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ASPHERICAL RELATIVE PRESENTATIONS

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A geometric hypothesis is presented under which the cohomology of a group G given by generators and defining relators can be computed in terms of a group H defined by a subpresentation. In the presence of this hypothesis, which is framed in terms of spherical pictures, one has that H is naturally embedded in G, and that the finite subgroups of G are determined by those of H. Practical criteria for the hypothesis to hold are given. The theory is applied to give simple proofs of results of Collins–Perraud and of Kanevskiĭ. In addition, we consider in detail the situation where G is obtained from H by adjoining a single new generator x and a single defining relator of the form $xaxbx^{\varepsilon}c$, where $a, b, c \in H$ and $|\varepsilon| = 1$.

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Introduction

There is a considerable amount of literature concerning the following situation (see, for example, **[2, 11]** and the references cited there). Let H be a group and construct a new group G as follows: adjoin a set of generators $\mathbf{x}$ to H; then factor the resulting free product $H * \langle \mathbf{x} \rangle$ by the normal closure N of a set $\mathbf{r}$ of cyclically reduced elements of $H * \langle \mathbf{x} \rangle - H$. We will say that G is *defined by the relative presentation* $\mathbf{P} = \langle H, \mathbf{x}; \mathbf{r} \rangle$.

In this paper we will consider relative presentations which are *orientable* and *aspherical*. The term "orientable" simply means that no element of $\mathbf{r}$ is a cyclic permutation of its inverse. The definition of "aspherical" is more complicated, and is expressed in terms of certain geometric objects called (spherical) *pictures*. These geometric objects are well-known for ordinary presentation (see, for example, **[3, 4, 12, 15, 23]**) but do not appear to have been used for relative presentations. The definition of asphericity requires that every non-empty spherical picture over the relative presentation contains a certain configuration called a *dipole*.

Now it turns out that if the relative presentation $\mathbf{P}$ is orientable and aspherical, then a considerable amount of group-theoretic information about the group G defined by $\mathbf{P}$ can be deduced. To be specific, the following results hold.

(0.1). *The natural homomorphisn $H \to G$ is injective (and so we can regard H as a subgroup of G).*

For $R \in \mathbf{r}$, write $R = \mathring{R}^{p(R)}$ where $\mathring{R}$ is not a proper power, and $p(R)$ is a positive integer. Let C_R be the subgroup of G generated by $\mathring{R}N$. Obviously C_R has order dividing $p(R)$.

(0.2). *C_R has order exactly $p(R)$ for all $R \in \mathbf{r}$.*

(0.3). *The homology and cohomology of G in dimensions $\geqq 3$ is determined by that of H and the subgroups $C_R\,(R \in \mathbf{r})$. To be specific, there are isomorphisms*

$$H_n(G,-) \cong H_n(H,-) \oplus \left(\bigoplus_{R \in \mathbf{r}} H_n(C_R,-) \right)$$

$$H^n(G,-) \cong H^n(H,-) \oplus \left(\prod_{R \in \mathbf{r}} H^n(C_R,-) \right)$$

for all $n \geqq 3$.

(0.4). *Any finite subgroup of G is contained in a conjugate of H or a conjugate of one of the subgroups $C_R(R \in \mathbf{r})$.*

The paper is divided into four sections, each of which is further subdivided. In the first section we discuss the general theory of aspherical relative presentations. In the second section we prove some results giving sufficient conditions for a relative presentation to be aspherical. In the third section we describe some applications of the results of the first two sections. In the fourth section we give an independent topological treatment of the theory of aspherical orientable relative presentations.

It is worth mentioning here, by way of illustration, one of our applications in Section 3 (Theorem 3.1, §3.1). (This theorem is one of the major results of our paper.)

Let $\mathbf{P} = \langle H, x;\, xa_1xa_2xa_3 \rangle$ *where a_1, a_2, a_3 are elements of H, not all the same. Then* $\mathbf{P}$ *is aspherical if and only if neither of the following conditions holds:*

(i) *For $i = 1, 2, 3$, $a_i a_{i+1}^{-1}$ has finite order p_i (subscripts* mod 3*), and*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} > 1.$$

(ii) *There exist $j \in \{1,2,3\}$, $p > 2$ and $0 \leqq k < p$ such that $sgp\{a_i a_{i+1}^{-1} : i = 1,2,3\}$ is finite cyclic with generator $a_j a_{j+1}^{-1}$ of order p, and $a_{j+1} a_{j+2}^{-1} = (a_j a_{j+1}^{-1})^k$ where either $k = 1$, or $p = k+2$, or $p = 2k+1$, or $p = 6$ and $k = 2, 3$.*

The headings of the various sections and subsections of the paper, along with some relevant comments, are given below.

1. *Asphericity*

1.1. Relative presentations

We introduce several definitions concerning relative presentations, and we fix our notation.

1.2. Pictures over relative presentations

As already mentioned, asphericity is defined in terms of pictures, and so the concept of picture is central to the rest of the paper.

1.3. Pictures over ordinary presentations

An ordinary presentation can be regarded as a special type of relative presentation. Specializing the material of § 1.2 gives the usual concept of "picture".

1.4. Dipoles and asphericity

This subsection contains the definition of a dipole, and the definition of an aspherical relative presentation.

1.5. Sequences, pictures and relation modules

The material in this subsection concerns ordinary presentations. Given an ordinary presentation **R**, we can define the concept of an *identity sequence over* **R**. Each identity sequence gives rise to a spherical picture over **R**, and conversely, every spherical picture gives rise to an identity sequence (unique up to a certain equivalence).

We also have the concept of the *relation module M* corresponding to the presentation **R**. There is a "natural" generating set of M. If we take a free module Φ with basis in one to one correspondence with this "natural" generating set, we get an epimorphism from Φ onto M, with kernel K say. Thus we have a presentation of M:

$$0 \rightarrow K \rightarrow \Phi \rightarrow M \rightarrow 0.$$

The computation of K relies on an analysis of identity sequences, or, what amounts to the same thing, an analysis of spherical pictures.

Most of the material in this subsection is well-known. It can be found, for example, in [3]. However, our treatment is slightly different, and is tailored to suit our purposes.

1.6. Lifting relative presentations

Given a relative presentation **P** there is an ordinary presentation $\tilde{\mathbf{P}}$ defining the same group. The interplay between pictures over **P** and pictures over $\tilde{\mathbf{P}}$ is discussed.

1.7. The main results concerning orientable aspherical relative presentations

If **P** is an orientable aspherical relative presentation then, using the results of the previous two subsections, we investigate identity sequences over the associated ordinary presentation $\tilde{\mathbf{P}}$. From this we deduce the results described at the start of this Introduction.

2. *Tests for asphericity*

2.1. Star-complexes

Star-complexes (or *star-graphs*, or *co-initial graphs*) of ordinary presentations are well-known, and have been used in several different contexts (see [6] and the references cited there). We define what we mean by the *star-complex of a relative presentation.*

2.2. Weight test

We give a "weight test" for asphericity. This amounts to trying to assign numbers ("weights") to the edges of the star-complex in such a way that certain conditions are satisfied. Our weight test generalizes work of Sieradski [25], Gersten [7] and Pride [20] for ordinary presentations. We remark that a weight test similar to ours has been introduced independently by Gersten [8] in unpublished work on equations over torsion-free groups.

2.3. Small cancellation conditions

We introduce small cancellation conditions $C(p), T(q)$ for relative presentations in a slightly non-standard way. Part of the definition makes use of the star-complex. A relative presentation satisfying $C(p)$, $T(q)$ with $1/p+1/q=1/2$ is aspherical.

2.4. New aspherical presentations from old

If we have an aspherical relative presentation $\langle H, \mathbf{x}; \mathbf{r}\rangle$, and if we replace each $R \in \mathbf{r}$ by a power $R^{n(R)}$ ($n(R)$ a positive integer) then the resulting relative presentation is aspherical.

We also obtain a "change of variables" theorem.

3. *Applications*

3.1. Some relative presentations with one defining relator (1)

3.2. Some relative presentations with one defining relator (2)

In these two subsections we discuss the asphericity of relative presentations of the forms $\langle H, x;\ xaxbxc\rangle$, $\langle H, x;\ xaxbx^{-1}c\rangle$ $(a,b,c \in H)$.

3.3. Quotients of free products ("generalized presentations")

A *generalized presentation* (or *quotient of a free product*) is an object $\langle H_i (i \in I);\ \mathbf{u}\rangle$, where the H_i are non-trivial groups, and $\mathbf{u}$ is a set of cyclically reduced elements of $* H_i$ of free product length at least 2. We can associate with $\mathbf{P}$ a relative presentation $\mathbf{P}_{aug}$. If $\mathbf{P}_{aug}$ is aspherical (and if no element of $\mathbf{u}$ is a cyclic permutation of its inverse) then information similar to (0.1)–(0.4) can be obtained concerning the group G defined by $\mathbf{P}$. For example, there are natural homomorphisms $H_i \to G (i \in I)$ which are injections. Also, the homology and cohomology of G in dimensions $\geqq 3$ can be determined.

3.4. Small cancellation quotients of free products (Theorem of Collins and Perraud)

The theorem of Collins and Perraud **[5]** is an almost immediate consequence of the results of §§ 2.3, 3.3.

We give some examples of small cancellation quotients of free products (including groups considered by Kanevskiĭ in **[16]**), to illustrate the $C(6)$ and $C(4)$, $T(4)$ cases. We also show that any $T(6)$ quotient of a free product actually satisfies $C(6)$.

3.5. Weight test for quotients of free products

3.6. LOG-presentations

These (ordinary) presentation have been discussed in **[1, 10]**. In **[1]**, the idea of a *shelling* of an LOG-presentation $\mathbf{L}$ relative to a certain core $\mathbf{K}$ was discussed. This gives rise to a relative presentation $\mathbf{L}//\mathbf{K}$. The proof of the main result of **[1]** demonstrates that $\mathbf{L}//\mathbf{K}$ is aspherical.

4. *Topological aspects*

We show how to construct a $K(G,1)$-space, where G is the group presented by an aspherical orientable relative presentation.

1. Asphericity

1.1. *Relative presentations*

A *relative presentation* $\mathbf{P}$ is a triple

$$\langle H, \mathbf{x}; \mathbf{r} \rangle \tag{1.1}$$

where H is a group, $\mathbf{x}$ is a set, and $\mathbf{r}$ is a set of words in the alphabet $H \cup \mathbf{x} \cup \mathbf{x}^{-1}$. Each element of $\mathbf{r}$ is assumed to be written in the form

$$x_1^{\varepsilon_1} h_1 x_2^{\varepsilon_2} h_2 \ldots x_n^{\varepsilon_n} h_n \tag{1.2}$$

where $x_i \in \mathbf{x}$, $\varepsilon_i = \pm 1$, and $h_i \in H$, and is assumed to be *cyclically reduced* in the sense that if $h_i = 1$ and $x_i = x_{i+1}$ (subscripts mod n), then $\varepsilon_i = \varepsilon_{i+1}$. The elements of $\mathbf{x} \cup \mathbf{x}^{-1}$ will be referred to as $\mathbf{x}$-*symbols*. The elements of H will sometimes be referred to as *coefficients*.

The words in $\mathbf{r}$ represent elements of the free product $H * \langle \mathbf{x} \rangle$. The *group* $G(\mathbf{P})$ *defined by* $\mathbf{P}$ is the quotient of $H * \langle \mathbf{x} \rangle$ by the normal closure of $\mathbf{r}$.

If s is a subset of $\mathbf{r}$ then we denote by $\mathbf{s}^*$ the set of all cyclic permutations of elements of $\mathbf{s} \cup \mathbf{s}^{-1}$ of the form (1.2), that is, all cyclic permuations which begin with an $\mathbf{x}$-symbol.

We define an operator $^-$ on $\mathbf{r}^*$ as follows. For $R \in \mathbf{r}$ write $R = Sh$ where $h \in H$ and S begins and ends with $\mathbf{x}$-symbols. We set

$$\bar{R} = S^{-1} h^{-1}.$$

Note that $\bar{\bar{R}} = R$, and that $\bar{R} \in \mathbf{r}^*$. It is not difficult to show that R is fixed by $^-$ if and only if R has the form

$$X h_1 X^{-1} h_2 \tag{1.3}$$

where X begins and ends with $\mathbf{x}$-symbols, and h_1, h_2 are elements of H each of order 2.

If R is an element of $\mathbf{r}^*$ then R can be written in the form $\mathring{R}^{p(R)}$ where $\mathring{R}$ is not a proper power, and $p(R)$ is a positive integer. We call $\mathring{R}$ the *root* of R, and $p(R)$ the *period*.

We will say that $\mathbf{P}$ is *slender* if for each $R \in \mathbf{r}$, $\{R\}^* \cap \mathbf{r} = \{R\}$. We will say that $\mathbf{P}$ is *orientable* if it is slender and if no element of $\mathbf{r}$ is a cyclic permutation of its inverse. We remark that an element $R \in \mathbf{r}$ is a cyclic permutation of its inverse if and only if some cyclic permuation of R has the form (1.3) The (easy) verification of this is left to the reader.

