a set S of involutions such that:

  (a) For $s \in S$, $w \in W$ we have $BsB.BwB \subset BwB \cup BswB$.
  (b) For $s \in S$, we have $sBs \not\subset B$.

The group W is called the **Weyl group** of the (B,N)-pair. Note that we write elements of W – instead of representatives of them in N – in expressions representing subsets of G when these expressions do not depend upon the chosen representative.

We will see in 3.1.3(v) that under the assumptions of 3.1.1 we have $S = \{w \in W - \{1\} \mid B \cup BwB \text{ is a group}\}$, thus S is determined by (B,N).

Proposition 3.1.2 *If* $\mathbf{G}$ *is a connected reductive group and* $\mathbf{T} \subset \mathbf{B}$ *is a pair of a maximal torus and a Borel subgroup, then* $(\mathbf{B}, N_{\mathbf{G}}(\mathbf{T}))$ *is a* (B,N)*-pair for* $\mathbf{G}$.

Proof We show first that $\mathbf{B} \cap N_{\mathbf{G}}(\mathbf{T}) = \mathbf{T}$. By 1.3.1(iii) we have $N_{\mathbf{B}}(\mathbf{T}) = C_{\mathbf{B}}(\mathbf{T}) \subset C_{\mathbf{G}}(\mathbf{T}) = \mathbf{T}$ (see 2.3.1(iii)). By definition $\mathbf{T}$ is normal in $N_{\mathbf{G}}(\mathbf{T})$. To prove (i) it remains to show that $\mathbf{B}$ and $N_{\mathbf{G}}(\mathbf{T})$ generate $\mathbf{G}$. Let Φ^+ be the positive subsystem defined by $\mathbf{B}$. By 2.3.1(vi), $\mathbf{B}$ contains all the $\mathbf{U}_\alpha$ $(\alpha \in \Phi^+)$. Since s_α conjugates $\mathbf{U}_\alpha$ to $\mathbf{U}_{s_\alpha(\alpha)} = \mathbf{U}_{-\alpha}$, the group generated by $\mathbf{B}$ and $N_{\mathbf{G}}(\mathbf{T})$ contains $\mathbf{T}$ and all the $\mathbf{U}_\alpha$ $(\alpha \in \Phi)$, thus by 2.3.1(v) this group is equal to $\mathbf{G}$.

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If Π is the basis defined by the ordering Φ^+, (ii) is obtained by taking for S the $\{s_\alpha \mid \alpha \in \Pi\}$.

(ii)(b) reflects that ${}^{s_\alpha}\mathbf{U}_\alpha = \mathbf{U}_{-\alpha}$ is not in $\mathbf{B}$.

It remains to show (ii)(a). Let $s = s_\alpha$, and write $\mathbf{B} = \mathbf{T}\prod_{\beta\in\Phi^+}\mathbf{U}_\beta$. As s normalises $\mathbf{T}$, as ${}^s\mathbf{U}_\beta = \mathbf{U}_{s_\alpha(\beta)}$ and as $s_\alpha(\beta) \in \Phi^+$ if $\beta \in \Phi^+ - \{\alpha\}$, we get $\mathbf{B}s\mathbf{B}w\mathbf{B} = \mathbf{B}s\mathbf{U}_\alpha w\mathbf{B}$. If $w^{-1}(\alpha) \in \Phi^+$ the right hand side is equal to $\mathbf{B}sw\mathbf{B}$. Otherwise we write it as $\mathbf{B}s\mathbf{U}_\alpha ssw\mathbf{B}$ where this time $(sw)^{-1}(\alpha) \in \Phi^+$. Let $\mathbf{B}_\alpha$ be the image by ϕ_α (see 2.3.1(ii)) of the Borel subgroup of $\mathbf{SL}_2$ of upper triangular matrices. If $c \neq 0$ we have in $\mathbf{SL}_2$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1/c & -a \\ 0 & -c \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix}$$

which taking images shows that $s\mathbf{U}_\alpha s \subset \operatorname{Im}\phi_\alpha = \mathbf{B}_\alpha \cup \mathbf{B}_\alpha s\mathbf{U}_\alpha$, whence $\mathbf{B}s\mathbf{U}_\alpha ssw\mathbf{B} \subset \mathbf{B}s\mathbf{U}_\alpha sw\mathbf{B} \cup \mathbf{B}sw\mathbf{B}$ where the first term in the right-hand side is $\mathbf{B}w\mathbf{B}$ since $(sw)^{-1}(\alpha) \in \Phi^+$. □

Theorem 3.1.3 *If G has a (B,N)-pair, then*

(i) $G = \coprod_{w\in W} BwB$ (**Bruhat decomposition**).

(ii) (W,S) *is a Coxeter group.*

(iii) *Condition (ii)(a) of 3.1.1 can be refined to*

$$BsB.BwB = \begin{cases} BswB & \text{if } l(sw) = l(w)+1, \\ BswB \cup BwB & \text{otherwise.} \end{cases}$$

(iv) *For any $t \in N(w)$ (see 2.1.2(ii)), we have $BtB \subset Bw^{-1}BwB$.*

(v) $S = \{w \in W - \{1\} \mid B \cup BwB$ *is a group*$\}$.

(vi) *We have $N_G(B) = B$.*

Proof Let us show (i). As B and N generate G, we have $G = \cup_i (BNB)^i$. Since $BNB = BWB$ we will get $G = BWB$ if we show that $BWBWB = BWB$. For this it is enough to show that $BwBWB \subset BWB$ for $w \in W$; writing $w = s_1 \ldots s_n$ with $s_i \in S$, since $BwB \subset Bs_1B \ldots Bs_nB$ it is enough to show $BsBWB \subset BWB$ for $s \in S$; but this results from 3.1.1(ii)(a). It remains to show that $BwB \neq Bw'B$ if $w \neq w'$. We show this by induction on $\inf(l(w), l(w'))$, where l is the length with respect to S; assume for instance that $l(w) \leq l(w')$. The start of the induction is $l(w) = 0$ and the result comes from $w' \notin B$. Otherwise, taking $s \in S$ such that $l(sw) < l(w)$, by induction $BswB$ is equal neither to $Bw'B$ nor to $Bsw'B$ thus $BswB \cap BsB.Bw'B = \emptyset$; as $BswB \subset BsB.BwB$ it follows that $BwB \neq Bw'B$.

For (ii), we use 2.2.9 with $D_s = \{w \in W \mid BsBwB = BswB\}$ (note that if $w \notin D_s$ then $BsBwB = BswB \coprod BwB$). Clearly $D_s \ni 1$.

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If $w,\ sw \in D_s$, then from $BsBwB = BswB$ and $BsBswB = BwB$ we get $BsBsBwB = BwB$ which is a contradiction since multiplying on the right by BwB the equality $BsBsB = BsB \coprod B$ (since $sBs \not\subset B$ by 3.1.1(ii)(b)), we get $BsBsBwB = BswB \coprod BwB$.

It remains to show for (ii) that $w \in D_s, ws' \notin D_s$ implies $ws' = sw$. The assumption $ws' \notin D_s$ implies $BsBws'B = Bsws'B \coprod Bws'B$; in particular $BsBws'$ meets $Bws'B$; multiplying on the right by $s'B$ it follows that $BsBwB$ meets $Bws'Bs'B \subset (BwB \coprod Bws'B)$ (this last inclusion follows from 3.1.1(ii)(a) reversed, which is obtained by taking inverses). Thus $BswB = BsBwB$ (since $w \in D_s$) is equal to $Bws'B$, or to BwB. The latter cannot happen since $w \neq sw$, thus $sw = ws'$ as was to be shown. We have also shown (iii) by the property of D_s given in the last sentence of 2.2.9.

Let us show (iv). If $w = s_1 \ldots s_k$ is a reduced expression, for all i we can write by (iii) $BwB = Bs_1 \ldots s_{i-1}Bs_iBs_{i+1} \ldots s_kB$ and similarly for $Bw^{-1}B$ whence

$$\begin{aligned} Bw^{-1}BwB &= Bs_k \ldots s_{i+1}Bs_iBs_{i-1} \ldots s_1Bs_1 \ldots s_{i-1}Bs_iBs_{i+1} \ldots s_kB \\ &\supset Bs_k \ldots s_{i+1}Bs_iBs_iBs_{i+1} \ldots s_kB \\ &\supset Bs_k \ldots s_{i+1}Bs_iBs_{i+1} \ldots s_kB \\ &\supset Bs_k \ldots s_{i+1}s_is_{i+1} \ldots s_kB \end{aligned}$$

whence the result.

(v) follows immediately from (iv), which implies that $B \cup BwB$ can be a group only if $|N(w)| \leq 1$, and from (iii) which implies that $B \cup BsB$ is a group.

(vi) also follows from (iv). For $g \in BwB$ we have ${}^gB = B \Leftrightarrow {}^wB = B \Leftrightarrow BwBw^{-1}B = B$ which by (iv) happens only for $w = 1$. □

In a group G with a (B,N)-pair, we call **Borel subgroups** the conjugates of B and **maximal tori** the conjugates of T; this fits the terminology for algebraic groups.

Corollary 3.1.4 *In a group G with a (B,N)-pair, every pair of Borel subgroups is conjugate to a pair of the form $(B, {}^wB)$ with $w \in W$; the intersection of two Borel subgroups contains a maximal torus.*

Proof Up to conjugacy, we may assume the given pair of Borel subgroups of the form $(B, {}^gB)$. By the Bruhat decomposition we may write $g = bwb'$ where $b, b' \in B$; thus the pair is equal to $(B, {}^{bw}B)$, which is conjugate to $(B, {}^wB)$. Since B and ${}^wB$ both contain T, the intersection of every conjugate pair also contains a maximal torus. □

Example 3.1.5 For m a matrix in $\mathbf{GL}_n$, let $m_{i,j}$ be the submatrix on the last lines $i, \ldots, n$ and first columns $1, \ldots, j$. Let w be a permutation matrix; then

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$m \in \mathbf{B}w\mathbf{B}$, where $\mathbf{B}$ is the Borel subgroup of upper triangular matrices, if and only if the matrices $m_{i,j}$ and $w_{i,j}$ have same rank for all i,j. Indeed,

- The ranks of $m_{i,j}$ are invariant by left or right multiplication of m by an upper triangular matrix.
- A permutation matrix w for the permutation σ is characterised by the ranks of $w_{i,j}$, given by $|\{k \leq j \mid \sigma(k) \geq i\}|$.

If $\{F'_i\}$ and $\{F''_i\}$ are two complete flags whose stabilisers are the Borel subgroups $\mathbf{B}'$ and $\mathbf{B}''$, then the permutation matrix w such that $(\mathbf{B}',\mathbf{B}'')$ is conjugate to $(\mathbf{B}, {}^w\mathbf{B})$ (the **relative position** of the two flags) is characterised by rank $w_{i,j} = \dim \frac{F'_i \cap F''_j}{(F'_{i-1} \cap F''_j)+(F'_i \cap F''_{j-1})}$.

3.2 Parabolic Subgroups of Coxeter Groups and of (B,N)-Pairs

Lemma 3.2.1 *Let (W,S) be a Coxeter system, let I be a subset of S, and let W_I be the subgroup of W generated by I, called a* **standard parabolic subgroup** *of W. Then (W_I,I) is a Coxeter system.*

An element $w \in W$ is said to be **reduced-I** *if it satisfies one of the equivalent conditions:*

(i) *For any $v \in W_I$, we have $l(wv) = l(w) + l(v)$.*
(ii) *For any $s \in I$, we have $l(ws) > l(w)$.*
(iii) *w has minimal length in the coset wW_I.*
(iv) *$N(w) \cap I = \emptyset$.*
(v) *$N(w) \cap \mathrm{Ref}(W_I) = \emptyset$.*

There is a unique reduced-I element in wW_I.

By exchanging left and right we have the notion of ***I*-reduced** element which satisfies the mirror properties. A subgroup of W conjugate to a standard parabolic subgroup is called a **parabolic subgroup**.

Proof A reduced expression in W_I is reduced in W by the exchange condition and then satisfies the exchange condition in W_I, thus (W_I,I) is a Coxeter system.

(iii)⇒(ii) since (iii) implies $l(ws) \geq l(w)$ when $s \in I$. Let us show that "not (iii)"⇒ "not (ii)". If w' does not have minimal length in $w'W_I$, then $w' = wv$ with $v \in W_I$ and $l(w) < l(w')$; adding one by one the terms of a reduced expression for v to w and applying at each stage the exchange condition, we find that w' has a reduced expression of the shape $\hat{w}\hat{v}$ where $\hat{w}$ (resp. $\hat{v}$) denotes

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a subsequence of the chosen reduced expression. As $l(\hat{w}) \leq l(w) < l(w')$, we have $l(\hat{v}) > 0$, thus w' has a reduced expression ending by an element of I, thus w' does not satisfy (ii).

(i)⇒(iii) is clear. Let us show "not (i)"⇒ "not (iii)". If $l(wv) < l(w) + l(v)$ then a reduced expression for wv has the shape $\hat{w}\hat{v}$ where $l(\hat{w}) < l(w)$. Then $\hat{w} \in wW_I$ and has a length smaller than that of w.

By 2.1.6(ii) property (ii) is equivalent to (iv).

It is clear that (v) implies (iv), and (i) applied to $v \in \mathrm{Ref}(W)$ implies (v) by 2.1.6(ii).

Finally, an element satisfying (i) is clearly unique in wW_I. □

Lemma 3.2.2 *Let I and J be two subsets of S. An element $w \in W$ is* **I-reduced-J** *if it satisfies one of the equivalent properties:*

(i) *w is both I-reduced and reduced-J.*
(ii) *w has minimal length in $W_I w W_J$.*
(iii) *Every element of $W_I w W_J$ can be written uniquely xwy with $x \in W_I$, $y \in W_J$, $l(x) + l(w) + l(y) = l(xwy)$ and xw is reduced-J.*

(iii) implies that in a double coset in $W_I \backslash W / W_J$ there is a unique I-reduced-J element, which has minimal length; by symmetry we can replace in condition (iii) the assumption that xw is reduced-J by the assumption that wy is I-reduced.

Proof We first show that two elements w, w' in the same double coset and satisfying (i) have the same length. Write $w' = xwy$ with $x \in W_I$ and $y \in W_J$; then $w'y^{-1} = xw$ and $x^{-1}w' = wy$; by the defining properties of I-reduced and reduced-J and using $l(y^{-1}) = l(y)$, $l(x^{-1}) = l(x)$ we get $l(w') + l(y) = l(x) + l(w)$ and $l(x) + l(w') = l(w) + l(y)$, whence $l(x) = l(y)$ and $l(w) = l(w')$. As clearly (ii)⇒(i) this common length must be the minimal length, thus (i)⇔(ii).

We now show (ii)⇒(iii). Assume w satisfies (ii); write an element $v \in W_I w W_J$ as xwy with $x \in W_I$, $y \in W_J$ and x of minimal possible length. By the exchange lemma a reduced expression for xwy is of the form $\hat{x}\hat{w}\hat{y}$ where $\hat{x}$ (resp. $\hat{w}$, $\hat{y}$) is a subsequence of a reduced expression for x (resp. w, y). Necessarily $\hat{w} = w$ otherwise w would not be of minimal length in its double coset. Then the minimal length assumption on x implies $\hat{x} = x$, whence $\hat{y} = y$, thus $l(x) + l(w) + l(y) = l(xwy)$. The element xw is reduced-J otherwise we can write $xw = v'y'$ where $v' \in W_I w W_J$, $y' \in W_J - \{1\}$ and $l(v') + l(y') = l(xw)$. Using what we just proved on w we can write $v' = x''wy''$ with $l(x'') + l(w) + l(y'') + l(y') = l(x) + l(w)$ which implies $l(x'') < l(x)$, contradicting the minimality of $l(x)$. Finally the decomposition xwy is unique since xw is the unique J-reduced element in its coset.

Finally, (iii)⇒(ii) is clear. □

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Note that not every decomposition xwy where w is I-reduced-J satisfies (iii); consider for instance the case $w = y = 1$, $I = J$ and x the longest element of W_I; thus the situation is not as good as in the I-reduced case.

In a group with a (B,N)-pair, we use the term **parabolic subgroups** for the subgroups containing a Borel subgroup.

Proposition 3.2.3 *Let G be a group with a (B,N)-pair. Then*

(i) *The (parabolic) subgroups containing B are the $P_I = BW_IB$ for some $I \subset S$.*

(ii) *Given two parabolic subgroups P_I and P_J, we have a* **relative Bruhat decomposition** *$G = \coprod_w P_IwP_J$ where w runs over the I-reduced-J elements. It follows a natural bijection $P_I\backslash G/P_J \xrightarrow{\sim} W_I\backslash W/W_J$.*

Proof Let us show (i). Let P be a subgroup containing B and let $w \in W$ be such that $BwB \subset P$. Since P is a group we get $Bw^{-1}BwB \subset P$, thus by 3.1.3(iv) we get $BtB \subset P$ for any $t \in N(w)$. If $w = s_1 \dots s_k$ is a reduced expression we get in particular $Bs_kB \subset P$, thus $s_1 \dots s_{k-1} \in P$ and by induction for each i we have $s_i \in P$. It follows that $P = BW_IB$ where I is the union of the elements of S appearing in any reduced expression of any w such that $BwB \subset P$. Conversely, for any $I \subset S$, using 3.1.1(ii)(a) we see that BW_IB is a group.

Let us show (ii). For any $w \in W$ we have $P_IwP_J = BW_IBwBW_JB = BW_IwW_JB$, the last equality by repeated application of 3.1.1(ii)(a) and of its right counterpart. Since, by Lemma 3.2.2 we can take I-reduced-J elements as representatives of the double cosets we see that the first assertion of (ii) is just the Bruhat decomposition. Conversely, any coset P_IgP_J is of the form P_IwP_J if $g \in BwB$ whence the last assertion of (ii). □

Remark 3.2.4 Using 3.2.3 we see that in the definition 1.3.5 of a parabolic subgroup the word "closed" can be omitted. Indeed a reductive group has a (B,N) pair, hence by 3.2.3 a subgroup containing a Borel subgroup is conjugate to some $\mathbf{B}W_I\mathbf{B}$, hence it is closed. In general, if $\mathbf{G}$ is a connected group and $\mathbf{P}$ is a subgroup containing a Borel subgroup, then $\mathbf{P}/R_u(\mathbf{G})$ contains a Borel subgroup of the reductive group $\mathbf{G}/R_u(\mathbf{G})$ hence it is closed, thus $\mathbf{P}$ is closed by continuity of the quotient morphism.

Example 3.2.5 In $\mathbf{GL}_n$, the parabolic subgroup $\mathbf{P}_J$ for $J \subset S$ containing the Borel subgroup of upper triangular matrices is the subgroup of upper block-triangular matrices where the blocks correspond to maximal intervals $[i,k]$ in $[1,n]$ such that $s_i,\dots,s_{k-1} \in J$.

Example 3.2.6 For the symplectic group $\mathbf{Sp}_{2n}$, as the stabiliser $\mathbf{B}$ of any complete isotropic flag $V_1 \subset \cdots \subset V_n$ in $\mathbf{Sp}_{2n}$ is a Borel subgroup, the stabiliser of any subflag is a parabolic subgroup. We thus get 2^n distinct parabolic subgroups containing $\mathbf{B}$. Since there are also 2^n subsets of S, they are the only parabolic

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subgroups containing **B**. As any isotropic flag may be completed to a complete one, we get the result that in general parabolic subgroups are the stabilisers of (complete or not) isotropic flags.

Lemma 3.2.7 (unicity in Bruhat decomposition) *Let* **G** *be a connected reductive group and* $\mathbf{B} = \mathbf{U} \rtimes \mathbf{T}$ *be a decomposition of a Borel subgroup* **B** *as in 1.3.1(ii), where* $\mathbf{U} = R_u(\mathbf{B})$. *Then* $\mathbf{B}w\mathbf{B}$ *has a direct product decomposition* $\mathbf{U} \times \mathbf{T}w \times \mathbf{U}_w$ *where* $\mathbf{U}_w := \prod_{\{\alpha\in\Phi^+ \mid w(\alpha)<0\}} \mathbf{U}_\alpha$.