1.2. *Pictures over relative presentations*

A (*generic*) *picture* $\mathbb{P}$ is a finite collection of pairwise disjoint discs $\{\Delta_1, \ldots, \Delta_m\}$ in the interior of a disc D^2, together with a finite collection of pairwise disjoint simple arcs $\{\alpha_1, \ldots, \alpha_n\}$ properly embedded in the closure of $D^2 - \bigcup\{\Delta_1, \ldots, \Delta_m\}$. By the *discs of* $\mathbb{P}$ we mean the discs $\{\Delta_1, \ldots, \Delta_m\}$ and not the ambient disc D^2. The boundary of $\mathbb{P}$ is the circle ∂D^2, denoted $\partial\mathbb{P}$. For $j \in \{1, \ldots, m\}$, the *corners of* Δ_j are closures of the connected components of $\partial\Delta_j - \bigcup\{\alpha_1, \ldots, \alpha_n\}$, where $\partial\Delta_j$ is the boundary of Δ_j. The *regions of* $\mathbb{P}$ are the closures of the connected components of $D^2 - (\bigcup\{\Delta_1, \ldots, \Delta_m\} \cup \bigcup\{\alpha_1, \ldots, \alpha_n\})$. An *inner region of* $\mathbb{P}$ is a simply connected region of $\mathbb{P}$ that does not meet $\partial\mathbb{P}$. The picture $\mathbb{P}$ is *non-trivial* if $m \geqq 1$, is *connected* if $\bigcup\{\Delta_1,\ldots,\Delta_m\} \cup \bigcup\{\alpha_1,\ldots,\alpha_n\}$ is connected, and is *spherical* if it is non-trivial and $(\bigcup\{\alpha_1,\ldots,\alpha_n\}) \cap \partial\mathbb{P} = \emptyset$.

Fix a relative presentation $\mathbf{P} = \langle H, \mathbf{x};\ \mathbf{r}\rangle$, and suppose that the picture $\mathbb{P}$ is *labelled*, in the following sense. Each arc is to be equipped with a normal orientation, indicated by a short arrow meeting the arc transversely, and labelled by an element of $\mathbf{x} \cup \mathbf{x}^{-1}$. Also, each corner of $\mathbb{P}$ is to be oriented anticlockwise (with respect to the ambient disc of $\mathbb{P}$) and labelled by an element of H.

If c is a corner of a disc Δ of the labelled picture $\mathbb{P}$, then we denote by $W(c)$ the word obtained by reading in anticlockwise order the labels on the arcs and corners meeting $\partial\Delta$ beginning with the label on the arc at the head (terminal point) of the anticlockwise oriented corner c. Here the understanding is that if we cross an arc labelled t in the direction of its normal orientation we read t, whereas if we cross in the opposite direction we read t^{-1}.

We say that the labelled picture $\mathbb{P}$ is a *picture over* $\mathbf{P}$ if the following two conditions are satisfied.

(1.4). For each corner c of $\mathbb{P}$, $W(c) \in \mathbf{r}^*$.

(1.5) If $h_1, h_2, \ldots, h_m$ is the sequence of corner labels encountered in a clockwise traversal of the boundary of an inner region of $\mathbb{P}$, then $h_1 h_2 \ldots h_m = 1$ in H.

A *based picture over* $\mathbf{P}$ is a picture over $\mathbf{P}$ with the following additional features.

(1.6) If Δ is a disc of the picture then there are distinguished points (basepoints) $0_1,\ldots,0_p$ in the interior of certain corners $c_1,\ldots,c_p$ of Δ. The words $W(c_1)$, $W(c_2),\ldots,W(c_p)$ are all equal to some element R of $\mathbf{r} \cup \bar{\mathbf{r}}$. Moreover, p is the period of R. We call R the *label on* Δ, and denote it by $W(\Delta)$.

(1.7) There is a distinguished point (basepoint) 0 on $\partial\mathbb{P}$ not lying on any arc of $\mathbb{P}$. If we travel around $\partial\mathbb{P}$ anticlockwise from 0 we will encounter a succession of arcs. Reading off the labels on these arcs will give a word $W(\mathbb{P})$, which we call the *label on* $\mathbb{P}$. (Note that $W(\mathbb{P})$ involves only $\mathbf{x}$-symbols.)

If $\mathbf{P}$ is a picture over $\mathbf{P}$ and if $\mathbf{x}_0$ is a subset of $\mathbf{x}$, then those arcs labelled by elements of $\mathbf{x}_0 \cup \mathbf{x}_0^{-1}$ will be called $\mathbf{x}_0$-arcs. Similarly, if $\mathbf{r}_0$ is a subset of $\mathbf{r}$, then those discs such that $W(c) \in \mathbf{r}_0^*$ (c a corner of the disc) will be called $\mathbf{r}_0$-*discs*.

1.3. *Pictures over ordinary presentations*

An ordinary presentation can be regarded as a relative presentation with $H = \{1\}$. More precisely, if we have an ordinary presentation $\mathbf{Q} = \langle \mathbf{x};\ \mathbf{r}\rangle$ ($\mathbf{r}$ a set of cyclically reduced words on $\mathbf{x}$) then we can think of it as the relative presentation $\mathbf{Q}_1 = \langle\{1\}, \mathbf{x};\ \mathbf{r}_1\rangle$ where $\mathbf{r}_1$ is the set of words R_1 obtained from words $R \in \mathbf{r}$ by inserting a 1 after each $\mathbf{x}$-symbol. Notice that in a picture over $\mathbf{Q}_1$ every corner is labelled by 1, and thus condition (1.5) is always satisfied. The labels on the corners can thus all be ignored, and we end up with the standard notion of a *picture over* $\mathbf{Q}$ ([**3, 4, 12, 15, 23**]). In general we will blur the distinction between $\mathbf{Q}$ and $\mathbf{Q}_1$, and between pictures over $\mathbf{Q}$ and pictures over $\mathbf{Q}_1$.

The following result is well-known (see [**3**, p. 190], or use Theorem V.1.1 and Lemma V.1.2 of [**19**] and dualise).

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Lemma 1.1. *A word W on* $\mathbf{x}$ *represents the identity of the group defined by* $\mathbf{Q}$ *if and only if there is a based picture over* $\mathbf{Q}$ *with boundary label W.*

1.4. *Dipoles and asphericity*

Let $\mathbf{P}$ be a relative presentation.

A *dipole* in a picture over $\mathbf{P}$ consists of a pair of corners c, c' of the picture together with an arc α joining the head of one corner with the tail of the other such that the following conditions hold:

(i) c and c' lie in the same region of the picture;

(ii) $W(c') = \overline{W(c)}$.

If $\mathbf{r}_0$ is a subset of $\mathbf{r}$ then the dipole is called an $\mathbf{r}_0$-*dipole* if $W(c) \in \mathbf{r}_0^*$. The discs on which the corners c, c' of the dipole lie are called the *discs of the dipole*. An important observation is that *if* $\mathbf{P}$ *is orientable then these discs are distinct.* (For otherwise, some cyclic permutation of an element of $\mathbf{r}^*$ would be equal to its image under $^-$, by (ii) above.)

A picture over $\mathbf{P}$ is *reduced* if it does not contain a dipole.

Our primary conceptual notion is the following.

Definition. A relative presentation $\mathbf{P}$ is *aspherical* if every connected spherical picture over $\mathbf{P}$ contains a dipole (that is, fails to be reduced).

Remark. It is technically convenient to define asphericity in terms of connected spherical pictures. Note however, that if $\mathbf{P}$ is aspherical then *every* spherical picture over $\mathbf{P}$ contains a dipole (consider a suitable connected spherical subpicture). This remark will be used often.

1.5. *Sequences, pictures and relation modules*

Consider an ordinary presentation

$$\mathbf{R} = \langle \mathbf{x};\, \mathbf{t} \rangle$$

where the elements of $\mathbf{t}$ are cyclically reduced. The group G defined by $\mathbf{R}$ is then (isomorphic to) F/N, where F is the free group on $\mathbf{x}$ and N is the normal closure of $\mathbf{t}$ in F.

We let $\mathbf{w}$ denote the set of all words (reduced or not) on $\mathbf{x}$. If $\mathbf{s}$ is a subset of $\mathbf{t}$ then we let $\mathbf{s}^{\mathbf{w}}$ denote the subset of $\mathbf{w}$ consisting of all words of the form

$$WS^{\varepsilon}W^{-1}\, (W \in \mathbf{w}, S \in \mathbf{s}, \varepsilon = \pm 1).$$

Two elements $W_1 T_1^{\varepsilon_1} W_1^{-1}$, $W_2 T_2^{\varepsilon_2} W_2^{-1}$ of $\mathbf{t}^{\mathbf{w}}$ will be said to be $\mathbf{R}$-*equivalent* if $T_1 = T_2, \varepsilon_1 = \varepsilon_2$, and $W_2^{-1} W_1 N \in sgp_G\{\mathring{T}N\}$.

We will be interested in finite sequences of elements of $\mathbf{t}^{\mathbf{w}}$. Let $\sigma=(C_1,\ldots,C_m)$ be such a sequence. We define the *inverse* σ^{-1} of σ to be $(C_m^{-1},\ldots,C_1^{-1})$. For $W\in\mathbf{w}$ we define the *conjugate* σ^W of σ by W to be $(WC_1W^{-1},\ldots,WC_mW^{-1})$. We define $\Pi\sigma$ to be the word obtained by freely reducing the product $C_1C_2\ldots C_m$. We say that σ is an *identity sequence* if $\Pi\sigma=1$.

Now let $\sigma'=(C'_1,\ldots,C'_n)$ be another sequence. We say that σ and σ' are **R**-*equivalent* if $m=n$, $\Pi\sigma=\Pi\sigma'$, and there is a permutation θ of $\{1,\ldots,m\}$ such that C'_λ is R-equivalent to $C_{\theta(\lambda)}$, $(\lambda=1,\ldots,m)$. The two sequences are said to be *equivalent* if one can be obtained from the other by a finite number of the following operations:

(I) Replace a sequence by an **R**-equivalent sequence.

(II) Delete two consecutive terms of a sequence if they are mutually inverse (as words).

(III) The reverse of (II).

We remark that the notion of equivalence of sequences could be phrased in terms of Peiffer exchanges, and collapses and expansions as defined on pp. 173–174 of [**3**]. However, we will not need this formulation here.

If $T\in\mathbf{t}$ and σ is a sequence then we define $\exp_T(\sigma)$ ("the exponent sum of T in σ") to be the number of terms of σ of the form WTW^{-1} $(W\in\mathbf{w})$ minus the number of terms of σ of the form $WT^{-1}W^{-1}$ $(W\in\mathbf{w})$. Note that if σ' is equivalent to σ then $\exp_T(\sigma')=\exp_T(\sigma)$ for all $T\in\mathbf{t}$.

We now relate sequences with pictures.

Let $\mathbb{P}$ be a based picture over **R** with discs $\Delta_1,\ldots,\Delta_m$ and basepoint 0. A *transverse path* in $\mathbb{P}$ is a path γ in $\mathbb{P}$ with the following properties; (a) γ interesects the arcs only finitely many times (moreover, if γ intersects an arc then it crosses if (and doesn't just touch it); (b) γ intersects $\partial\mathbb{P}\cup\Delta_1\cup\cdots\cup\Delta_m$ is a subset of the basepoints. Since we will only ever consider transverse paths we will from now on drop the use of the adjective "transverse", and simply refer to *paths*.

If we travel along a path γ from its initial point to its terminal point then we will cross various arcs, and we can read off the labels on these arcs, giving a word $W(\gamma)$, the *label on* γ.

A *spray* over **P** is a sequence $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_m)$ of simple paths satisfying the following: (a) for $\lambda=1,\ldots,m$, γ_λ starts at 0 and ends at a basepoint of $\Delta_{\theta(\lambda)}$, where θ is a permutation of $\{1,\ldots,m\}$ (depending on γ); (b) for $1\leqq\lambda<\mu\leqq m$, γ_λ and γ_μ intersect only at 0; (c) travelling around 0 anticlockwise in $\mathbb{P}$ we encounter the paths in the order $\gamma_1,\gamma_2,\ldots,\gamma_m$. The *sequence* $\sigma(\gamma)$ *associated with* γ is

$$(W(\gamma_1)W(\Delta_{\theta(1)})W(\gamma_1)^{-1},\ldots,W(\gamma_m)W(\Delta_{\theta(m)})W(\gamma_m)^{-1}).$$

The proofs of the following two lemmas can be obtained by suitably modifying the discussion in [**23**]. (The modifications required arise from the fact that, unlike in [**23**], the discs in our pictures have several basepoints, determined by the periods of their labels.)

Lemma 1.2. *If γ, γ' are two sprays over $\mathbb{P}$ then $\sigma(\gamma)$ and $\sigma(\gamma')$ are* **R***-equivalent.*

Lemma 1.3. *Given a sequence σ there is a based picture $\mathbb{P}$ and a spray γ over $\mathbb{P}$ such that the label on $\mathbb{P}$ is $\Pi\sigma$, and $\sigma(\gamma)=\sigma$. (We say that the picture $\mathbb{P}$* represents *the sequence σ.)*

We now relate identity sequences with relation modules.

Let M denote the *relation module* of $\mathbf{R}$. Thus M is the left $\mathbb{Z}G$-module with underlying abelian group N/N' and G-action given by

$$WN \cdot UN' = WUW^{-1}N' \ (W \in F, U \in N').$$

Let Φ be the free left $\mathbb{Z}G$-module with basis e_T $(T \in \mathbf{t})$. There is an epimorphism

$$\phi: \Phi \to M, e_T \mapsto TN' \, (T \in \mathbf{t}). \tag{1.8}$$

Observe that the elements $(1-\mathring{T}N)e_T$ $(T \in \mathbf{t})$ lie in the kernel of this homomorphism. We call these the *trivial elements.*

If $\sigma=(W_1 T_1^{\varepsilon_1} W_1^{-1}, \ldots, W_m T_m^{\varepsilon_m} W_m^{-1})$ is a sequence of elements of $\mathbf{t}^{\mathbf{w}}$ then $\operatorname{eval}(\sigma)$ will denote the element

$$\varepsilon_1 W_1 N e_{T_1} + \cdots + \varepsilon_m W_m N e_{T_m}$$

of Φ. Note that if σ' is equivalent to σ then $\operatorname{eval}(\sigma)-\operatorname{eval}(\sigma')$ lies in the submodule of Φ generated by the trivial elements.

Lemma 1.4. *Let Σ be a set of identity sequences such that every identity sequence is equivalent to a product of conjugates of elements of $\Sigma \cup \Sigma^{-1}$. Then* $\operatorname{Ker}\phi$ *is generated by the trivial elements together with the elements* $\operatorname{eval}(\sigma)$ $(\sigma \in \Sigma)$.

Proof. Let K be the submodule of Φ generated by the trivial elements and the elements $\operatorname{eval}(\sigma)$ $(\sigma \in \Sigma)$. Certainly $K \subseteq \operatorname{Ker}\phi$.

To show the reverse inclusion, let

$$\xi = \varepsilon_1 W_1 N e_{T_1} + \cdots + \varepsilon_m W_m N e_{T_m}$$

belong to $\operatorname{Ker}\phi$. Let $C_\lambda = W_\lambda T_\lambda^{\varepsilon_\lambda} W_\lambda^{-1}$ $(\lambda = 1, \ldots, m)$. Then $C_1 C_2 \ldots C_m \in N'$, so there exist elements $D_1, D_2, \ldots, D_{2n-1}, D_{2n}$ of $\mathbf{t}^{\mathbf{w}}$ such that

$$\sigma = (C_1, C_2, \ldots, C_m, D_1, D_2, D_1^{-1}, D_2^{-1}, \ldots, D_{2n-1}, D_{2n}, D_{2n-1}^{-1}, D_{2n}^{-1})$$

is an identity sequence. By assumption, σ is equivalent to a product $\sigma' = \sigma_1^{\delta_1 V_1} \ldots \sigma_k^{\delta_k V_k}$ where $\sigma_i \in \Sigma$, $V_i \in \mathbf{w}$, $\delta_i = \pm 1$ $(i = 1, \ldots, k)$. By a previous remark

$$\operatorname{eval}(\sigma) - \operatorname{eval}(\sigma') \in K.$$

Since $\mathrm{eval}(\sigma)=\xi$, and $\mathrm{eval}(\sigma')\in K$, the result follows.

1.6. *Lifting relative presentations*

Let $\mathbf{P}=\langle H,\mathbf{x};\mathbf{r}\rangle$ be a relative presentation. We obtain an ordinary presentation $\tilde{\mathbf{P}}$ defining the same group G as follows.

Let $\mathbf{Q}=\langle \mathbf{a};\mathbf{s}\rangle$ be an ordinary presentation of H. Then there is a homomorphism ϕ from the free group on $\mathbf{a}$ onto H with kernel the normal closure of $\mathbf{s}$. For each $h\in H$ we choose an element of $\phi^{-1}(h)$, represented by a freely reduced word in $\mathbf{a}$. Now ϕ extends to a homomorphism from the free group on $\mathbf{a}\cup\mathbf{x}$ to $H*\langle\mathbf{x}\rangle$ in the obvious way, and the lifting of elements of H decribed above induces a lifting of elements of $H*\langle\mathbf{x}\rangle$. In particular, for each $R\in\mathbf{r}$ we have its lift $\tilde{R}$ (a cyclically reduced word on $\mathbf{a}\cup\mathbf{x}$). We let

$$\tilde{\mathbf{P}}=\langle \mathbf{a},\mathbf{x};\mathbf{s},\tilde{\mathbf{r}}\rangle$$

where $\tilde{\mathbf{r}}=\{\tilde{R}:R\in\mathbf{r}\}$.

Lemma 1.5. *If P is orientable and aspherical, then every picture over $\tilde{\mathbf{P}}$ having at least one $\tilde{\mathbf{r}}$-disc and having no $\mathbf{x}$-arcs meeting the boundary of the picture, contains an $\tilde{\mathbf{r}}$-dipole.*

Proof. Let $\tilde{\mathbb{P}}$ be a picture over $\tilde{\mathbf{P}}$, and consider an $\tilde{\mathbf{r}}$-disc of $\tilde{\mathbb{P}}$. Between two successive $\mathbf{x}$-arcs meeting this $\tilde{\mathbf{r}}$-disc, there is a succession of $\mathbf{a}$-arcs; let W be the word obtained by reading these arcs (anticlockwise). Now erase these $\mathbf{a}$-arcs and label the corner between the two successive $\mathbf{x}$-arcs by $\phi(W)\in H$. We let $\tilde{\mathbb{P}}/\mathbf{Q}$ denote the picture over $\mathbf{P}$ obtained by removing all $\mathbf{a}$-arcs and $\mathbf{s}$-discs, and labelling corners between $\mathbf{x}$-arcs as above. The condition (1.5) for pictures over $\mathbf{P}$ is satisfied by Lemma 1.1.

Now suppose that $\tilde{\mathbb{P}}$ contains at least one $\tilde{\mathbf{r}}$-disc, and that no $\mathbf{x}$-arc of $\tilde{\mathbb{P}}$ meets $\partial\tilde{\mathbb{P}}$. Then $\tilde{\mathbb{P}}/\mathbf{Q}$ is a spherical picture over $\mathbf{P}$; since $\mathbf{P}$ is aspherical, $\tilde{\mathbb{P}}/\mathbf{Q}$ contains a dipole. Since $\mathbf{P}$ is orientable, this dipole arises from an $\tilde{\mathbf{r}}$-dipole in $\tilde{\mathbb{P}}$.

We remark that without the orientability assumption, this result is false. For example, if $\mathbf{P}=\langle H,x:[x,a]^2\rangle$, where H is cyclic of order 2 generated by a, then it can be shown that $\mathbf{P}$ is aspherical. However $\mathbf{P}$ is not orientable, and the picture over $\tilde{\mathbf{P}}$ shown in Figure 1 contains no dipole.

The passage from pictures over $\tilde{\mathbf{P}}$ to pictures over $\mathbf{P}$ is reversible, in the following sense.

Lemma 1.6. *If $\mathbb{P}$ is a connected spherical picture over* $\mathbf{P}$, *then there is a picture $\tilde{\mathbb{P}}$ over $\tilde{\mathbf{P}}$ with $\tilde{\mathbb{P}}/\mathbf{Q}=\mathbb{P}$.*

Proof. For each inner region Σ of $\mathbb{P}$, Lemma 1.1 provides a picture $\tilde{\Sigma}$ over $\mathbf{Q}$ with boundary label equal to the product of the words in $\mathbf{a}$ which represent the corner labels of Σ. The one remaining region of $\mathbb{P}$ is an annulus. Replace each corner label in this region by a succession of $\mathbf{a}$-arcs reading the representative word (anticlockwise around

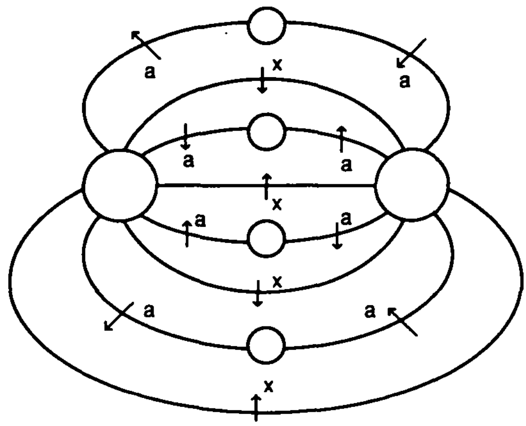

FIGURE 1

the ambient disc). These **a**-arcs extend radially to the boundary, giving the required picture $\tilde{\mathbb{P}}$.

A connected spherical picture $\mathbb{P}$ over **P** is defined to be *strictly spherical* if the product of the corner labels in the annular region (taken in anticlockwise order) defines the identity in H. The relative presentation **P** is *weakly aspherical* if each strictly spherical picture over **P** contains a dipole.

Lemma 1.7. *If* **P** *is weakly aspherical and if the natural map of* H *into* G *is an embedding, then* **P** *is aspherical.*

Proof. Given a connected spherical picture $\mathbb{P}$ over **P**, construct a lifted picture $\tilde{\mathbb{P}}$ over $\tilde{\mathbf{P}}$ as in Lemma 1.6. Note that the boundary label on $\tilde{\mathbb{P}}$ is a word W in **a** where $\phi(W)$ is equal to the product of the outer corner labels. By Lemma 1.1, $\phi(W)$ is in the kernel of $H \to G$, and hence is trivial in H. Thus $\mathbb{P}$ is strictly spherical, and so contains a dipole.

We remark that the converse of this result holds. This will follow from Theorem 1.1, Corollary 1 (§ 1.7) in the case where **P** is orientable. The distinction between asphericity and weak asphericity will be useful in § 3.1 below.

1.7. *The main results concerning orientable aspherical relative presentations*

Throughout this section we will assume, without further comment, that $\mathbf{P} = \langle H, \mathbf{x}; \mathbf{r}\rangle$ is an orientable aspherical relative presentation. We let $\tilde{\mathbf{P}} = \langle \mathbf{a}, \mathbf{x}; \mathbf{s}, \tilde{\mathbf{r}}\rangle$ where $\langle \mathbf{a}; \mathbf{s}\rangle$ is a presentation of H and $\tilde{\mathbf{r}}$ is a lift of **r** (as in § 1.6). We let **w** denote the set of words on $\mathbf{a} \cup \mathbf{x}$. If σ is a sequence of elements of $(\mathbf{s} \cup \tilde{\mathbf{r}})^{\mathbf{w}}$ then we let $d(\sigma)$ denote the number of terms of σ which belong to $\tilde{\mathbf{r}}^{\mathbf{w}}$. The group defined by **P** will be denoted by G.

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Theorem 1.1. *If σ is a sequence where $d(\sigma)>0$ and where $\Pi\sigma$ is a word on* $\mathbf{a}$, *then σ is equivalent to a sequence σ' with $d(\sigma')=d(\sigma)-2$.*

Proof. Let $\tilde{\mathbb{P}}$ be a based picture representing σ (see Lemma 1.3). Then $\tilde{\mathbb{P}}$ contains an $\tilde{\mathbf{r}}$-dipole (Lemma 1.5). By considering a spray over $\tilde{\mathbb{P}}$ whose first two paths go to appropriate basepoints of the discs of an $\tilde{\mathbf{r}}$-dipole we find (using Lemma 1.2) that σ is $\tilde{\mathbf{P}}$-equivalent to a sequence whose first two terms are in $\tilde{\mathbf{r}}^{\mathbf{w}}$ and are mutually inverse. By deleting these first two terms we obtain the required sequence σ'.

Corollary 1. *The natural homomorphism $H\to G$ is injective.*

Proof. Let W be a freely reduced word in $\mathbf{a}$ which defines the identity of G. Then $W=\Pi\sigma$ for some sequence σ. It follows from Theorem 1.1 and induction that $W=\Pi\tau$ for some sequence τ with $d(\tau)=0$. Thus W defines the identity in the group $H*\langle\mathbf{x}\rangle$ given by the presentation $\langle\mathbf{x},\mathbf{a};\,\mathbf{s}\rangle$, and therefore defines the identity of H.

Corollary 2. *Every identity sequence over $\tilde{\mathbf{P}}$ is equivalent to an identity sequence all of whose terms belong to* $\mathbf{s}^{\mathbf{w}}$.