Proof Notice first that $\mathbf{U}_w$ is a group; since if in 2.3.1(vii) α and β are sent to negative roots by w, then the same holds for $\lambda\alpha + \mu\beta$. We have $\mathbf{U} = \mathbf{U}'\mathbf{U}_w$ where $\mathbf{U}' = \prod_{\{\alpha\in\Phi^+ \mid w(\alpha)>0\}} \mathbf{U}_\alpha$ thus ${}^w\mathbf{U}' \subset \mathbf{U}$; thus $\mathbf{B}w\mathbf{B} = \mathbf{U}\mathbf{T}w\mathbf{U}'\mathbf{U}_w = \mathbf{U}\mathbf{T}w\mathbf{U}_w$. It remains to be shown that the decomposition is unique; that is, if $u\mathbf{T}wu' = \mathbf{T}w$ with $u \in \mathbf{U}, u' \in \mathbf{U}_w$ then $u = u' = 1$. The condition implies ${}^wu' \in \mathbf{B}$. But ${}^w\mathbf{U}_w \cap \mathbf{B} = 1$ since all $\mathbf{U}_\alpha$ in ${}^w\mathbf{U}_w$ are for negative α. Thus $u' = 1$, whence $u = 1$. □

The next proposition says that the decomposition of **G** in **Bruhat cells** $\mathbf{B}w\mathbf{B}$ is a stratification (the closure of a stratum is a union of strata).

Proposition 3.2.8 *Let* **G** *be a connected reductive group and* $\mathbf{B} = \mathbf{U} \rtimes \mathbf{T}$ *be a decomposition of a Borel subgroup* **B** *as in 3.2.7. Then the Zariski closure of* $\mathbf{B}w\mathbf{B}$ *in* **G** *is given by* $\overline{\mathbf{B}w\mathbf{B}} = \coprod_{v\leq w} \mathbf{B}v\mathbf{B}$, *where* $\leq$ *is the* **Bruhat–Chevalley order** *on* w, *given by* $v \leq w$ *if a reduced expression of* v *is a subsequence of a reduced expression of* w.

Reference See Chevalley (1994, Proposition 6). □

3.3 Closed Subsets of a Crystallographic Root System

In this section, Φ will be a reduced crystallographic root system in the $\mathbb{Q}$-vector space V, and Π will be a basis of Φ; we denote by Φ^+ the corresponding positive subsystem and by (W,S) the corresponding Coxeter system, where $S = \{s_\alpha\}_{\alpha\in\Pi}$.

Definition 3.3.1 A subset $\Psi \subset \Phi$ is:

(i) **closed** if $\alpha,\beta \in \Psi, \alpha+\beta \in \Phi \Rightarrow \alpha+\beta \in \Psi$.
(ii) **symmetric** if $\Psi = -\Psi$.

The intersection of two closed subsets is clearly closed.

Lemma 3.3.2 *The reduced crystallographic root systems of rank 2 are* $A_1\times A_1$, A_2, $C_2 = B_2$ *and* G_2.

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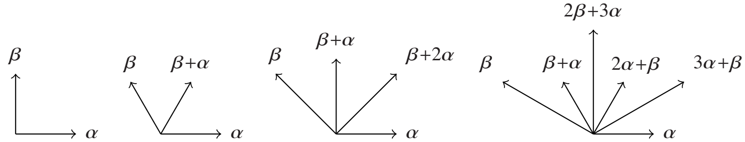
Here is a picture of their positive roots:



Proof Let Φ be crystallographic of rank 2 with Weyl group W. Let $\Pi = \{\alpha, \beta\}$. Choosing a W-invariant scalar product $(\,.\,,.\,)$ as in 2.2.3, we have $\alpha^\vee(\beta)\beta^\vee(\alpha) = 4\frac{(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)} = 4\cos^2\theta$ where θ is the angle between α and β. Since $\alpha, \beta \in \Pi$ we have $(\alpha, \beta) \leq 0$ thus $\pi/2 \leq \theta \leq \pi$ and the integrality of $4\cos^2\theta$ implies that $4\cos^2\theta \in \{0, 1, 2, 3\}$ thus $\pi - \theta \in \{\pi/2, \pi/3, \pi/4, \pi/6\}$. Except for $A_1 \times A_1$, the ratio of the lengths of α and β is implied by the equation $\alpha^\vee(\beta)\beta^\vee(\alpha) = 4\cos^2\theta$. For instance if $4\cos^2\theta = 2$ the only integral solution, up to exchanging α and β, is $\alpha^\vee(\beta) = -1$ and $\beta^\vee(\alpha) = -2$ whence $2(\beta, \beta) = (\alpha, \alpha)$. For $A_1 \times A_1$ the ratio of the lengths is not determined, we have chosen 1 in the picture. □

Corollary 3.3.3 *For a crystallographic root system Φ and a positive subsystem Φ^+, we have:*

(i) *If $\alpha, \beta \in \Phi$, $\alpha \neq -\beta$ and $(\alpha, \beta) < 0$, then $\alpha + \beta \in \Phi$.*
(ii) *If $\alpha, \beta \in \Phi$ and $\alpha + n\beta \in \Phi$ for $n \in \mathbb{N}$, then $\alpha + m\beta \in \Phi$ for all $0 \leq m \leq n$.*
(iii) *If $\alpha_1, \ldots, \alpha_k \in \Phi^+$ and $\alpha = \alpha_1 + \cdots + \alpha_k \in \Phi^+$, then if $k > 1$ we have $\alpha - \alpha_i \in \Phi^+$ for some i.*
(iv) *If $\Psi \subset \Phi$ is closed, $\alpha, \beta \in \Psi$, $\alpha \neq -\beta$ and $n\alpha + m\beta \in \Phi$ for some $n, m > 0$, then $n\alpha + m\beta \in \Psi$.*

Proof For (i), by the argument in the proof of 3.3.2 about possible integral solutions, up to exchanging α and β we have $\alpha^\vee(\beta) = -1$, whence $\alpha + \beta = s_\alpha(\beta) \in \Phi$. For (iii) as $(\alpha, \alpha) > 0$ we must have $(\alpha, \alpha_i) > 0$ for some i thus by (i) $\alpha - \alpha_i \in \Phi$.

For (ii), by (iii) either $\alpha + (n-1)\beta$ or $n\beta$ is in Φ, and since Φ is reduced, $n\beta \notin \Phi$ if $n \neq 1$, whence the result by induction on n.

For (iv) we may assume both α and β positive (they are for some order since $\alpha \neq -\beta$), and then we apply (iii) and induction on $n + m$. □

Corollary 3.3.4 *If $\Psi \subset \Phi$ is closed and symmetric, it is a root subsystem.*

Proof For $\alpha, \beta \in \Psi$, we have to show that $s_\alpha(\beta) \in \Psi$. This is true if $\beta = \pm\alpha$ since Ψ is symmetric. Otherwise, replacing α by $-\alpha$ if necessary we have $s_\alpha(\beta) = \beta + n\alpha$ for some $n \in \mathbb{N}^*$; then Corollary 3.3.3(iv) gives the result. □

Proposition 3.3.5 *If Ψ is closed and $\Psi \cap -\Psi = \emptyset$, there exists a positive subsystem Φ^+ such that $\Psi \subset \Phi^+$.*

Proof We first show by induction on $k > 0$ that 0 is not the sum of k elements of Ψ. This is clear for $k = 1$. If $0 = \alpha_1 + \cdots + \alpha_k$ then $0 < (\alpha_1,\alpha_1) = (-\alpha_1,\alpha_2 + \cdots + \alpha_k)$ thus there exists $i \neq 1$ such that $(\alpha_1,\alpha_i) < 0$. Using $\alpha_1 \neq -\alpha_i$ (since $-\alpha_i \notin \Psi$ by assumption) and 3.3.3(i) we get $\alpha_1 + \alpha_i \in \Phi$ thus $\alpha_1 + \alpha_i \in \Psi$, thus the sum is the sum of $k-1$ elements, a contradiction.

We now build by induction on k a sequence γ_k of elements of Ψ such that $\gamma_k \in \Psi$ is the sum of k elements of Ψ. We start with γ_1 equal to an arbitrary element of Ψ. If there is $\alpha \in \Psi$ such that $(\gamma_k,\alpha) < 0$ we set $\gamma_{k+1} = \alpha + \gamma_k \in \Psi$. For $i < j$ we have $\gamma_i \neq \gamma_j$, otherwise $\gamma_j - \gamma_i$ would be a zero sum of elements of Ψ, thus by finiteness the sequence must stop on some γ_k such that $(\gamma_k,\alpha) \geq 0$ for any $\alpha \in \Psi$. The linear form $(\gamma_k, .)$ almost defines an order as in 2.2.4. We need to modify it on $\gamma_k^\perp$. But $\gamma_k^\perp \cap \Psi \subset \gamma_k^\perp \cap \Phi$ satisfies the same assumptions as the proposition and we may iterate the construction on this subspace. □

For $I \subset S$ we set $\Pi_I := \{\alpha \in \Pi \mid s_\alpha \in I\}$ and $\Phi_I = \Phi \cap \mathbb{Q}\Pi_I$; it is clearly a root subsystem with basis Π_I, since when decomposed on Π a root of Φ_I involves only elements of Π_I.

It is clear that Φ_I is closed and symmetric and that $\Phi^+ - \Phi_I$ and $\Phi^+ \cup \Phi_I$ are closed.

Example 3.3.6 There exist closed and symmetric subsystems which are not of the form Φ_I; for instance the long roots in a system B_2 form a system of type $A_1 \times A_1$, and the long roots in G_2 form a system of type A_2. See also 11.2.7.

Lemma 3.3.7 *If $s_\alpha \in W_I$ for $\alpha \in \Phi$, then $\alpha \in \Phi_I$.*

Proof Elements of W_I are the product of some s_β for $\beta \in \Pi_I$, thus they fix $\Pi_I^\perp$. Thus s_α fixes $\Pi_I^\perp$, which implies that $\alpha \in \mathbb{Q}\Pi_I \cap \Phi = \Phi_I$. □

We say that Ψ is a **parabolic** subset of Φ if Ψ is closed and $\Psi \cup -\Psi = \Phi$.