Corollary 3. *If σ is an identity sequence then* $\exp_{\tilde{R}}(\sigma)=0$ *for all $\tilde{R}\in\tilde{\mathbf{r}}$.*

Corollary 4. *If $\tilde{R}\in\tilde{\mathbf{r}}$ then $\tilde{R}$ defines an element of order precisely $p(\tilde{R})$ in G.*

Proof. Suppose $\tilde{R}$ defines an element of order l in G. Then $l\,|\,p(\tilde{R})$. Now $\mathring{\tilde{R}}^{l}=\Pi\tau$ for some sequence τ. Then the sequence σ consisting of $\tilde{R}^{-1}$ followed by $p(\tilde{R})/l$ copies of τ is an identity sequence. By Corollary 3 we have

$$-1+(p(\tilde{R})/l)\exp_{\tilde{R}}(\tau)=\exp_{\tilde{R}}(\sigma)=0.$$

Thus $l=p(\tilde{R})$.

Corollary 4 is a "relative" version of Proposition 1 of [14] (compare also with Proposition 2.7 of [5]).

Let M be the relation module corresponding to the presentation $\tilde{\mathbf{P}}$ of G. (Thus $M=N/N'$, where N is the normal closure of $\mathbf{s}\cup\tilde{\mathbf{r}}$ in the free group on $\mathbf{a}\cup\mathbf{x}$.) Also, let M_H be the relation module corresponding to the presentation $\langle\mathbf{a};\,\mathbf{s}\rangle$ of H, and, for $R\in\mathbf{r}$, let M_R be the relation module corresponding to the presentation $\langle b_R;\, b_R^{p(\tilde{R})}\rangle$ of the cyclic group C_R of order $p(\tilde{R})$. By Corollary 1, H is embedded into G by the natural homomorphism, and by Corollary 4, C_R is embedded into G by the homomorphism defined by $b_R\mapsto\tilde{R}N$. We can therefore regard $\mathbb{Z}G$ as a (free) right $\mathbb{Z}H$-module, and also as a (free) right $\mathbb{Z}C_R$-module ($R\in\mathbf{r}$), and we can form the induced modules

$$\mathbb{Z}G\otimes_{\mathbb{Z}H}M_H,\ \mathbb{Z}G\otimes_{\mathbb{Z}C_R}M_R\ (R\in\mathbf{r}). \tag{1.9}$$

Theorem 1.2. *M is isomorphic to the direct sum of the modules in (1.9).*

Proof. Let Φ_H be the free left $\mathbb{Z}G$-module with basis $\{e_S: S \in \mathbf{s}\}$, and for $R \in \mathbf{r}$, let Φ_R be the free left $\mathbb{Z}G$-module $\mathbb{Z}Ge_R$ of rank 1. Let

$$\Phi = \Phi_H \oplus \left(\bigoplus_{R \in \mathbf{r}} \Phi_R \right).$$

By Corollary 2 above and Lemma 1.4 (§ 1.5) the kernel of the epimorphism $\phi: \Phi \to M$ (as in (1.8)) is

$$K_H \oplus \left(\bigoplus_{R \in \mathbf{r}} \mathbb{Z}G(1 - \mathring{\tilde{R}}N)e_R \right),$$

where K_H is the submodule of Φ_H generated by $\{\mathrm{eval}(\sigma): \sigma$ is an identity sequence of elements of $\mathbf{s}^{\mathbf{w}}\}$.

Now Φ_H/K_H is isomorphic to the submodule of M generated by $\{SN': S \in \mathbf{s}\}$. This submodule is in turn isomorphic to $\mathbb{Z}G \otimes_{\mathbb{Z}H} M_H$ (see [**22**, Lemma 2] in this regard). Also

$$\frac{\Phi_R}{\mathbb{Z}G(1 - \mathring{\tilde{R}}N)e_R} \cong \mathbb{Z}G \otimes_{\mathbb{Z}C_R} M_R \quad (R \in \mathbf{r}).$$

The result follows.

Theorem 1.3. *For any left $\mathbb{Z}G$-module A, and any right $\mathbb{Z}G$-module B we have*

$$H^n(G, A) \cong H^n(H, A) \oplus \left(\prod_{R \in \mathbf{r}} H^n(sgp_G\{\mathring{\tilde{R}}N\}, A) \right)$$

$$H_n(G, B) \cong H_n(H, B) \oplus \left(\bigoplus_{R \in \mathbf{r}} H_n(sgp_G\{\mathring{\tilde{R}}N\}, B) \right)$$

for all $n \geqq 3$.

This follows from Theorem 1.2 using fairly standard arguments concerning dimension shifting and Shapiro's lemma. (For similar calculations, see the proof of Theorem 2 of [**23**].)

Theorem 1.4. *Any finite subgroup of G is contained in a conjugate of H or in a conjugate of one of the cyclic subgroups $sgp_G\{\tilde{R}N\}$ $(R \in \mathbf{r})$.*

This follows from Theorem 1.3 and a result due to Serre (quoted in [**14**]).

2. Tests for asphericity

Throughout this section $\mathbf{P}=\langle H,\mathbf{x};\mathbf{r}\rangle$ will be an orientable relative presentation.

2.1. *Star complexes*

The *star-complex* $\mathbf{P}^{\mathrm{st}}$ of $\mathbf{P}$ is a graph (in the sense of Serre [**24**]) whose edges are labelled by elements of the coefficient group H. The definition is as follows.

The vertex and edge sets are $\mathbf{x}\cup\mathbf{x}^{-1}$, $\mathbf{r}^*$ respectively. For $R\in\mathbf{r}^*$, write $R=Sh$ where $h\in H$ and S begins and ends with $\mathbf{x}$-symbols. The initial and terminal functions are given by: $\iota(R)$ is the first symbol of S, $\tau(R)$ is the inverse of the last symbol of S. The inversion function on edges is given by the operator $^-$ defined in § 1.1. By remarks in § 1.1, $\bar{R}\neq R$ for all $R\in\mathbf{r}^*$, since $\mathbf{P}$ is orientable. The labelling function is defined by $\lambda(R)=h^{-1}$, and is extended to paths in the obvious way. Note that $\lambda(\bar{R})=\lambda(R)^{-1}$.

Lemma 2.1. *Let $c_1,\ldots,c_k$ be the sequence of corners encountered in an anticlockwise traverse of an inner region Σ of a picture $\mathbb{P}$ over $\mathbf{P}$. Then:* (i) *The sequence of edges $W(c_1),\ldots,W(c_k)$ is a cycle in P^{st}*; (ii) *$W(c_j)$, $W(c_{j+1})$ (subscripts* mod k*) is a backtracking if and only if there is an arc α joining c_j to c_{j+1} in the boundary of Σ such that α, c_j and c_{j+1} together form a dipole.*

Proof. (i) If an arc α meets the corners c_j and c_{j+1} in the boundary of Σ, then it must join the head of c_{j+1} to the tail of c_j (or vice versa). From this it follows that

$$\tau(W(c_j))=\iota(W(c_{j+1}))(=\text{the label on }\alpha).$$

The claim (ii) is clear.

The cycle in (i) will be referred to as the *cycle supported by the inner region* Σ. By the condition (1.5) for pictures, this cycle has trivial label in H. A non-empty cyclically reduced cycle in $\mathbf{P}^{\mathrm{st}}$ will be called *admissible* if it has trivial label in H. Each inner region of a reduced picture over $\mathbf{P}$ supports an admissible cycle in $\mathbf{P}^{\mathrm{st}}$.

2.2. *Weight test*

A *weight function* θ on $\mathbf{P}^{\mathrm{st}}$ is a real valued function on the set of edges of $\mathbf{P}^{\mathrm{st}}$ which satisfies $\theta(\bar{R})=\theta(R)$ for all $R\in\mathbf{r}^*$. The *weight of a path* is the sum of the weights of the constituent edges.

A weight function θ on $\mathbf{P}^{\mathrm{st}}$ is *weakly aspherical* if the following two conditions are satisfied.

(2.1) Let $R\in\mathbf{r}$, say $R=x_1^{\varepsilon_1}h_1\ldots x_n^{\varepsilon_n}h_n$ as in (1.2). Then

$$\sum_{i=1}^{n}\left(1-\theta(x_i^{\varepsilon_i}h_i\ldots x_n^{\varepsilon_n}h_n x_1^{\varepsilon_1}h_1\ldots x_{i-1}^{\varepsilon_{i-1}}h_{i-1})\right)\geqq 2$$

(2.2) Each admissible cycle in $\mathbf{P}^{\mathrm{st}}$ has weight at least 2.

A weakly aspherical weight function on $\mathbf{P}^{\mathrm{st}}$ is *aspherical* if each edge of $\mathbf{P}^{\mathrm{st}}$ has non-negative weight.

Theorem 2.1. (i) *If* $\mathbf{P}^{\mathrm{st}}$ *admits a weakly aspherical weight function, then* $\mathbf{P}$ *is weakly aspherical.*

(ii) *If* $\mathbf{P}^{\mathrm{st}}$ *admists an aspherical weight function then* $\mathbf{P}$ *is aspherical.*

Proof. Suppose that $\mathbb{P}$ is a reduced strictly spherical picture over $\mathbf{P}$. In the presence of a weakly aspherical weight function θ, we derive a contradiction.

Shrink each disc of $\mathbb{P}$ to a vertex, and identify $\partial\mathbb{P}$ to a point to obtain a tesselation T of the 2-sphere. Let n_0, n_1, n_2 be the number of vertices, edges, faces of T respectively. Thus n_0 is equal to the number of discs of $\mathbb{P}$, n_1 is equal to the number of arcs, and n_2 is equal to the number of regions. Denote the set of corners of $\mathbb{P}$ by C.

Clearly

$$2n_1 = \sum_{c \in C} 1.$$

Summing over all vertices of T, the condition (2.1) implies

$$2n_0 \leqq \sum_{c \in C} (1 - \theta(W(c))).$$

Now, as remarked in §2.1, each inner region of $\mathbb{P}$ supports an admissible cycle in $\mathbf{P}^{\mathrm{st}}$. Moreover, since $\mathbb{P}$ is strictly aspherical, the outer annular region also supports an admissible cycle in $\mathbf{P}^{\mathrm{st}}$. Thus, summing over all faces of T, the condition (2.2) implies

$$2n_2 \leqq \sum_{c \in C} \theta(W(c)).$$

We now obtain the following contradiction:

$$2 = n_0 - n_1 + n_2 \leqq \tfrac{1}{2} \sum_{c \in C} ((1 - \theta(W(c))) - 1 + \theta(W(c))) = 0.$$

(ii) Suppose that $\mathbb{P}$ is a reduced connected spherical picture over $\mathbf{P}$, and that $\mathbf{P}^{\mathrm{st}}$ admits an aspherical weight function θ. Proceed as in (i), with the exception that the outer region of $\mathbb{P}$ need not support an admissible cycle; it does support a cycle of non-negative weight, however. In this case we have the following consequence of (2.2):

$$2(n_2 - 1) \leqq \sum_{c \in C} \theta(W(c)).$$

This leads to the contradiction:

$$2=n_0-n_1+n_2\leqq\frac{1}{2}\left(2+\sum_{c\in C}((1-\theta(W(c)))-1+\theta(W(c)))\right)=1.$$

Remark. The present weight test generalizes work of Sieradski [**25**], Gersten [**7**] and Pride [**20**] concerning ordinary presentations (and more generally 2-complexes). Aspherical weight functions similar to ours have been introduced by Gersten in unpublished work on equations over torion-free groups [**8**].

2.3. *Small cancellation conditions*

Let k be a positive integer. A *k-wheel* over **P** is a (non-trivial) connected picture $\mathbb{W}$ over **P** which has discs $\{\Delta_0,\Delta_1,\dots,\Delta_k\}$, and which satisfies:

(i) each arc of $\mathbb{W}$ meets a disc Δ_j for some $j\in\{1,\dots,k\}$;

(ii) each arc of $\mathbb{W}$ either meets Δ_0 or $\partial\mathbb{W}$;

(iii) each disc of $\mathbb{W}$ has a corner which lies in a region of $\mathbb{W}$ that meets $\partial\mathbb{W}$. The disc Δ_0 is the *hub* of the k-wheel.

A typical k-wheel is depicted in Figure 2.

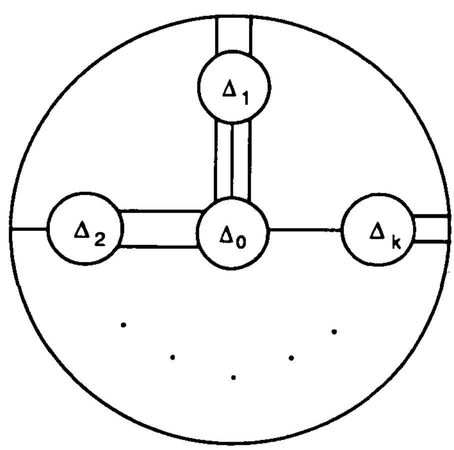

FIGURE 2

Definition. Let p be a positive integer. Then **P** *satisfies* $C(p)$ if there are no reduced k-wheels over **P** for $k<p$.