Proposition 3.3.8

(i) *A parabolic subset is conjugate to a parabolic subset containing Φ^+; such a subset is of the form $\Phi^+ \cup \Phi_I$ for some $I \subset S$.*

(ii) *A parabolic subset is a set of the form $\{\alpha \mid \lambda(\alpha) \geq 0\}$ for some linear form λ on V.*

Proof For the first part of (i) it is equivalent to show that a parabolic subset Ψ contains some positive subsystem. Choose such a positive subsystem Φ^+ such that $|\Psi \cap \Phi^+|$ is maximal. We show by contradiction that $\Phi^+ \subset \Psi$. Otherwise let Π be the basis of Φ defining Φ^+; there must exist $\alpha \in \Pi, \alpha \notin \Psi$, thus $-\alpha \in \Psi$. Since $\alpha \notin \Psi$ we have $s_\alpha(\Psi \cap \Phi^+) \subset \Phi^+$; applying s_α again we get $\Psi \cap \Phi^+ \subset s_\alpha(\Phi^+)$. But then the positive subsystem $s_\alpha(\Phi^+)$ contains $-\alpha$ thus satisfies $|\Psi \cap s_\alpha(\Phi^+)| > |\Psi \cap \Phi^+|$, a contradiction.

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We now assume that $\Psi \supset \Phi^+$. Let $I = \{s_\alpha \mid -\alpha \in \{-\Pi \cap \Psi\}\}$. Let us show that $\Psi \cap \Phi^- = \Phi_I^-$.

We first show that $\Phi_I^- \subset \Psi$. Note that by 2.2.8 applied to the basis $-\Pi_I$ of Φ_I^- any root in Φ_I^- is a sum of elements of $-\Pi_I$. We show by induction on k that a root in Φ_I^- sum of k roots in $-\Pi_I$ is in Ψ. It is true by assumption when $k = 1$; in general by 3.3.3(iii) we may write the root as $\alpha + \beta$ where $\alpha \in -\Pi_I$ and $\beta \in \Phi_I^-$ sum of $k-1$ roots in $-\Pi_I$; by induction $\beta \in \Psi$ and as $\alpha \in \Psi$ and Ψ is closed $\alpha + \beta \in \Psi$.

We finally show the reverse inclusion by induction. Let $\gamma \in \Psi \cap \Phi^-$ be the sum of k roots of $-\Pi$, and write it $\gamma = \alpha + \beta$ where $\alpha \in -\Pi$ and $\beta \in \Phi$ is the sum of $k-1$ roots in $-\Pi$. As $-\beta \in \Phi^+ \subset \Psi$ we get $\alpha = \gamma + (-\beta) \in \Psi$ whence $\alpha \in -\Pi \cap \Psi = -\Pi_I$. Thus $-\alpha \in \Psi$ whence $\beta = \gamma + (-\alpha) \in \Psi$, and we conclude since by induction $\beta \in \Phi_I^-$.

Conversely the fact that for any $I \subset S$ the set $\Phi^+ \cup \Phi_I$ is parabolic is a consequence of the proof of (ii) below.

We now show (ii). It is clear that a subset of the form $\{\alpha \mid \lambda(\alpha) \geq 0\}$ is parabolic. It is thus sufficient to show that $\Phi^+ \cup \Phi_I$ is of this form. Take any x such that $\langle x, \alpha \rangle = 0$ if $\alpha \in \Phi_I$ and $\langle x, \alpha \rangle > 0$ if $\alpha \in \Psi - \Phi_I$. Such an x exists: the projection of Φ^+ on $\Phi_I^\perp$ lies in a half-space, and we may take x in this half-space, orthogonal to the hyperplane which delimits it. It is clear that by construction x has the required properties. □

A consequence of 3.3.8 is that the complement of a parabolic subset is closed.

Subgroups of Maximal Rank and Quasi-closed Sets

In the remainder of this chapter $\mathbf{G}$ is a connected reductive algebraic group, $\mathbf{T}$ is a maximal torus of $\mathbf{G}$, and Φ is the set of roots of $\mathbf{G}$ relative to $\mathbf{T}$. For $\Psi \subset \Phi$, we set $\mathbf{G}_\Psi^* := \langle \mathbf{U}_\alpha \mid \alpha \in \Psi \rangle$ and $\mathbf{G}_\Psi := \langle \mathbf{T}, \mathbf{U}_\alpha \mid \alpha \in \Psi \rangle$. These are closed connected subgroups by 1.1.3 and $\mathbf{G}_\psi^*$ is a normal subgroup of $\mathbf{G}_\Psi$.

Definition 3.3.9 A subset $\Psi \subset \Phi$ is called **quasi-closed** if $\mathbf{G}_\Psi^*$ does not contain any $\mathbf{U}_\alpha$ with $\alpha \in \Phi - \Psi$.

We get an equivalent definition by replacing $\mathbf{G}_\Psi^*$ with $\mathbf{G}_\Psi$, since $\mathbf{G}_\Psi / \mathbf{G}_\Psi^*$ is a quotient of $\mathbf{T}$ and thus is a torus. Hence any $\mathbf{U}_\alpha \subset \mathbf{G}_\Psi$ is in the kernel of this quotient, and is thus in $\mathbf{G}_\Psi^*$.

Proposition 3.3.10 *A closed and connected subgroup* $\mathbf{H} \subset \mathbf{G}$ *containing* $\mathbf{T}$ *is equal to* $\mathbf{G}_\Psi$ *with* $\Psi = \{\alpha \in \Phi \mid \mathbf{U}_a \subset \mathbf{H}\}$*; the set* Ψ *is quasi-closed.*

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Proof By 2.3.1(iv) **H** is generated by **T** and the $\mathbf{U}_\alpha$ it contains. The subset $\Psi \subset \Phi$ of those α is quasi-closed by definition. □

Let Ψ, Ψ' be quasi-closed; it is clear that $\Psi \cap \Psi'$ is quasi-closed (since $\mathbf{G}_{\Psi\cap\Psi'}$ is a subgroup of both $\mathbf{G}_\Psi$ and $\mathbf{G}_{\Psi'}$); actually one sees that $\mathbf{G}_{\Psi\cap\Psi'} = (\mathbf{G}_\Psi \cap \mathbf{G}_{\Psi'})^0$ by applying 2.3.1(iv) to the right-hand side.

Definition 3.3.11 A connected linear algebraic group **P** has a **Levi decomposition** if there is a closed subgroup $\mathbf{L} \subset \mathbf{P}$ such that $\mathbf{P} = R_u(\mathbf{P}) \rtimes \mathbf{L}$. The group **L** is called a **Levi subgroup** of **P** (or a **Levi complement**).

A Levi complement is clearly reductive.

Proposition 3.3.12 *Let $\Psi \subset \Phi$ be quasi-closed, and let $\Psi_s = \{\alpha \in \Psi \mid -\alpha \in \Psi\}$ and $\Psi_u = \{\alpha \in \Psi \mid -\alpha \notin \Psi\}$. Then Ψ_s and Ψ_u are quasi-closed and $\mathbf{G}_\Psi$ has a Levi decomposition $\mathbf{G}_\Psi = \mathbf{G}^*_{\Psi_u} \rtimes \mathbf{G}_{\Psi_s}$ where $\mathbf{G}^*_{\Psi_u} = R_u(\mathbf{G}_\Psi)$. In particular $\mathbf{G}_\Psi$ is reductive if and only if Ψ is symmetric.*

Proof We first show that Ψ_s is quasi-closed. As the intersection of two quasi-closed sets is quasi-closed, it is enough to show that $-\Psi$ is quasi-closed. This results from the existence of the opposition automorphism of **G** which acts by -1 on $X(\mathbf{T})$; see Example 2.4.9.

As a connected group normalised by **T** the group $R_u(\mathbf{G}_\Psi)$ is – by Theorem 2.3.1(v) – of the form $\mathbf{G}^*_{\Psi'}$ for a subset $\Psi' \subset \Psi$ that we may assume quasi-closed. We have $\Psi' \subset \Psi_u$, otherwise there is $\alpha \in \Psi_s \cap \Psi'$, thus $\mathbf{U}_{-\alpha} \subset \mathbf{G}_\Psi$ thus normalises $R_u(\mathbf{G}_\Psi)$ thus $[\mathbf{U}_{-\alpha}, \mathbf{U}_\alpha] \subset R_u(\mathbf{G}_\Psi)$ which is a contradiction since this commutator set contains non-unipotent elements by Theorem 2.3.1(ii).

To show $\Psi' = \Psi_u$ it is thus enough to show $\Psi - \Psi' \subset \Psi_s$. If $\alpha \in \Psi - \Psi'$, then $\mathbf{U}_\alpha \cap R_u(\mathbf{G}_\Psi) = 1$ since this intersection is normalised by **T** thus contains the whole $\mathbf{U}_\alpha$ if not trivial. Thus, in the quotient $\mathbf{G}_\Psi \to \mathbf{L}'$, where $\mathbf{L}'$ is the reductive group $\mathbf{G}_\Psi / R_u(\mathbf{G}_\Psi)$, the group $\mathbf{U}_\alpha$ maps injectively to a root subgroup of $\mathbf{L}'$. Let $\mathbf{U}'$ be the root subgroup of $\mathbf{L}'$ corresponding to the opposed root and $\mathbf{U}''$ its preimage. Any element of $\mathbf{U}''$ is unipotent since, its image being unipotent, its semi-simple part is in $R_u(\mathbf{G}_\Psi)$ so is trivial. Hence $\mathbf{U}''$ is a unipotent subgroup normalised by **T**, so is a product of certain root subgroups and must contain $\mathbf{U}_{-\alpha}$, thus $-\alpha \in \Psi$ and $\alpha \in \Psi_s$.

It also follows from the proof that $\mathbf{G}_{\Psi_s}$ maps injectively to $\mathbf{L}'$, thus $\mathbf{G}_{\Psi_s}$ is a Levi complement of $R_u(\mathbf{G}_\Psi)$. □

Proposition 3.3.13 *A closed subset is quasi-closed.*

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Proof Let $\Psi \subset \Phi$ be a closed subset and let Ψ_s and Ψ_u be as in 3.3.12. It is clear that Ψ_s is closed. Note that if $\alpha \in \Psi$, $\beta \in \Psi_u$ and $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Psi_u$, otherwise $\alpha + \beta \in \Psi_s$ whence $-\alpha - \beta \in \Psi_s$ thus $\alpha + (-\alpha - \beta) = -\beta \in \Psi$ which contradicts $\beta \in \Psi_u$. In particular Ψ_u is closed. By 3.3.5 there exists a positive subsystem such that $\Psi_u \subset \Phi^+$.

Lemma 3.3.14 *If Ψ is a closed subset of a positive subsystem Φ^+ of Φ, then Ψ is quasi-closed and $\mathbf{G}^*_\Psi = \prod_{\alpha\in\Psi} \mathbf{U}_\alpha$ where the product is taken in an arbitrary order.*

In the situation of the lemma we will write $\mathbf{U}_\Psi$ for $\mathbf{G}^*_\Psi$.

Proof By 2.3.1(vii) and 3.3.3(iv) $\prod_{\alpha\in\Psi} \mathbf{U}_\alpha$ is a group, thus equal to $\mathbf{G}^*_\Psi$. □

We deduce that Ψ_u is quasi-closed. In addition, if $\alpha \in \Psi_s$, $\beta \in \Psi_u$ and $\alpha + \beta \in \Phi$, using the fact that $\alpha + \beta \in \Psi_u$ and 3.3.3(iv), we get $n\alpha + m\beta \in \Psi_u$ for $n, m \geq 1$ such that $n\alpha + m\beta \in \Phi$. Thus $\mathbf{G}^*_{\Psi_u}$ is normalised by $\mathbf{G}_{\Psi_s}$.

Since Ψ_s is closed and symmetric, it is a root subsystem by 3.3.4. Let Π_s be its basis corresponding to the positive subsystem $\Psi_s \cap \Phi^+$. Note that $\mathbf{G}_{\Psi_s}$ is already generated by $\mathbf{T}$ and $\mathbf{U}_\alpha$ such that $\alpha \in \pm\Pi_s$; indeed $\langle \mathbf{U}_\alpha, \mathbf{U}_{-\alpha}\rangle$ contains a representative of s_α by 2.3.1(iii), thus $\mathbf{G}_{\Psi_s}$ contains W_{Ψ_s}, and every root of Ψ_s is in the orbit of Π_s by 2.2.8, whence the result by the remark above 2.3.3. We show now that $\mathbf{G}_{\Psi_s} = \mathbf{U}_{\Psi_s^+} W_{\Psi_s} \mathbf{T}\mathbf{U}_{\Psi_s^+}$. For that it is enough to show that the right-hand side is a group. Since it is stable by left translation by $\mathbf{T}$ and by any $\mathbf{U}_\alpha$ for $\alpha \in \Psi_s^+$ it is enough to see it is stable by left translation by $\mathbf{U}_{-\alpha}$ for $\alpha \in \Pi_s$. Decomposing $\mathbf{U}_{\Psi_s^+} = \mathbf{U}_{\Psi_s^+ - \{\alpha\}}\mathbf{U}_\alpha$, and using that by 2.3.1(vii) $\mathbf{U}_{-\alpha}$ normalises $\mathbf{U}_{\Psi_s^+ - \{\alpha\}}$ since α is simple, it is enough to see that $\mathbf{U}_\alpha W_{\Psi_s}\mathbf{T}\mathbf{U}_{\Psi_s^+}$ is stable by left translation by $\mathbf{U}_{-\alpha}$. The Bruhat decomposition $\langle \mathbf{T}, \mathbf{U}_\alpha, \mathbf{U}_{-\alpha}\rangle = \mathbf{U}_\alpha\mathbf{T} \cup \mathbf{U}_\alpha s_\alpha \mathbf{T}\mathbf{U}_\alpha$ shows that $\mathbf{U}_{-\alpha}\mathbf{U}_\alpha \subset \mathbf{U}_\alpha\mathbf{T} \cup \mathbf{U}_\alpha s_\alpha\mathbf{T}\mathbf{U}_\alpha$. We just need to consider the second term

$$\mathbf{U}_\alpha s_\alpha \mathbf{T}\mathbf{U}_\alpha W_{\Psi_s}\mathbf{T}\mathbf{U}_{\Psi_s^+} = \bigcup_{w\in\Psi_s} \mathbf{U}_\alpha s_\alpha \mathbf{U}_\alpha w\mathbf{T}\mathbf{U}_{\Psi_s^+}.$$

If $w^{-1}(\alpha) \in \Psi^+$, then $\mathbf{U}_\alpha w\mathbf{T} = w\mathbf{T}\mathbf{U}_{w^{-1}(\alpha)}$ and the term has the right form. Otherwise, letting $\beta = -w^{-1}(\alpha) \in \Psi_s^+$ we get

$$\begin{aligned}\mathbf{U}_\alpha s_\alpha w\mathbf{T}\mathbf{U}_{w^{-1}(\alpha)}\mathbf{U}_{\Psi_s^+} &= \mathbf{U}_\alpha s_\alpha w\mathbf{T}\mathbf{U}_{-\beta}\mathbf{U}_\beta\mathbf{U}_{\Psi_s^+-\{\beta\}}\\ &\subset \mathbf{U}_\alpha s_\alpha w\mathbf{T}(\mathbf{U}_\beta \cup \mathbf{U}_\beta s_\beta \mathbf{U}_\beta)\mathbf{U}_{\Psi_s^+-\{\beta\}}\\ &= \mathbf{U}_\alpha s_\alpha w\mathbf{T}\mathbf{U}_{\Psi_s^+} \cup \mathbf{U}_\alpha s_\alpha w\mathbf{T}\mathbf{U}_\beta s_\beta \mathbf{U}_{\Psi_s^+}.\end{aligned}$$

We just need to consider the rightmost term. Since $s_\alpha w\mathbf{U}_\beta\mathbf{T} = s_\alpha\mathbf{U}_{-\alpha}w\mathbf{T} = \mathbf{U}_\alpha s_\alpha w\mathbf{T}$ we get the result.

Let us now show that Ψ_s is quasi-closed. Let γ be such that $\mathbf{U}_\gamma \subset \mathbf{G}_{\Psi_s}$; since $\Psi_s = -\Psi_s$, we may assume $\gamma \in \Phi^+$, thus $\mathbf{U}_\gamma \subset \mathbf{B}$. As each term $\mathbf{U}_{\Psi_s^+} w\mathbf{T}\mathbf{U}_{\Psi_s^+}$ is in a unique Bruhat cell of $\mathbf{G}$, we must have $\mathbf{U}_\gamma \subset \mathbf{T}\mathbf{U}_{\Psi_s^+}$. By Lemma 3.3.14 Ψ_s^+ is quasi-closed, thus $\gamma \in \Psi_s^+$.

We have seen that $\mathbf{G}_\Psi$ has a semi-direct product decomposition $\mathbf{G}^*_{\Psi_u} \rtimes \mathbf{G}_{\Psi_s}$. It follows that Ψ is quasi-closed since if $\alpha \notin \Psi_u$ and $\mathbf{U}_\alpha \subset \mathbf{G}_\Psi$ then $\mathbf{U}_\alpha$ maps isomorphically to the quotient $\mathbf{G}_{\Psi_s}$ thus $\alpha \in \Psi_s$. □

Conversely any quasi-closed subset of Φ is closed apart from some exceptions in characteristics 2 and 3; see Borel and Tits (1965, 3.8). This can be shown by proving that in other characteristics the group $\langle \mathbf{U}_\alpha, \mathbf{U}_\beta \rangle$ contains all $\mathbf{U}_{n\alpha+m\beta}$ for $n, m \in \mathbb{N}$ such that $n\alpha + m\beta \in \Phi$, using the explicit values of the coefficients in the proof of 2.3.1(vii). For a combinatorial description of quasi-closed subsets in these characteristics, see Malle and Testerman (2011, Corollary 13.7).

3.4 Parabolic Subgroups and Levi Subgroups

Proposition 3.4.1 *The parabolic subgroup* $\mathbf{P}_I = \mathbf{B}W_I\mathbf{B}$ (*see 3.2.3*) *has a Levi decomposition* $\mathbf{P}_I = R_u(\mathbf{P}_I) \rtimes \mathbf{L}_I$ *where* $R_u(\mathbf{P}_I) = \mathbf{U}_{\Phi^+-\Phi_I}$ *and* $\mathbf{L}_I = \langle \mathbf{T}, \{\mathbf{U}_\alpha\}_{\alpha\in\Phi_I} \rangle$ *is reductive. We have* $\mathbf{P}_I = N_{\mathbf{G}}(R_u(\mathbf{P}_I))$.

Proof The set $\Psi = \Phi^+ \cup \Phi_I$ is quasi-closed since it is closed by Proposition 3.3.8. The proposition is then a consequence of 3.3.12 if we show that $\mathbf{P}_I = \mathbf{G}_\Psi$. We have $\mathbf{P}_I \supset \mathbf{G}_\Psi$ since $\mathbf{P}_I \supset \mathbf{U}_\alpha$ for $\alpha \in \Phi^+$ since $\mathbf{P}_I \supset \mathbf{B}$, and by 3.2.7 $\mathbf{P}_I$ contains all $\mathbf{U}_\alpha$ for $\alpha \in \Phi^-$ which change sign by some element of W_I, thus contains all $\mathbf{U}_\alpha$ for $\alpha \in \Phi_I^-$. Conversely $\mathbf{G}_\Psi$ contains $\mathbf{U}_\alpha$ and $\mathbf{U}_{-\alpha}$ for all $\alpha \in \Pi_I$, hence $\mathbf{G}_\Psi$ contains a representative of s_α in $\langle \mathbf{U}_\alpha, \mathbf{U}_{-\alpha} \rangle$ hence $\mathbf{G}_\Psi$ contains W_I, thus contains $\mathbf{P}_I$.

Finally, $N_{\mathbf{G}}(R_u(\mathbf{P}_I))$ contains $\mathbf{P}_I$ thus $\mathbf{B}$, thus is some parabolic subgroup $\mathbf{P}_J$. If $J \supsetneq I$ we have $R_u(\mathbf{P}_J) = \mathbf{U}_{\Phi^+-\Phi_J} \subsetneq R_u(\mathbf{P}_I)$ which contradicts that $\mathbf{P}_J$ normalises $R_u(\mathbf{P}_I)$ since $R_u(\mathbf{P}_J)$ is the largest normal connected unipotent subgroup of $\mathbf{P}_J$. □

We will say that $\mathbf{P}_I$ (resp. $\mathbf{L}_I$) is a **standard** parabolic subgroup (resp. Levi subgroup) of $\mathbf{G}$.