Definition. Let q be a positive integer. Then **P** *satisfies* $T(q)$ if there are no admissible cycles in $\mathbf{P}^{st}$ of length l for $3\leqq l<q$.

Theorem 2.2. *If* **P** *satisfies* $C(p)$, $T(q)$ *where* $1/p+1/q=1/2$ *then* **P** *is aspherical.*

Proof. Suppose that $\mathbb{P}$ is a reduced connected spherical picture over **P**. Observe that each inner region of $\mathbb{P}$ has at least two corners since the relators of **P** are cyclically reduced. Remove all inner regions of $\mathbb{P}$ that contain just two corners by identifying the two bounding arcs to a single arc; denote the modified picture by $\mathbb{P}^*$. The identification process is depicted in Figure. 3. (Note that the discs Δ and Δ' need not be distinct).

FIGURE 3

The labels on the arcs and corners of $\mathbb{P}$ involved in the identification process are eliminated in the passage to $\mathbb{P}^*$. However the corners of $\mathbb{P}^*$ remain labelled by coefficients; by Lemma 2.1(i), each inner region of $\mathbb{P}^*$ supports an admissible cycle of length at least 3, and hence of length at least q, by $T(q)$. If there are k incidences of arcs of $\mathbb{P}^*$ on a disc Δ of $\mathbb{P}^*$, then one can easily use the structure of the picture $\mathbb{P}$ near Δ to construct a reduced k-wheel with Δ as hub. By $C(p)$ then, there are at least p incidence of arcs on each disc Δ of $\mathbb{P}^*$.

Shrink each disc of $\mathbb{P}^*$ to a vertex, and identify $\partial\mathbb{P}$ to a point, to obtain a tesselation T of the two-sphere containing n_0 vertices, n_1 edges, and n_2 faces. The $C(p)$ condition gives that each vertex of T supports at least p incidences of edges, which implies

$$pn_0 \leqq 2n_1.$$

The $T(q)$ condition gives that all faces of T but one (the "outer" face) have at least q boundary edges, which implies that

$$q(n_2-1) \leqq 2n_1.$$

This produces the following contradiction:

$$\begin{aligned} 2 = n_0 - n_1 + n_2 &\leqq \frac{2n_1}{p} - n_1 + \frac{2n_1}{q} + 1 \\ &= 2n_1\left(\frac{1}{p}+\frac{1}{p}\right) - n_1 + 1 \\ &= 1. \end{aligned}$$

2.4. *News aspherical presentations from old*

For each $R \in \mathbf{r}$ let $n(R)$ be a positive integer. Let

$$\hat{\mathbf{P}} = \langle H, \mathbf{x};\ R^{n(R)}\ (R \in \mathbf{r})\rangle.$$

Theorem 2.3 (proper powers). *If* $\mathbf{P}$ *is aspherical then so is* $\hat{\mathbf{P}}$.

Proof. Let $\hat{\mathbb{P}}$ be a (connected) spherical picture over $\hat{\mathbf{P}}$. Convert to a spherical picture over $\mathbf{P}$ as follows. Select a disc Δ of $\hat{\mathbb{P}}$. For some $R \in \mathbf{r}$ there are corners $c_1, \ldots, c_{n(R)}$ of Δ such that $W(c_j) = (Sh)^{n(R)}$ $(j = 1, \ldots, n(R))$; here $h \in H$, S begins and ends with $\mathbf{x}$-symbols, and Sh is either R or $\bar{R}$. Fracture Δ at each of the corners c_j, and break Δ up into $n(R)$ smaller discs $\Delta_1, \ldots, \Delta_{n(R)}$, so that c_j is now a corner of Δ_j and $W(c_j) = Sh$. The process is depicted in Figure 4.

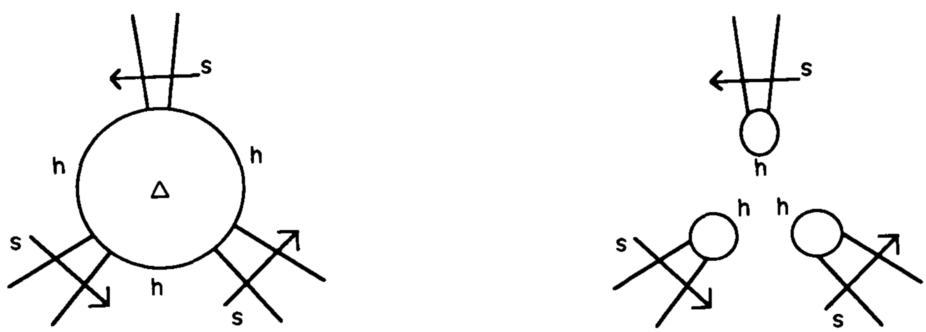

FIGURE 4

The new picture $\mathbb{P}_1$ has the same arcs as $\hat{\mathbb{P}}$, but will generally not qualify as a picture over either $\mathbf{P}$ or $\hat{\mathbf{P}}$: the condition (1.4) will typically be violated. The labelled and oriented picture $\mathbb{P}_1$ does satisfy the condition (1.5) however. For the regions of $\mathbb{P}_1$ are just the regions of $\hat{\mathbb{P}}$, except for a single region Σ which is a union of one or more regions of $\hat{\mathbb{P}}$, together with the "junction" that was created when Δ was fractured. If Σ is not inner to $\mathbb{P}_1$, then there is nothing to prove. Otherwise, the sequence of coefficients obtained in a traverse of the boundary of Σ is a concatenation of $n(R)$ coefficient sequences taken from inner regions of $\hat{\mathbb{P}}$, and so has trivial product in H, as desired.

Continue in this fashion to obtain a finite sequence of spherical pictures

$$\hat{\mathbb{P}} = \mathbb{P}_0, \mathbb{P}_1, \ldots, \mathbb{P}_t = \mathbb{P}$$

where $\mathbb{P}_{j+1}$ is obtained from $\mathbb{P}_j$ by fracturing a disc of $\mathbb{P}_j$ with label in $R^{n(R)} (R \in \mathbf{r})$, where each $\mathbb{P}_j$ satisfies the condition (1.5), and where $\mathbb{P}$ qualifies as a picture over $\mathbf{P}$, the condition (1.4) being satisfied for $\mathbf{r}$.

While $\mathbb{P}$ need not be connected, it is spherical and so contains a dipole. Since no relator of $\mathbf{r}$ is repeated, this implies that $\hat{\mathbb{P}}$ contains a dipole, and the proof is complete.

Our next result concerns a "change of variables".

Let $\{t_x : x \in \mathbf{x}\}$ be a set in 1:1 correspondence with $\mathbf{x}$. For each $x \in \mathbf{x}$, select $h_{x,1}$ and $h_{x,-1}$ in H, and select $v_x \in \{1, -1\}$. Rewrite each word in $H \cup \mathbf{x} \cup \mathbf{x}^{-1}$ using the substitutions

$$x^{\varepsilon} = h^{\varepsilon}_{x,\varepsilon} t_x^{\varepsilon v_x} h^{\varepsilon}_{x,-\varepsilon}$$

where $\varepsilon = \pm 1$. Let $\mathbf{r}'$ denote the set of words obtained from $\mathbf{r}$ by substituting as above, cyclically permuting so as to begin with a $\mathbf{t}$-symbol, and finally multiplying adjacent elements of H. We note that the words in $\mathbf{r}'$ are cyclically reduced. Let

$$\mathbf{P}' = \langle H, \mathbf{t}; \mathbf{r}' \rangle.$$

Theorem 2.4 (change of variables). *$\mathbf{P}$ is (weakly) aspherical if and only if $\mathbf{P}'$ is (weakly) aspherical.*

Proof. It suffices to prove one direction. Convert a picture $\mathbb{P}$ over $\mathbf{P}$ to a picture $\mathbb{P}'$ over $\mathbf{P}'$ as follows. Replace each label $x^\varepsilon \in \mathbf{x} \cup \mathbf{x}^{-1}$ on each arc of $\mathbb{P}$ by $t_x^{\varepsilon \nu_x}$. For a corner c of a disc of $\mathbb{P}$, if

$$\iota(W(c)) = x^\varepsilon \text{ and } \tau(W(c)) = y^\delta$$

(where ι and τ are the initial and terminal maps of $\mathbf{P}^{\mathrm{st}}$; x^ε and y^δ are $\mathbf{x}$-symbols), then replace the label λ on c by

$$h_{y,\delta}^{-\delta} \lambda h_{x,\varepsilon}^{\varepsilon}.$$

One can now check that this new picture $\mathbb{P}'$ is a picture over $\mathbf{P}'$. Moreover, a dipole in $\mathbb{P}'$ gives rise to a dipole in $\mathbb{P}$. The result follows easily.

3. Applications

3.1. *Some relative presentations with one defining relator (1)*

A relative presentation of the form

$$\langle H, x; xa_1 xa_2 \ldots xa_n \rangle$$

$(a_1, a_2, \ldots, a_n \in H)$ is orientable and satisfies $T(4)$. If $n \geqq 4$ and if all the a's are distinct then it also satisfies $C(4)$, and so the presentation will be aspherical in this case (Theorem 2.2). However, when $n = 3$ the presentation satisfies $C(4)$ only when $a_1 = a_2 = a_3$. Nevertheless, we can still investigate the asphericity of the presentation by using the weight test (Theorem 2.1). This is our aim in this section.

Theorem 3.1. *Let a_1, a_2 and a_3 be elements of a group H such that $\{a_1, a_2, a_3\}$ contains at least two elements. The relative presentation $\langle H, x; xa_1 xa_2 xa_3 \rangle$ is aspherical if and only if neither of the following conditions holds:*

(i) *For $i = 1, 2, 3$, $a_i a_{i+1}^{-1}$ has finite order $p_i > 0$ (subscripts* mod 3*), and*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} > 1.$$

(ii) *There exist* $j \in \{1,2,3\}$, $p > 2$, *and* $0 \leqq k < p$ *such that* $sgp\{a_i a_{i+1}^{-1}: i = 1,2,3\}$ *is finite cyclic with generator* $a_j a_{j+1}^{-1}$ *of order* p, *and* $a_{j+1} a_{j+2}^{-1} = (a_j a_{j+1}^{-1})^k$ *where either*

(a) $k = 1$.

(b) $p = k+2$ *or* $2k+1$, *or*

(c) $p = 6$ *and* $k = 2$ *or* 3.

The proof of this result is broken up into several pieces. In order to fix and simplify notation, we work with a relative presentation

$$\mathbf{P}_1 = \langle H, x;\ xaxbxc \rangle$$

where $\{a, b, c\}$ is a subset of H consisting of at least two distinct elements.

Theorem 3.2. *If one of* ab^{-1}, bc^{-1}, ca^{-1} *has infinite order then* $\mathbf{P}_1$ *is aspherical.*

Theorem 3.3. *Suppose* ab^{-1}, bc^{-1}, ca^{-1} *have finite orders* p, q, r *respectively. Then* $\mathbf{P}_1$ *is aspherical if*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leqq 1$$

and $sgp\{ab^{-1}, bc^{-1}, ca^{-1}\}$ *is not generated by any one of* ab^{-1}, bc^{-1}, ca^{-1}.

Theorem 3.4. *If* ab^{-1}, bc^{-1}, ca^{-1} *have finite orders* p, q, r *respectively, where*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1,$$

then P_1 *is not aspherical.*

The Theorems 3.2, 3.3 and 3.4 cover the majority of Theorem 3.1. After proving these results we will consider the remaining cases, in which $\text{sgp}\{ab^{-1}, bc^{-1}, ca^{-1}\}$ is finite cyclic, generated by one of ab^{-1}, bc^{-1}, ca^{-1}.

Before proving the above theorems we make some preliminary remarks.

The star-complex of $\mathbf{P}_1$ has the form

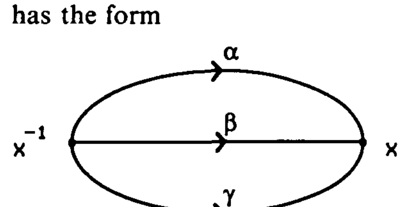

FIGURE 5

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where the edges α, β, γ are labelled by a, b, c respectively. We let $A=\alpha\beta^{-1}$, $B=\beta\gamma^{-1}$, $C=\gamma\alpha^{-1}$. A word in A, B, C will be said to be *special* if it is non-empty, cyclically reduced, and none of its cyclic permutations has a subword $(AB)^{\pm 1}$, $(BC)^{\pm 1}$ or $(CA)^{\pm 1}$. Then each non-empty cyclically reduced path ρ in $\mathbf{P}_1^{\mathrm{st}}$ starting at x^{-1} can be expressed as a special word $W(\rho)$ in A, B, C. Moreover, ρ is admissible if and only if $W(\rho)$ is in the kernel of the homomorphism λ from the free group on A, B, C to H given by

$$A\mapsto ab^{-1}, B\mapsto bc^{-1}, C\mapsto ca^{-1}.$$

We note the following fact for future use.