Proposition 3.4.2 *Let* $\mathbf{P}$ *be a parabolic subgroup of* $\mathbf{G}$ *containing* $\mathbf{T}$.

(i) *There is a unique Levi subgroup of* $\mathbf{P}$ *containing* $\mathbf{T}$.
(ii) *Two Levi subgroups of* $\mathbf{P}$ *are conjugate by a unique element of* $R_u(\mathbf{P})$.

Proof The existence of a Levi subgroup containing $\mathbf{T}$ results from 3.4.1, since $\mathbf{P}$ is conjugate to some $\mathbf{P}_I$ and all maximal tori of $\mathbf{P}_I$ are conjugate in $\mathbf{P}_I$. Conversely, we may assume $\mathbf{P} = \mathbf{P}_I$; any Levi subgroup of $\mathbf{P}_I$ containing $\mathbf{T}$ is a $\mathbf{G}_\Psi$ for some $\Psi \subset \Phi^+ \cup \Phi_I$ by Proposition 3.3.10. Since any $\mathbf{U}_\alpha$ where $\alpha \in \Phi^+ - \Phi_I$ is in $R_u(\mathbf{P}_I)$, we must have $\Psi \subset \Phi_I$, thus $\mathbf{L} \subset \mathbf{L}_I$, thus there must be equality.

Two Levi subgroups $\mathbf{L}, \mathbf{L}'$ of $\mathbf{P}$ are conjugate in $\mathbf{P}$, since by (i) an element which conjugates a maximal torus $\mathbf{T}$ of $\mathbf{L}$ into $\mathbf{L}'$ conjugates $\mathbf{L}$ to $\mathbf{L}'$. Modulo $\mathbf{L}$, we can choose the conjugating element u in $R_u(\mathbf{P})$. The unicity of u is equivalent to $R_u(\mathbf{P}) \cap N_{\mathbf{P}}(\mathbf{L}) = 1$. Assume $u \in R_u(\mathbf{P}) \cap N_{\mathbf{P}}(\mathbf{L})$; then for any $l \in \mathbf{L}$ we have $[u,l] \in R_u(\mathbf{P}) \cap \mathbf{L} = 1$, thus $u \in C_{\mathbf{P}}(\mathbf{L})$; but $C_{\mathbf{P}}(\mathbf{L}) \subset C_{\mathbf{G}}(\mathbf{T}) = \mathbf{T}$ thus $u = 1$. □

Proposition 3.4.3 *The* $\mathbf{G}$*-conjugacy classes of Levi subgroups of parabolic subgroups of* $\mathbf{G}$ *are in bijection with the* W*-orbits of subsets of* S*, which are themselves in bijection with the* W*-conjugacy classes of parabolic subgroups of* W*.*

Proof Since all parabolic subgroups are conjugate to a $\mathbf{P}_I$, we may assume that we consider a Levi subgroup of some $\mathbf{P}_I$. Since by 3.4.2 such a Levi subgroup is $\mathbf{G}$-conjugate to $\mathbf{L}_I$, the question becomes that of finding when $\mathbf{L}_J$ is $\mathbf{G}$-conjugate to $\mathbf{L}_I$ for two subsets I and J of S. If $\mathbf{L}_J = {}^g\mathbf{L}_I$ for some $g \in \mathbf{G}$, then, since ${}^{g^{-1}}\mathbf{T}$ and $\mathbf{T}$ are two maximal tori of $\mathbf{L}_I$, there exists $l \in \mathbf{L}_I$ such that ${}^{g^{-1}}\mathbf{T} = {}^l\mathbf{T}$ and $gl \in N_{\mathbf{G}}(\mathbf{T})$ also conjugates $\mathbf{L}_I$ to $\mathbf{L}_J$; so the $\mathbf{G}$-conjugacy classes of $\mathbf{L}_I$ are the same as the $W(\mathbf{T})$-conjugacy classes. Since $\mathbf{L}_I = \mathbf{G}_{\Phi_I}$, the element $w \in W$ conjugates $\mathbf{L}_I$ to $\mathbf{L}_J$ if and only if ${}^w\Phi_I = \Phi_J$. Since any two bases of Φ_I are conjugate by an element of W_I (see 2.2.6), we may assume that ${}^wI = J$ whence the first part of the statement. To see the second part it is enough to see that if $w \in N_W(W_I)$ then ${}^w\Phi_I = \Phi_I$. This results from Lemma 3.3.7. □

The proof above shows that $N_{\mathbf{G}}(\mathbf{L}_I)/\mathbf{L}_I$ is isomorphic to $N_W(W_I)/W_I$.

Proposition 3.4.4 *Let* $\mathbf{L}$ *be a Levi subgroup of a parabolic subgroup* $\mathbf{P}$*. Then* $R(\mathbf{P}) = R_u(\mathbf{P}) \rtimes R(\mathbf{L})$.

Proof The quotient $\mathbf{P}/(R(\mathbf{L})R_u(\mathbf{P}))$ is isomorphic to $\mathbf{L}/R(\mathbf{L})$, so is semisimple. So $R(\mathbf{P}) \subset R(\mathbf{L})R_u(\mathbf{P})$. But $R(\mathbf{L})R_u(\mathbf{P})$ is connected, solvable and normal in $\mathbf{P}$ as the inverse image of a normal subgroup of the quotient $\mathbf{P}/R_u(\mathbf{P}) \simeq \mathbf{L}$, whence the reverse inclusion. □

We will now characterise parabolic subgroups in terms of roots.

Proposition 3.4.5 *A closed subgroup* $\mathbf{P}$ *of* $\mathbf{G}$ *containing* $\mathbf{T}$ *is parabolic if and only if* $\mathbf{P} = \mathbf{G}_\Psi$ *for some parabolic subset* Ψ*.*

Proof We have seen that a parabolic subset Ψ is conjugate under W to $\Phi^+ \cup \Phi_I$; thus $\mathbf{G}_\Psi$ is conjugate under W to $\mathbf{P}_I$. Conversely, assume that $\mathbf{P}$ is a parabolic subgroup. It contains a Borel subgroup containing $\mathbf{T}$, thus up to conjugacy by W it contains $\mathbf{B}$ (see Proposition 2.3.3), thus is of the form $\mathbf{P}_I$. □

We now give an important property of Levi subgroups.

Proposition 3.4.6 *Let* $\mathbf{L}$ *be a Levi subgroup of a parabolic subgroup of* $\mathbf{G}$*; then* $\mathbf{L} = C_\mathbf{G}(Z(\mathbf{L})^0)$.

Proof We may assume that $\mathbf{L} = \mathbf{L}_I$. Then by 2.3.4(i) the group $Z(\mathbf{L})$ is the intersection of the kernels of the roots in Φ_I. The group $C_\mathbf{G}(Z(\mathbf{L})^0)$ is connected as it is the centraliser of a torus – $Z(\mathbf{L})^0$ is diagonalisable by 1.2.1(ii) and is a torus by 1.2.3(i) thus 1.3.3(iii) applies. It is normalised by $\mathbf{T}$ because it contains $\mathbf{T}$, hence by 2.3.1(iv) it is generated by $\mathbf{T}$ and the $\mathbf{U}_\alpha$ it contains. If $\mathbf{U}_\alpha \subset C_\mathbf{G}(Z(\mathbf{L})^0)$ then α is trivial on $(\bigcap_{\alpha \in \Phi_I} \operatorname{Ker} \alpha)^0$. This identity component has finite index in $\bigcap_{\alpha \in \Phi_I} \operatorname{Ker} \alpha$, hence some multiple $n\alpha$ of α is trivial on $\bigcap_{\alpha \in \Phi_I} \operatorname{Ker} \alpha$. With the notation of 1.2.12, this can be rewritten as $n\alpha \in (\langle \Phi_I \rangle^\perp_\mathbf{T})^\perp_{X(\mathbf{T})}$. But $(\langle \Phi_I \rangle^\perp_\mathbf{T})^\perp_{X(\mathbf{T})} / \langle \Phi_I \rangle$ is a torsion group (see 1.2.13). This implies that $\alpha \in \langle \Phi_I \rangle \otimes \mathbb{Q}$, which in turn yields $\alpha \in \Phi_I$ by the definition of Φ_I. This proves that $C_\mathbf{G}(Z(\mathbf{L})^0) \subset \mathbf{L}$. The reverse inclusion is obvious. □

The next proposition is a kind of converse.

Proposition 3.4.7 *For any torus* $\mathbf{S}$*, the group* $C_\mathbf{G}(\mathbf{S})$ *is a Levi subgroup of some parabolic subgroup of* $\mathbf{G}$.

Proof Let $\mathbf{T}$ be a maximal torus containing $\mathbf{S}$. As the group $C_\mathbf{G}(\mathbf{S})$ is connected by 1.3.3(iii) and contains $\mathbf{T}$, by 2.3.1(iv) we have $C_\mathbf{G}(\mathbf{S}) = \langle \mathbf{T}, \mathbf{U}_\alpha \mid \mathbf{U}_\alpha \subset C_\mathbf{G}(\mathbf{S}) \rangle$. As $\mathbf{S}$ acts by α on $\mathbf{U}_\alpha$ (see 2.3.1(i)), we have

$$\mathbf{U}_\alpha \subset C_\mathbf{G}(\mathbf{S}) \Leftrightarrow \alpha|_\mathbf{S} = 0,$$

where 0 is the trivial element of $X(\mathbf{S})$. Let us choose a total order on $X(\mathbf{S})$; that is, a structure of ordered $\mathbb{Z}$-module. As $X(\mathbf{S})$ is a quotient of $X(\mathbf{T})$ (see 1.2.4) there exists a total order on $X(\mathbf{T})$ compatible with the chosen order on $X(\mathbf{S})$; that is, such that for $x \in X(\mathbf{T})$ we have $x \geq 0 \Rightarrow x|_\mathbf{S} \geq 0$. This implies that the set $\Psi = \{\alpha \in \Phi \mid \alpha > 0 \text{ or } \alpha|_\mathbf{S} = 0\}$ is also equal to $\{\alpha \in \Phi \mid \alpha|_\mathbf{S} \geq 0\}$. This last definition implies that Ψ is closed, so (see 3.3.13 and 3.3.10) Ψ is also the set of α such that $\mathbf{U}_\alpha \subset \mathbf{G}_\Psi$. Since Ψ is parabolic, it follows then from 3.4.5 that $\mathbf{G}_\Psi$ is a parabolic subgroup, of which $C_\mathbf{G}(\mathbf{S})$ is a Levi complement. □

We now study the intersection of two parabolic subgroups. First note that by 3.1.4 the intersection of two parabolic subgroups always contains some maximal torus of $\mathbf{G}$.

Proposition 3.4.8 *Let* $\mathbf{P}$ *and* $\mathbf{P}'$ *be two parabolic subgroups of* $\mathbf{G}$ *and let* $\mathbf{L}$ *and* $\mathbf{L}'$ *be respective Levi subgroups of* $\mathbf{P}$ *and* $\mathbf{P}'$ *containing the same maximal torus* $\mathbf{T}$ *of* $\mathbf{G}$. *Let* $\mathbf{U} = R_u(\mathbf{P})$ *and* $\mathbf{U}' = R_u(\mathbf{P}')$. *Then*

(i) *The group* $(\mathbf{P} \cap \mathbf{P}').\mathbf{U}$ *is a parabolic subgroup of* $\mathbf{G}$ *which has the same intersection as* $\mathbf{P}'$ *with* $\mathbf{L}$, *and it has* $\mathbf{L} \cap \mathbf{L}'$ *as a Levi subgroup.*

(ii) *The group* $\mathbf{P} \cap \mathbf{P}'$ *is connected, as well as* $\mathbf{L} \cap \mathbf{L}'$ *and we have the Levi decomposition*

$$\mathbf{P} \cap \mathbf{P}' = ((\mathbf{L} \cap \mathbf{U}').(\mathbf{L}' \cap \mathbf{U}).(\mathbf{U} \cap \mathbf{U}')) \rtimes (\mathbf{L} \cap \mathbf{L}')$$

where the right-hand side is a direct product of varieties – the decomposition of an element of $\mathbf{P} \cap \mathbf{P}'$ *as a product of four terms is unique. On the right-hand side the last factor is a Levi subgroup of* $\mathbf{P} \cap \mathbf{P}'$ *and the first 3 factors form a decomposition of* $R_u(\mathbf{P} \cap \mathbf{P}')$.

Proof Let Φ be the roots of $\mathbf{G}$ with respect to $\mathbf{T}$ and define subsets $\Psi, \Psi' \subset \Phi$ by $\mathbf{P} = \mathbf{G}_\Psi$ and $\mathbf{P}' = \mathbf{G}_{\Psi'}$.

Let us show first that for any $\alpha \in \Phi$, either $\mathbf{U}_\alpha$ or $\mathbf{U}_{-\alpha}$ is in the group $(\mathbf{P} \cap \mathbf{P}') \cdot \mathbf{U}$ (it is a group since $\mathbf{P}$ normalises $\mathbf{U}$). By the remarks before 3.3.11 and by 3.3.12 we have $(\mathbf{P} \cap \mathbf{P}')^0 \cdot \mathbf{U} = \mathbf{G}_{\Psi \cap \Psi'} \cdot \mathbf{G}^*_{\Psi_u}$, with the notation of 3.3.12. If neither α nor $-\alpha$ is in Ψ_u, they are both in Ψ in which case since one of them is in Ψ', one of them is in $\Psi \cap \Psi'$. Hence $(\Psi \cap \Psi') \cup \Psi_u$ is a parabolic set; indeed, this set is closed as the sum of an element of Ψ and an element of Ψ_u which is a root is in Ψ_u – see the beginning of the proof of 3.3.13. Proposition 3.4.5 then shows that $(\mathbf{P} \cap \mathbf{P}')^0.\mathbf{U}$ is a parabolic subgroup of $\mathbf{G}$, equal to $\mathbf{G}_{(\Psi \cap \Psi') \cup \Psi_u}$. Then $(\mathbf{P} \cap \mathbf{P}')\mathbf{U}$, containing a parabolic subgroup is connected, hence equal to $(\mathbf{P} \cap \mathbf{P}')^0.\mathbf{U}$.

Now $(\mathbf{P} \cap \mathbf{P}').\mathbf{U} = (\mathbf{P} \cap \mathbf{P}').\mathbf{G}^*_{\Psi_u - \Psi'}$ since $\Psi' \cap \Psi_u \subset \Psi \cap \Psi'$. The set $\Psi_u - \Psi'$ is closed as the intersection of the closed subsets Ψ_u and the complement $-\Psi'_u$ of Ψ', hence the product $(\mathbf{P} \cap \mathbf{P}').\mathbf{G}^*_{\Psi_u - \Psi'}$ is a direct product of varieties as the intersection is a unipotent subgroup normalised by $\mathbf{T}$ containing no $\mathbf{U}_\alpha$, see 2.3.1(v). As the product is connected, each term is. Thus $\mathbf{P} \cap \mathbf{P}'$ is connected equal to $\mathbf{G}_{\Psi \cap \Psi'}$. The groups $(\mathbf{P} \cap \mathbf{P}').\mathbf{U}$ and $\mathbf{P} \cap \mathbf{P}'$ have both $(\mathbf{L} \cap \mathbf{L}')^0$ as a Levi subgroup since $((\Psi \cap \Psi') \cup \Psi_u)_s = (\Psi \cap \Psi')_s = \Psi_s \cap \Psi'_s$ – indeed if $\alpha \in \Psi_u$ then $-\alpha \notin \Psi$ thus $-\alpha \notin (\Psi \cap \Psi') \cup \Psi_u$.

The decomposition $\Psi \cap \Psi' = (\Psi_s \cap \Psi'_s) \coprod (\Psi_s \cap \Psi'_u) \coprod (\Psi_u \cap \Psi'_s) \coprod (\Psi_u \cap \Psi'_u)$ shows that $\mathbf{P} \cap \mathbf{P}' = \langle \mathbf{L} \cap \mathbf{L}', \mathbf{L} \cap \mathbf{U}', \mathbf{L}' \cap \mathbf{U}, \mathbf{U} \cap \mathbf{U}' \rangle$. Using that $\mathbf{U} \cap \mathbf{U}'$ is normal in $\mathbf{P} \cap \mathbf{P}'$, then that $\mathbf{L} \cap \mathbf{L}'$ normalises $\mathbf{L} \cap \mathbf{U}'$ and $\mathbf{L}' \cap \mathbf{U}$, we get

$$\mathbf{P} \cap \mathbf{P}' = (\mathbf{L} \cap \mathbf{L}').\langle \mathbf{L} \cap \mathbf{U}', \mathbf{L}' \cap \mathbf{U} \rangle.(\mathbf{U} \cap \mathbf{U}').$$

Further, the commutator of an element of $\mathbf{L} \cap \mathbf{U}'$ with an element of $\mathbf{L}' \cap \mathbf{U}$ is in $\mathbf{U} \cap \mathbf{U}'$, thus

$$\mathbf{P} \cap \mathbf{P}' = (\mathbf{L} \cap \mathbf{L}').(\mathbf{L} \cap \mathbf{U}').(\mathbf{L}' \cap \mathbf{U}).(\mathbf{U} \cap \mathbf{U}'),$$

Write now $x = lu'uv \in \mathbf{P} \cap \mathbf{P}'$, where $l \in \mathbf{L} \cap \mathbf{L}'$, $u' \in \mathbf{L} \cap \mathbf{U}'$, $u \in \mathbf{L}' \cap \mathbf{U}$, $v \in \mathbf{U} \cap \mathbf{U}'$. Then lu' is the image of x by the projection $\mathbf{P} \to \mathbf{L}$ and l (resp. u) is the image of lu' (resp. uv) by the morphism $\mathbf{P}' \to \mathbf{L}'$. Thus the decomposition of x is unique, and the product map $(\mathbf{L} \cap \mathbf{L}') \times (\mathbf{L} \cap \mathbf{U}') \times (\mathbf{L}' \cap \mathbf{U}) \times (\mathbf{U} \cap \mathbf{U}') \to \mathbf{P} \cap \mathbf{P}'$ is an isomorphism of varieties; the four terms are connected since the product is. In particular $\mathbf{L} \cap \mathbf{L}'$ is connected. □

Proposition 3.4.9

(i) *Let* $\mathbf{P}$ *and* $\mathbf{P}'$ *be two parabolic subgroups of* $\mathbf{G}$ *such that* $\mathbf{P}' \subset \mathbf{P}$*, then* $R_u(\mathbf{P}') \supset R_u(\mathbf{P})$ *and for any Levi subgroup* $\mathbf{L}'$ *of* $\mathbf{P}'$*, there exists a unique Levi subgroup* $\mathbf{L}$ *of* $\mathbf{P}$ *such that* $\mathbf{L} \supset \mathbf{L}'$.

(ii) *Let* $\mathbf{L}$ *be a Levi subgroup of a parabolic subgroup* $\mathbf{P}$ *of* $\mathbf{G}$ *and* $\mathbf{L}'$ *be a closed subgroup of* $\mathbf{L}$*. Then the following are equivalent:*

   (a) $\mathbf{L}'$ *is a Levi subgroup of a parabolic subgroup of* $\mathbf{L}$.

   (b) $\mathbf{L}'$ *is a Levi subgroup of a parabolic subgroup of* $\mathbf{G}$.

Proof Let us prove (i); given a maximal torus $\mathbf{T}$ of $\mathbf{L}'$ there is by 3.4.2(i) a unique Levi subgroup $\mathbf{L}$ of $\mathbf{P}$ containing $\mathbf{T}$. Then by 3.4.8(ii) the group $\mathbf{L}' \cap \mathbf{L}$ is a Levi subgroup of $\mathbf{P}' = \mathbf{P} \cap \mathbf{P}'$ thus $\mathbf{L} \cap \mathbf{L}' = \mathbf{L}'$. Also $R_u(\mathbf{P})$ is contained in all Borel subgroups of $\mathbf{P}$, thus in $\mathbf{P}'$, whence $R_u(\mathbf{P}) \subset R_u(\mathbf{P}')$.