If X, Y are distinct elements of $\{A,B,C\}$ and if $XY^n\in\operatorname{Ker}\lambda$ for some n, then $sgp\{ab^{-1},bc^{-1},ca^{-1}\}$ is cyclic, generated by $\lambda(Y)$. (3.1)

Since the natural homomorphism of H into the group defined by $\mathbf{P}_1$ is injective **[18]**, it follows from Lemma 1.7 and Theorem 2.1(i) that $\mathbf{P}_1$ is aspherical if there is a weakly aspherical weight function on $\mathbf{P}_1^{\mathrm{st}}$. Sufficient conditions for a weight function θ on $\mathbf{P}_1^{\mathrm{st}}$ to be weakly aspherical are the following:

$$\theta(A)+\theta(B)+\theta(C)=2 \tag{3.2}$$

and

$$\theta(W)\geqq 2\ (W \text{ a special word in } \operatorname{Ker}\lambda). \tag{3.3}$$

Proof of Theorem 3.2. We suppose that ab^{-1} has infinite order, and we exhibit a weight function satisfying (3.2) and (3.3).

Case 1. B or C belongs to $\operatorname{Ker}\lambda$.

Suppose without loss of generality that $B\in\operatorname{Ker}\lambda$. We claim that any special word W in $\operatorname{Ker}\lambda$ must involve B. For if W involved only A and C then, since $\lambda(A)=\lambda(C)^{-1}$ (using $\lambda(ABC)=1$), $\lambda(W)$ would be a non-zero power of $\lambda(A)=ab^{-1}$, and so $\lambda(W)\neq 1$.

It now suffices to put $\theta(\alpha)=-1, \theta(\beta)=\theta(\gamma)=1$.

Case 2. B^2 or C^2 belongs to $\operatorname{Ker}\lambda$.

Suppose without losing generality that $B^2\in\operatorname{Ker}\lambda$ (and that B, $C\notin\operatorname{Ker}\lambda$). We claim that any special word W in $\operatorname{Ker}\lambda$ must involve at least two occurrences of letters in $\{B^{\pm 1},C^{\pm 1}\}$. For otherwise, up to cyclic permutation and inversion, W would be of the form A^nB^{-1} or A^nC^{-1} $(n>0)$. If $A^nB^{-1}\in\operatorname{Ker}\lambda$ then $\lambda(A)^{2n}=1$, contradicting the fact that $\lambda(A)$ has infinite order. If $A^nC^{-1}\in\operatorname{Ker}\lambda$ then we obtain the contradiction $\lambda(A)^{2(n+1)}=1$.

It now suffices to define $\theta(\alpha)=\theta(\beta)=0$ and $\theta(\gamma)=1$.

Case 3. One of AB^{-1}, AC^{-1}, BC^{-1} is in $\operatorname{Ker}\lambda$.

If $BC^{-1} \in \mathrm{Ker}\,\lambda$ then $\lambda(B)=\lambda(C)$ and, since $\lambda(ABC)=1$, we have $\lambda(A)=\lambda(B)^{-2}$, which means that $\lambda(B)$ has infinite order. Consequently, all three of AB^{-1}, AC^{-1}, BC^{-1} have the form $XY^{-1}(X \neq Y)$ with $\lambda(X)$ of infinite order. Thus, by symmetry, it suffices to deal with the case when $AB^{-1} \in \mathrm{Ker}\,\lambda$.

If $AB^{-1} \in \mathrm{Ker}\,\lambda$ then $\lambda(A)=\lambda(B)$ and $\lambda(C)=\lambda(A)^{-2}$. Thus $\lambda(A)$, $\lambda(B)$, $\lambda(C)$ all have infinite order, and so any special word W in $\mathrm{Ker}\,\lambda$ must involve at least one of A, B. We claim that the total number of A's and B's in W must be at least 2. For otherwise, up to cyclic permutation and inversion W would have one of the forms $AC^{-n}, BC^{-n} (n>0)$. Suppose $AB^{-n} \in \mathrm{Ker}\,\lambda$. Then $\lambda(A)^2=\lambda(C)^{2n}=\lambda(A)^{-4n}$, contradicting the fact that $\lambda(A)$ has infinite order. Similarly $BC^{-n} \notin \mathrm{Ker}\,\lambda$.

It follows from the above that if we put $\theta(\alpha)=\theta(\gamma)=0$, $\theta(\beta)=1$, then (3.2) and (3.3) are satisfied.

Case 4. No special word of length less than 3 is in $\mathrm{Ker}\,\lambda$.

Put $\theta(\alpha)=\theta(\beta)=\theta(\gamma)=\frac{1}{3}$. (Alternatively, note that $\mathbf{P}_1$ satisfies $C(3)$, $T(6)$.)

For the proof of Theorem 3.3 we need the following.

Lemma 3.1. *Suppose $sgp\{ab^{-1}, bc^{-1}, ca^{-1}\}$ is not generated by any one of $ab^{-1}, bc^{-1}, ca^{-1}$, and suppose that ab^{-1}, bc^{-1}, ca^{-1} have finite orders p, q, r respectively. Then a weight function θ on $\mathbf{P}_1^{st}$ is weakly aspherical if it satisfies (3.2) and:*

$$\theta(A) \geqq \frac{2}{p}, \theta(B) \geqq \frac{2}{q}, \theta(C) \geqq \frac{2}{r}; \tag{3.4}$$

if X, Y are distinct elements of $\{A, B, C\}$ and if W is a special word involving both X and Y at least twice then $\theta(W) \geqq 2$. (3.5)

Proof. We show that (3.3) holds. Let W be a special word in $\mathrm{Ker}\,\lambda$. If W involves only one of A, B, C then $\theta(W) \geqq 2$ by (3.4). If W involves exactly two of A, B, C then $\theta(W) \geqq 2$ by (3.5) and (3.1). If W involves all three of A, B, C then $\theta(W) \geqq 2$ by (3.2) and the fact that $\theta(A)$, $\theta(B)$, $\theta(C) \geqq 0$.

Proof of Theorem 3.3. We exhibit a weight function θ satisfying (3.2), (3.4), (3.5).

Case 1. $p, q, r \geqq 3$.

Put $\theta(a)=\theta(\beta)=\theta(\gamma)=\frac{1}{3}$. (Alternatively, note that $\mathbf{P}_1$ satisfies $C(3)$ and $T(6)$.)

Case 2. $p=2$, $q=3$, $r \geqq 6$.

Put $\theta(\alpha)=\frac{1}{3}$, $\theta(\beta)=\frac{2}{3}$, $\theta(\gamma)=0$.

Case 3. $p=2$, q, $r \geqq 4$.

Put $\theta(\alpha)=\theta(\beta)=\frac{1}{2}$, $\theta(\gamma)=0$.

Proof of Theorem 3.4. We exhibit a spherical picture $\mathbb{P}$ over $\mathbf{P}_1$ without dipoles. We can assume, without loss of generality, that $p \leqq q \leqq r$.

Case 1. $p=1$.

Take $\mathbb{P}$ to be the following picture with $2d$ discs, where $d = \mathrm{lcm}(q,r)$

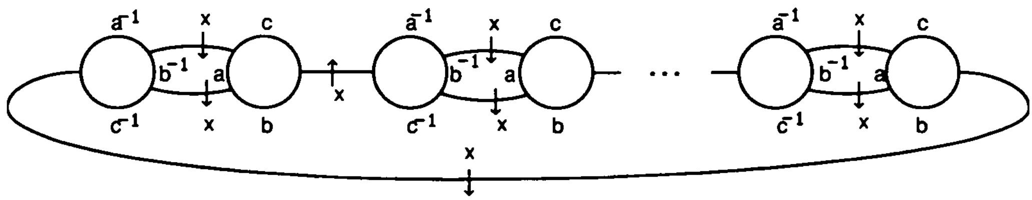

FIGURE 6

Case 2. $p=2$.

We indicate how to construct a suitable picture $\mathbb{P}$.

Start with a regular tesselation of the sphere where each vertex has valence q and each region has r sides, and project this tesselation onto the plane. (The vertices and edges of this tesselation will *not* form of $\mathbb{P}$, but will be used to indicate where to place various parts of $\mathbb{P}$.)

In the middle of each edge of the tesselation place a configuration thus:

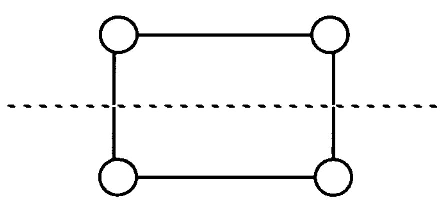

FIGURE 7

Then add additional arcs to "box in" each vertex of the tesselation.

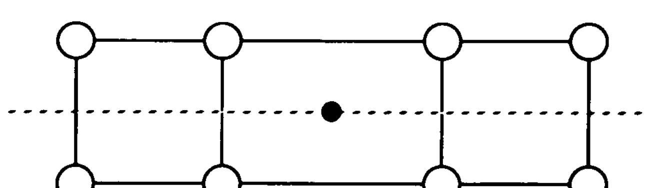

FIGURE 8

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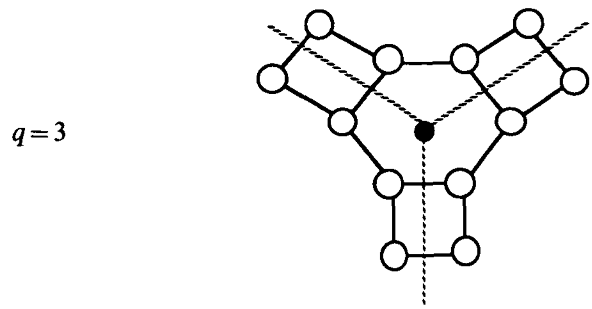

FIGURE 9

Finally, rub out all the vertices and edges of the tesselation, and label the arcs and corners of the remaining configuration to obtain a spherical picture $\mathbb{P}$ (there is essentially only one way to do this).

The case $q=3$, $r=4$ is depicted below. (We have left in the edges and vertices of the tesselation for illustrative purposes.)

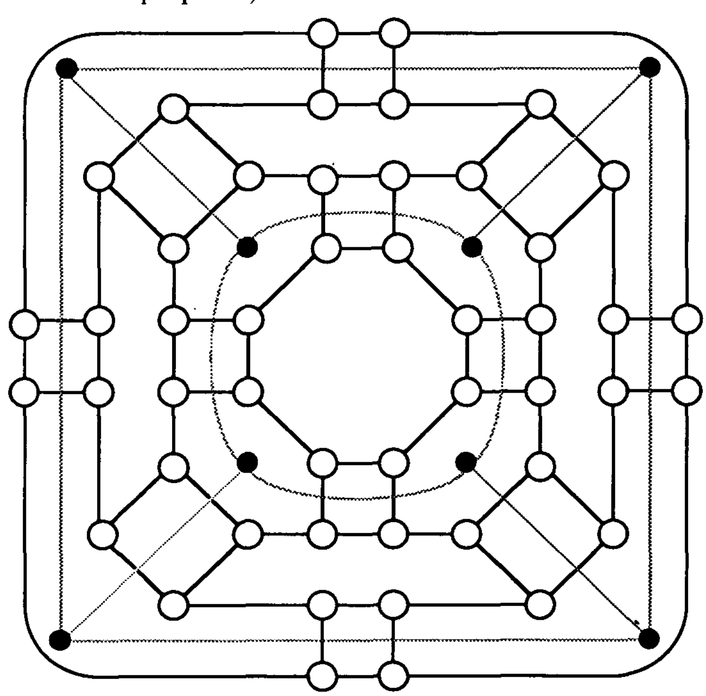

FIGURE 10

To complete the proof of Theorem 3.1 we now suppose that $\text{sgp}\{ab^{-1}, bc^{-1}, ca^{-1}\}$ is finite cyclic generated by ab^{-1} of order p, and we write $bc^{-1}=(ab^{-1})^k$ where $0\leqq k<p$. We may assume that $p>2$ and $k\neq 0$, $p-1$ (otherwise Theorem 3.4 applies).

If $k=1$ then

$$(ab^{-1})^{-1}(bx)(ab^{-1})=(bx)^{-2}$$

in the group G defined by $\mathbf{P}_1$, and so bx represents an element of order dividing $(-2)^p-1$ in G. However, bx is not conjugate in G to any element of H (in fact, bx does not even belong to the normal closure of H in G), so $\mathbf{P}_1$ is not aspherical, by Theorem 1.4.

If $p=k+2$ then

$$(a^{-1}b)(xa)(a^{-1}b)^{-1}=(xa)^{-2} \text{ in } G,$$

while if $p=2k+1$ then

$$(c^{-1}a)(xc)(c^{-1}a)^{-1}=(xc)^{-2} \text{ in } G.$$

In both cases, arguments similar to those in the previous paragraph shows that $\mathbf{P}_1$ is not aspherical.

Suppose that $p=6$ and $k=2$ or 3. The substitution $t^{-1}=bx$ show that

$$G=H \underset{ab^{-1}=s}{*} K$$

where K is the group given by the ordinary presentation $\langle s,t;\, s^6, t^2s^{-1}ts^k\rangle$. A computer assisted coset enumeration establishes that K is finite (of order 342). It follows from this, using Theorems 1.4 and 2.4, that $\mathbf{P}_1$ is not aspherical.

Now suppose that $k\neq 1$, $p\neq k+1$, $2k+1$, and that if $p=6$ then $k\geqq 4$. We will show that $\mathbf{P}_1$ is aspherical. A cyclically reduced loop in $\mathbf{P}_1^{\mathrm{st}}$ of length less than 6 is labelled by an element of the form $(ab^{-1})^l$, where $|l|$ is one of 1, 2, k, $2k$, $k+1$, $2(k+1)$, $k-1$, $2k+1$, $k+2$. If none of these labels is trivial then define a weight function θ on $\mathbf{P}_1^{\mathrm{st}}$ by $\theta(\alpha)=\theta(\beta)=\theta(\gamma)=\frac{1}{3}$ so that (3.2) and (3.3) hold. (Alternatively, observe that $\mathbf{P}_1$ satisfies $C(3)$ and $T(6)$.) Otherwise, our assumptions imply that either $p=2k$ and $k\geqq 4$, or $p=2(k+1)$ and $k\geqq 3$. Using the change of variable $t^{-1}=axb$ and appealing to Theorem 2.4, if necessary, allows us to concentrate on showing that $\mathbf{P}_1$ is aspherical when $p=2k$ and $k\geqq 4$.

Define a weight function θ on $\mathbf{P}_1^{\mathrm{st}}$ by $\theta(\alpha)=0$, $\theta(\beta)=\theta(\gamma)=\frac{1}{2}$. Then (3.2) is satisfied. To see that (3.3) holds, first observe that none of the special words A, A^2, A^3, C, C^2, C^3, CA^{-1}, CA^{-2}, C^2A^{-1} is in $\mathrm{Ker}\lambda$, and so any special word in $\mathrm{Ker}\lambda$ not involving B has length at least 4, and thus has weight at least 2. Next, observe that none of B, AB^{-1}, BC^{-1} is in $\mathrm{Ker}\lambda$, and so any special word in $\mathrm{Ker}\lambda$ involving just one occurrence of B has length at least 3, and thus has weight at least 2. Finally, note that a special word involving two or more occurrences of B clearly has weight at least 2.

3.2. *Some relative presentations with one defining relator* (2)

In this section we augment the considerations of §3.1 by discussing relative presentations $\mathbf{P}_2$ of the form

$$\langle H, x;\, xaxbx^{-1}c\rangle$$

where $a, b, c \in H$, and $b \neq 1 \neq c$. Observe that such presentations are orientable.

Theorem 3.5. $\mathbf{P}_2$ *is aspherical except possibly when b and c have finite orders p, q and either* $1/p + 1/q > 1/2$, *or* $a^{-1}ba = c^k$ *for some k, or* $aca^{-1} = b^k$ *for some k.*

Proof. The star complex of $\mathbf{P}_2$ has the form

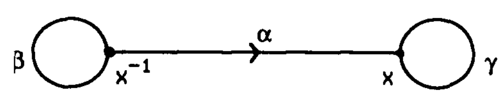

FIGURE 11

where the edges α, β, γ are labelled by a, b, c respectively. For ρ a path in $\mathbf{P}_2^{st}$ and v any one of α, β, γ we define $L_v(\rho)$ to be the total number of occurrences of v and v^{-1} in ρ.

To show that $\mathbf{P}_2$ is aspherical, it suffices to find a non-negative weight function θ on $\mathbf{P}_2^{st}$ such that $\theta(\alpha) + \theta(\beta) + \theta(\gamma) = 1$, and such that the weight of every admissible path is at least 2.

Case 1. b and c have infinite order.

Then $L_\alpha(\rho) \geqq 2$ for every admissible path ρ, and so it suffices to define θ by

$$\theta(\alpha) = 1, \theta(\beta) = \theta(\gamma) = 0.$$

Case 2. b has finite order, c has infinite order.

Then $L_\beta(\rho) \geqq 2$ for admissible path ρ. For otherwise, up to cyclic permutation and inversion, ρ would be one of β, $\beta\alpha\gamma^n\alpha^{-1}$ $(n \neq 0)$. But β is certainly not admissible. Moreover, if $\beta\alpha\gamma^n\alpha^{-1}$ were admissible then the element b (of finite order) would be conjugate in H to the element c^n of infinite order, a contradiction.

We define θ by $\theta(\alpha) = \theta(\gamma) = 0$, $\theta(\beta) = 1$.

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Case 3. b, c have finite order p, q respectively, $1/p+1/q \leqq 1/2$, no relations of the form $a^{-1}ba=c^k$, $aca^{-1}=b^k$ hold.

We define θ by

$$\theta(\alpha)=1-\left(\frac{2}{p}+\frac{2}{q}\right), \theta(\beta)=\frac{2}{p}, \theta(\gamma)=\frac{2}{q}.$$

Let ρ be an admissible path. Certainly if $L_\alpha(\rho)=0$ then $\theta(\rho) \geqq 2$.

Suppose $L_\alpha(\rho) \geqq 2$. We claim that $L_\beta(\rho)$, $L_\gamma(\rho) \geqq 2$. This is obvious if $L_\alpha(\rho) \geqq 4$, and if $L_\alpha(\rho)=2$ then it follows from the fact that no relations of the form $a^{-1}ba=c^k$, $aca^{-1}=b^k$ hold. We now see that

$$\theta(\rho) \geqq 2\left(1-\left(\frac{2}{p}+\frac{2}{q}\right)\right)+2\frac{2}{p}+2\frac{2}{q} \geqq 2.$$

3.3. *Quotients of free products* ("*generalized presentations*")

Consider a *generalized presentation* **[13]**

$$\mathbf{P}=\langle H_i (i \in I);\ \mathbf{u}\rangle. \tag{3.6}$$

Here the H_i are non-trivial groups, and $\mathbf{u}$ is a set of cyclically reduced elements of $H={*}_{i \in I} H_i$ of free product length at least 2. The *group G defined by* **P** is (isomorphic to) the quotient of H by the normal closure K of $\mathbf{u}$ in H.

We denote the set of cyclic permutations of elements of $\mathbf{u} \cup \mathbf{u}^{-1}$ by $\mathbf{u}^*$. We can assume without loss of generality that if $U \in \mathbf{u}^*$ then no cyclic permutation of $U^{\pm 1}$ except for U itself, belongs to $\mathbf{u}^*$. If $U \in \mathbf{u}$ then we may write $U=\mathring{U}^{p(U)}$ where $\mathring{U}$ is not a proper power and $p(U)$ is a positive integer ($\mathring{U}$ is the *root* of U, and $p(U)$ is the *period*).

Let x_i $(i \in I)$ be collection of new symbols. If U is an element $\mathbf{u}$, say $U=u_1 u_2 \dots u_m$ in normal form ($u_\lambda \in H_{i_\lambda}$, $\lambda=1, \dots, m$), then let U_{aug} denote the element

$$x_{i_1} u_1 x_{i_1}^{-1} 1\, x_{i_2} u_2 x_{i_2}^{-1} 1 \dots x_{i_m} u_m x_{i_m}^{-1} 1$$

of $H * \langle x_i\ (i \in I)\rangle$. (Here 1 is the identity of H.) Define $\mathbf{P}_{\text{aug}}$ to be the relative presentation

$$\langle H, x_i\ (i \in I);\ \mathbf{u}_{\text{aug}}\rangle,$$

where $\mathbf{u}_{\text{aug}}=\{U_{\text{aug}}\colon U \in \mathbf{u}\}$.

We make the following assumptions.

$$\textit{No element of } \mathbf{u} \textit{ is a cyclic permutation of its inverse.} \tag{3.7}$$

$$\mathbf{P}_{\text{aug}} \textit{ is aspherical.} \tag{3.8}$$

Lemma 3.2. *The natural homomorphisms $H_i \to G$ $(i \in I)$ are injective. If $U \in \mathbf{u}$ then $\mathring{U}$ defines an element of order $p(U)$ in G.*

Proof. Let ϕ: $H \mapsto G$, ϕ_{aug}: $H * X \to G_{\text{aug}}$ be the natural epimorphisms (where X is the free group on $\{x_i;\, i \in I\}$ and G_{aug} is the group defined by $\mathbf{P}_{\text{aug}}$). Let ψ be the automorphism of $H * X$ given by

$$h_i \mapsto x_i h_i x_i^{-1} (h_i \in H_i),\ x_i \mapsto x_i$$

for each $i \in I$. Then there is an induced isomorphism

$$\psi^*\colon G * X \to G_{\text{aug}},$$

where $\psi^*(\phi * id) = \phi_{\text{aug}}\psi$. By Theorem 1.1, Corollary 1, the restriction of ϕ_{aug} to H is injective. Let $h \in H_i$, and suppose $\phi(h_i) = 1$. Then $(\phi * id)(x_i^{-1} h_i x_i) = 1$, so $\phi_{\text{aug}}\psi(x_i^{-1} h_i x_i) = 1$. Thus $\phi_{\text{aug}}(h_i) = 1$, which implies that $h_i = 1$, as required.

The second statement follows from the fact that $\phi_{\text{aug}}(\psi(\mathring{U}))$ has order $p(U)$ by Theorem 1.1, Corollary 4.

For each $i \in I$ choose a presentation $\langle \mathbf{y}_i; \mathbf{s}_i \rangle$ of H_i. Then there is a homomorphism ϕ_i from the free group on $\mathbf{y}_i$ onto H_i with kernel the normal closure of $\mathbf{s}_i$. A *lift* of H_i is a choice of exactly one element of $\phi_i^{-1}(h)$ for each $h \in H_i$. (We assume that the chosen element of $\phi_i^{-1}(h)$ is expressed in freely reduced form for each $h \in H_i$). We then get an induced lift of $H = *_{i \in I} H_i$ via $*_{i \in I}\, \phi_i$. Let $\tilde{\mathbf{u}}$ be the lift of $\mathbf{u}$, and let

$$\tilde{\mathbf{P}} = \langle \mathbf{y}_i (i \in I);\ \mathbf{s}_i (i \in I), \tilde{\mathbf{u}} \rangle.$$

Then $\tilde{\mathbf{P}}$ is an ordinary presentation defining the same group G as $\mathbf{P}$.

The lift of H also induces a lift $\tilde{\mathbf{P}}_{\text{aug}}$ of $\mathbf{P}_{\text{aug}}$:

$$\tilde{\mathbf{P}}_{\text{aug}} = \langle \mathbf{y}_i (i \in I), x_i (i \in I);\ \mathbf{s}_i (i \in I), \tilde{\mathbf{u}}_{\text{aug}} \rangle.$$

Let $\mathbf{s} = \bigcup_{i \in I} \mathbf{s}_i$, and let $\mathbf{w}$, $\mathbf{w}_{\text{aug}}$ denote the set of words in $\bigcup_{i \in I} \mathbf{y}_i$, $\bigcup_{i \in I} (\mathbf{y}_i \cup \{x_i\})$, respectively.

Lemma 3.3. *If σ is an identity sequence over $\tilde{\mathbf{P}}$ then σ is equivalent to a sequence all of whose terms are in $\mathbf{s}^{\mathbf{w}}$.*

Proof. The proof is by induction on the number $d(\sigma)$ of terms of σ not in $\mathbf{s}^{\mathbf{w}}$. If $d(\sigma) = 0$ the result holds.

Suppose $d(\sigma) > 0$. Let $\tilde{\mathbb{P}}$ be a spherical picture representing σ (Lemma 1.3). We can convert $\tilde{\mathbb{P}}$ to a spherical picture $\tilde{\mathbb{P}}_{\text{aug}}$ over $\tilde{\mathbf{P}}_{\text{aug}}$ as follows.

(a) For each arc labelled by an element $y \in \mathbf{y}_i \cup \mathbf{y}_i^{-1}$ replace it by three parallel arcs.

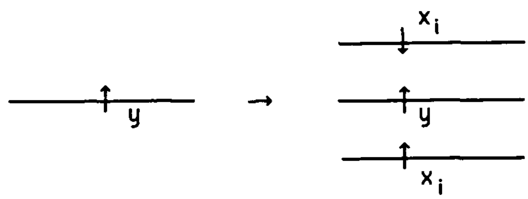

FIGURE 12

(b) If, when reading around a disc we have two successive arcs labelled by x_i, x_i^{-1}, then perform a bridge move to cancel them.