Let us show (ii); if $\mathbf{L}'$ is a Levi subgroup of the parabolic subgroup $\mathbf{P}_\mathbf{L}$ of $\mathbf{L}$, then $\mathbf{P}_\mathbf{L} R_u(\mathbf{P})$ is a parabolic subgroup of $\mathbf{G}$; indeed it is a group since $\mathbf{L}$, thus $\mathbf{P}_\mathbf{L}$, normalises $R_u(\mathbf{P})$ and it clearly contains either $\mathbf{U}_\alpha$ or $\mathbf{U}_{-\alpha}$ for any $\alpha \in \Phi$. Thus $\mathbf{L}'$ is a Levi subgroup of $\mathbf{P}_\mathbf{L} R_u(\mathbf{P})$, since $R_u(\mathbf{P}_\mathbf{L}).R_u(\mathbf{P})$ is unipotent normal in $\mathbf{P}_\mathbf{L}.R_u(\mathbf{P})$. We have shown that (a) implies (b).

Conversely, let $\mathbf{P}'$ be a parabolic subgroup of $\mathbf{G}$ with $\mathbf{L}'$ as a Levi subgroup. By 3.4.8(ii) we have $\mathbf{P} \cap \mathbf{P}' = \mathbf{L}'.(\mathbf{L} \cap \mathbf{U}').(\mathbf{U} \cap \mathbf{U}')$ thus $(\mathbf{L} \cap \mathbf{U}') \rtimes \mathbf{L}'$ is a Levi decomposition of $\mathbf{L} \cap \mathbf{P}'$, and this last group is a parabolic subgroup of $\mathbf{L}$ by 3.4.5. □

When $\mathbf{L}$ is a Levi subgroup of some parabolic subgroup of $\mathbf{G}$ we will say (improperly) "$\mathbf{L}$ is a **Levi subgroup of G**" which is justified by statement (ii) of 3.4.9.

Proposition 3.4.10 *Let* $\mathbf{H}$ *be a closed connected reductive subgroup of* $\mathbf{G}$ *of maximal rank. Then:*

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(i) *The Borel subgroups of* $\mathbf{H}$ *are the* $\mathbf{B} \cap \mathbf{H}$ *where* $\mathbf{B}$ *is a Borel subgroup of* $\mathbf{G}$ *containing a maximal torus of* $\mathbf{H}$.

(ii) *The parabolic subgroups of* $\mathbf{H}$ *are the* $\mathbf{P} \cap \mathbf{H}$, *where* $\mathbf{P}$ *is a parabolic subgroup of* $\mathbf{G}$ *containing a maximal torus of* $\mathbf{H}$.

(iii) *If* $\mathbf{P}$ *is a parabolic subgroup of* $\mathbf{G}$ *containing a maximal torus of* $\mathbf{H}$, *the Levi subgroups of* $\mathbf{P} \cap \mathbf{H}$ *are the* $\mathbf{L} \cap \mathbf{H}$ *where* $\mathbf{L}$ *is a Levi subgroup of* $\mathbf{P}$ *containing a maximal torus of* $\mathbf{H}$.

Proof Let $\mathbf{T}$ be a maximal torus of $\mathbf{H}$; by assumption, it is also a maximal torus of $\mathbf{G}$. Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$, and let $\mathbf{B} = \mathbf{U}.\mathbf{T}$ be the corresponding semi-direct product decomposition. The Borel subgroup $\mathbf{B}$ defines an order on the root system Φ (resp. $\Phi_{\mathbf{H}}$) of $\mathbf{G}$ (resp. $\mathbf{H}$) with respect to $\mathbf{T}$. The group $\mathbf{U} \cap \mathbf{H}$ is normalised by $\mathbf{T}$, so is connected and equal to the product of the $\mathbf{U}_\alpha$ it contains, that is those $\mathbf{U}_\alpha$ such that α is positive and in $\Phi_{\mathbf{H}}$, so $(\mathbf{U} \cap \mathbf{H}).\mathbf{T} = \mathbf{B} \cap \mathbf{H}$ is a Borel subgroup of $\mathbf{H}$. This gives (i) since all Borel subgroups of $\mathbf{H}$ are conjugate under $\mathbf{H}$.

Let us prove (ii). If $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$ containing $\mathbf{T}$, it contains a Borel subgroup $\mathbf{B}$ containing $\mathbf{T}$, so its intersection with $\mathbf{H}$ contains the Borel subgroup $\mathbf{B} \cap \mathbf{H}$ of $\mathbf{H}$ and thus is a parabolic subgroup. Conversely, let $\mathbf{Q}$ be a parabolic subgroup of $\mathbf{H}$ containing $\mathbf{T}$, and let x be a vector of $X(\mathbf{T}) \otimes \mathbb{Q}$ defining $\mathbf{Q}$ as in 3.3.8(ii). Then x defines a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. It remains to show that $\mathbf{P} \cap \mathbf{H} = \mathbf{Q}$. The group $\mathbf{P} \cap \mathbf{H}$ is a parabolic subgroup of $\mathbf{H}$ by the first part. It is generated by $\mathbf{T}$ and the $\mathbf{U}_\alpha$ it contains. But $\mathbf{U}_\alpha \subset \mathbf{P} \cap \mathbf{H}$ if and only if $\alpha \in \Phi_{\mathbf{H}}$ and $\langle \alpha, x \rangle \geq 0$; that is, if and only if $\mathbf{U}_\alpha \subset \mathbf{Q}$ by definition of x.

Similarly, the Levi subgroup of $\mathbf{Q}$ containing $\mathbf{T}$ is the intersection of the Levi subgroup of $\mathbf{P}$ containing $\mathbf{T}$ with $\mathbf{H}$, as it is generated by $\mathbf{T}$ and the $\mathbf{U}_\alpha$ with $\alpha \in \Phi_{\mathbf{H}}$ orthogonal to x, whence (iii). □

3.5 Centralisers of Semi-Simple Elements

Proposition 3.5.1 *Let* $s \in \mathbf{G}$ *be a semi-simple element, and let* $\mathbf{T}$ *be a maximal torus containing* s *and* Φ *be the set of roots of* $\mathbf{G}$ *relative to* $\mathbf{T}$; *then*

(i) *The identity component* $C_{\mathbf{G}}(s)^0$ *is generated by* $\mathbf{T}$ *and the* $\mathbf{U}_\alpha$ *for* $\alpha \in \Phi$ *such that* $\alpha(s) = 1$. *It is a connected reductive subgroup of* $\mathbf{G}$ *of maximal rank.*

(ii) $C_{\mathbf{G}}(s)$ *is generated by* $C_{\mathbf{G}}(s)^0$ *and the elements* $n \in N_{\mathbf{G}}(\mathbf{T})$ *such that* ${}^n s = s$.

Proof (i) is an immediate consequence of 2.3.1(iv), and of the fact that the corresponding set of α is closed and symmetric – see 3.3.12 and 3.3.13.

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Let us prove (ii). Let $\mathbf{B} = \mathbf{U} \rtimes \mathbf{T}$ be the Levi decomposition of a Borel subgroup of $\mathbf{G}$ and let $g \in C_{\mathbf{G}}(s)$; by Lemma 3.2.7 the element g has a unique decomposition $g = unv$ with $n \in N_{\mathbf{G}}(\mathbf{T})$, $u \in \mathbf{U}$ and $v \in \mathbf{U}_w$ where w is the image of n in $W(\mathbf{T})$. As s normalises $\mathbf{U}$, $\mathbf{U}_w$ and $N_{\mathbf{G}}(\mathbf{T})$, this decomposition is invariant under conjugation by s, so each of u, n and v also centralises s. Writing again a unique decomposition of the form $u = \prod_{\alpha>0} u_\alpha$ we see that the α must satisfy $\alpha(s) = 1$ so $u \in C_{\mathbf{G}}(s)^0$, and the same argument applies to v. Thus we get (ii). □

Remark 3.5.2 The Weyl group $W^0(s)$ of $C_{\mathbf{G}}(s)^0$ is thus the group generated by the reflections s_α for which $\alpha(s) = 1$. It is a normal subgroup of the Weyl group of $C_{\mathbf{G}}(s)$ which is $W(s) = \{w \in W(\mathbf{T}) \mid {}^w s = s\}$. The quotient $W(s)/W^0(s)$ is isomorphic to the quotient $C_{\mathbf{G}}(s)/C_{\mathbf{G}}(s)^0$.

Proposition 3.5.3 *If $x = su$ is the Jordan decomposition of an element of* $\mathbf{G}$*, where s is semi-simple and u unipotent, then $x \in C_{\mathbf{G}}(s)^0$.*

Proof Let $\mathbf{B} = \mathbf{U} \rtimes \mathbf{T}$ be a Levi decomposition of a Borel subgroup containing x, where $\mathbf{T}$ is a maximal torus of $\mathbf{B}$ containing s, and write $u = \prod u_\alpha$ (with $u_\alpha \in \mathbf{U}_\alpha$ where $\mathbf{U} = \prod \mathbf{U}_\alpha$). Then for any root α such that $u_\alpha \neq 1$, we have $\alpha(s) = 1$ which implies that $\mathbf{U}_\alpha \subset C_{\mathbf{G}}(s)^0$, whence the result as $s \in \mathbf{T} \subset C_{\mathbf{G}}(s)^0$. □

Examples 3.5.4

(i) All centralisers in $\mathbf{GL}_n$ are connected. Indeed, the centraliser in the variety of all matrices is an affine space, thus its intersection with $\mathbf{GL}_n$ is an open subspace of an affine space, which is always connected.

(ii) In the group $\mathbf{SL}_n$, centralisers of semi-simple elements are connected. Indeed such an element is conjugate to an element $s = \mathrm{diag}(t_1, \ldots, t_n)$ of the torus $\mathbf{T}$ of diagonal matrices where we may assume, in addition, that equal t_i are grouped in consecutive blocks, thereby defining a partition π of n. The elements of $W(\mathbf{T})$ (permutation matrices) which centralise s are products of generating reflections s_α which centralise s; that is, $W(s) = W^0(s)$ showing that the centraliser of s is connected.

(iii) We finish with an example of a semi-simple element whose centraliser is not connected. Let $s \in \mathbf{PGL}_2$ be the image of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; in characteristic different from 2, $C_{\mathbf{PGL}_2}(s)$ has two connected components, consisting respectively of the images of the matrices of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and of the form $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$.

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Notes

A classic reference about (B,N)-pairs is Bourbaki (1968, Chapter IV). A detailed study of closed and quasi-closed subsets and reductive and parabolic subgroups is in Borel and Tits (1965). A detailed study of the centralisers of semi-simple elements can be found in, for example, Deriziotis (1984).