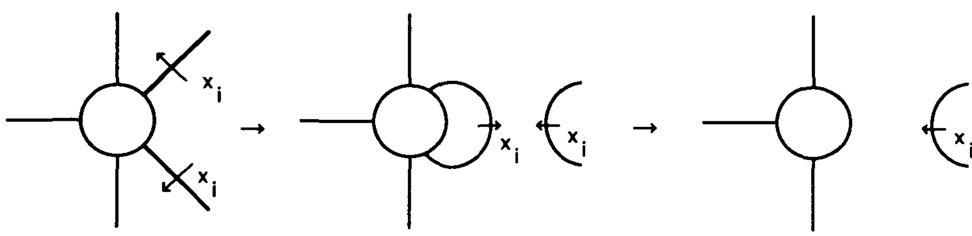

FIGURE 13

(c) Remove any "floating" x_i-arcs.

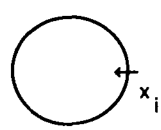

FIGURE 14

Now $\tilde{\mathbb{P}}_{\mathrm{aug}}$ contains a $\tilde{\mathbf{u}}_{\mathrm{aug}}$-dipole (Lemma 1.5). There is therefore a spray γ in $\tilde{\mathbb{P}}_{\mathrm{aug}}$ such that the first two terms of the sequence associated with γ are in $\hat{\mathbf{u}}^{w_{\mathrm{aug}}}_{\mathrm{aug}}$, and are mutually inverse. Now recover $\tilde{\mathbb{P}}$ from $\tilde{\mathbb{P}}_{\mathrm{aug}}$ by rubbing out all the x_i-arcs. Then γ is a spray in $\tilde{\mathbb{P}}$, and in the associated sequence the first two terms are mutually inverse elements of $\tilde{\mathbf{u}}^{w}$. Deleting these two terms gives a sequence σ' equivalent to σ, with $d(\sigma') = d(\sigma) - 2$.

Let M be the relation module corresponding to the presentation $\mathbf{P}$. For $i \in \boldsymbol{I}$, let M_i be the submodule of M generated by the cosets of the elements of $\mathbf{s}_i$, and let M_{H_i} be the relation module corresponding to the presentation $\langle \mathbf{y}_i; \mathbf{s}_i \rangle$ of H_i. Finally, for $U \in \mathbf{u}$, let M_U denote the relation module of the standard presentation of the cyclic group C_U of order $p(U)$.

Now the M_i $(i \in I)$ generate their direct sum in M, as is easily seen by considering their images under the standard embedding of M into a free module [**9**, p. 119]. Moreover,

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$$M_i \cong \mathbb{Z}G \otimes_{\mathbb{Z}H_i} M_{H_i} \, (i \in I)$$

(see [22, Lemma 2]).

Using the above remarks, together with arguments based on Lemma 3.3, we can obtain the following results (cf. Theorems 1.2, 1.3, 1.4).

Theorem 3.6. $M \cong (\oplus_{i \in I} \mathbb{Z}G \otimes_{\mathbb{Z}H_i} M_{H_i}) \oplus (\oplus_{U \in \mathbf{u}} \mathbb{Z}G \otimes_{\mathbb{Z}C_U} M_U)$.

Theorem 3.7. *For any left $\mathbb{Z}G$-module A, and any right $\mathbb{Z}G$-module B we have*

$$H^n(G,A) \cong \left(\prod_{i \in I} H^n(H_i, A)\right) \oplus \left(\prod_{U \in \mathbf{u}} H^n(sgp\{\mathring{U}K\}, A)\right)$$

$$H_n(G,B) \cong \left(\bigoplus_{i \in I} H_n(H_i, B)\right) \oplus \left(\bigoplus_{U \in \mathbf{u}} H_n(sgp\{\mathring{U}K\}, B)\right)$$

for all $n \geqq 3$.

Theorem 3.8. *Any finite subgroup of G is contained in a conjugate of some H_i $(i \in I)$ or in a conjugate of some $sgp\{\mathring{U}K\}$ $(U \in \mathbf{u})$.*

In order to make practical use of the above results, we need ways of telling when a generalized presentation satisfies (3.7) and (3.8). Verification that (3.7) holds is of course just a simple matter of inspection. In the next two sections we will give some sufficient conditions for (3.8) to hold. These make use of the *star-complex* of a generalized presentation, which we now define.

Let $\mathbf{P}$ be as in (3.6). The inversion operator $^{-1}$ is an involution on $\mathbf{u}^*$, and this operation has no fixed points (since we are assuming that the elements of $\mathbf{u}$ are cyclically reduced). We will need another involution $^{-}$ on $\mathbf{u}^*$. Let $U \in \mathbf{u}$, say

$$U = u_1 u_2 \dots u_m \tag{3.9}$$

in normal form $(u_\lambda \in H_{u_\lambda}, \lambda = 1, \dots, m)$. Then we define $\bar{U}$ to be

$$u_1^{-1} u_m^{-1} \dots u_2^{-1}.$$

(Thus, if we write $U = u_1 V$, then $\bar{U} = u_1^{-1} V^{-1}$.) The necessary and sufficient conditions for $^{-}$ to have no fixed points is that (3.7) holds.

Assuming that (3.7) holds, we can define the star-complex $\mathbf{P}^{st}$ of $\mathbf{P}$ to be the labelled 1-complex specified as follows.

Vertex set: $I \times \{-1, 1\}$.

Edge set: $\mathbf{u}^* \times \{-1, 1\}$.

Operations: Let $U \in \mathbf{u}^*$ and suppose the first letter of U lies in H_i and the last lies in H_j. Then

$$\iota(U, 1) = (i, 1),\ \tau(U, 1) = (j, 1),\ (U, 1)^{-1} = (U^{-1}, 1)$$

$$\iota(U, -1) = \tau(U, -1) = (i, -1),\ (U, -1)^{-1} = (\bar{U}, -1).$$

Labelling: An edge of the form $(U, 1)$ is labelled by $1 \in H$. An edge of the form $(U, -1)$ is labelled by the inverse of the first letter of U.

The important fact about $\mathbf{P}^{\mathrm{st}}$ is that it is isomorphic (as a labelled 1-complex) to $\mathbf{P}^{\mathrm{st}}_{\mathrm{aug}}$, where the isomorphism is specified as follows. Let U be an element of $\mathbf{u}^*$ as in (3.9). Then the edges $(U, 1)$, $(U, -1)$ of $\mathbf{P}^{\mathrm{st}}$ are mapped to the edges

$$x_{i_1} u_1 x_{i_1}^{-1} 1 x_{i_2} u_2 x_{i_2}^{-1} 1 \ldots x_{i_m} u_m x_{i_m}^{-1} 1,\ x_{i_1}^{-1} 1 x_{i_2} u_2 x_{i_2}^{-1} 1 \ldots x_{i_m} u_m x_{i_m}^{-1} 1 x_{i_1} u_1$$

of $\mathbf{P}^{\mathrm{st}}_{\mathrm{aug}}$, respectively.

3.4. *Small cancellation quotients of free products* (*Theorem of Collins and Perraud*)

We use the notation of the previous section.

Let W be an element of H, say $W = w_1 w_2 \ldots w_n$ in normal form. An *initial segment* of W is a string $w_1 w_2 \ldots w_t$ with $0 < t \leqq n$. A *semi-initial segment* of W is a string $w_1 w_2 \ldots w_{t-1} h$ where $0 < t \leqq n$ and where h lies in the same factor of H as w_t. Let W_1 be an initial segment of W, and let W' be what is left after removing W_1. Let W_2 be an initial segment of W', and let W'' be what is left after removing W_2 and so on. Continuing this way, we get a *factorization* $W_1 W_2 \ldots W_l$ of W.

Let $U \in \mathbf{u}^*$, and let $U_1 U_2 \ldots U_l$ be a factorization of U. We call this a *factorization into pieces* if, for $\lambda = 1, \ldots, l$, U_λ is a semi-initial segment of some element of $\mathbf{u}^*$ different from $U_\lambda U_{\lambda+1} \ldots U_l U_1 \ldots U_{\lambda-1}$.

C(p): If $U_1 U_2 \ldots U_l$ is a factorization of an element of $\mathbf{u}^*$ into pieces, then $l \geqq p$.

T(q): There is no admissible path in $\mathbf{P}^{\mathrm{st}}$ of length m, with $3 \leqq m < q$.

Note that, because of the isomorphism between $\mathbf{P}^{\mathrm{st}}$ and $\mathbf{P}^{\mathrm{st}}_{\mathrm{aug}}$ (see § 3.3) we have:

$$\mathbf{P} \textit{ satisfies } T(q) \textit{ if and only if } \mathbf{P}_{\mathrm{aug}} \textit{ satisfies } T(q). \tag{3.10}$$

The following result can also be proved (the proof is left as an exercise for the reader).

If $\mathbf{P}$ *satisfies* $C(p)$, *then* $\mathbf{P}_{\text{aug}}$ *satisfies* $C(p)$. (3.11)

We now deduce from (3.10), (3.11) and Theorem 2.2 that if $\mathbf{P}$ satisfies $C(p)$,

$$T(q)\left(\frac{1}{p}+\frac{1}{q}=\frac{1}{2}\right)$$

then (3.10) holds, and we can apply Theorems 3.6, 3.7, 3.8. This gives a proof of results of Collins and Perraud [**5**].

We now give a few specific examples illustrating the above.

Example 1. (Kanevskiĭ's groups [**16**], and generalizations).

Let A_i $(i \in I)$ be a collection of cyclic groups of order 2, with A_i generated by a_i. Let T be a set of ordered triples of elements of I with the properties:

(a) if $(i, j, k) \in T$ then all three of i, j, k are distinct;

(b) if two triples in T have more than one element in common, then the triples coincide.

Let ϕ be a function from T to $\{2, 3, 4, \ldots\}$, and let

$$U_{(i,j,k)} = (a_i a_j a_k)^{\phi(i,j,k)} ((i, j, k) \in T).$$

Let

$$\mathbf{K} = \langle A_i \ (i \in I);\ U_{(i,j,k)} \ ((i, j, k) \in T)\rangle.$$

Then (3.7) obviously holds for $\mathbf{K}$, and (3.8) holds because $\mathbf{K}$ satisfies $C(6)$.

The case when $\mathrm{Im}\phi = \{2\}$ was studied by Kanevskiĭ [**16**] (see also [**17, 21**]). Kanevskiĭ was interested in the torsion in the group defined by the presentation. His result (and, in fact, a more general result than his) follows from Theorem 3.8.

Example 2. Let

$$\mathbf{P} = \langle A, B, C, D;\ a_\lambda b_\lambda c_\lambda d_\lambda \ (\lambda \in \Lambda)\rangle$$

where $a_\lambda \in A - \{1\}$, $b_\lambda \in B - \{1\}$ and so on $(\lambda \in \Lambda)$. Suppose that: (i) $a_\lambda \neq a_\mu, a_\mu^2, a_\mu^{-2} (\lambda \neq \mu)$; (ii) a_λ does not have order 3 $(\lambda \in \Lambda)$; $a_\lambda^{\varepsilon_\lambda} a_\mu^{\varepsilon_\mu} a_\nu^{\varepsilon_\nu} \neq 1$ (λ, μ, ν distinct elements of Λ, ε_λ, ε_μ, $\varepsilon_\nu \in \{1, -1\}$), and similarly for the b's, c's and d's. Then (3.7) holds for $\mathbf{P}$, and (3.8) holds because $\mathbf{P}$ satisfies $C(4)$ and $T(4)$.

We do not give any example illustrating the $C(3)$, $T(6)$ case. The reason is the following:

Theorem 3.9. *Any $T(6)$ generalized presentation satisfies $C(6)$.*

Proof. Let $\mathbf{P}$ be as in (3.6) and suppose that $\mathbf{P}$ satisfies $T(6)$. We claim that in any factorization of an element of $\mathbf{u}^*$ into pieces, each term of the factorization has length 1. For otherwise there would be distinct edges $e=(U,1)$, $f=(V,1)$ of $\mathbf{P}^{\mathrm{st}}$ with $\iota(e)=\iota(f)$, $\tau(e)=\tau(f)$. Then $(ef^{-1})^2$ would be an admissible path of length 4, contradicting $T(6)$.

It now suffices to show that each element U of $\mathbf{u}$ has length at least 6. Write $U=u_1u_2\dots u_m$ in normal form, as in (3.9). Now the edges

$$(u_i\dots u_m u_1\dots u_{i-1},1)\ i=1,\dots,m$$

of $\mathbf{P}^{\mathrm{st}}$ constitute an admissible path ρ. If $m=3$, 4, 5 then we get an immediate contradiction to $T(6)$, while if $m=2$ we get a contradiction by considering ρ^2.

3.5. *Weight test for quotients of free products*

We continue the notation of § 3.3.

The isomorphism between $\mathbf{P}^{\mathrm{st}}$ and $\mathbf{P}^{\mathrm{st}}_{\mathrm{aug}}$ allows us to formulate a weight test for $\mathbf{P}^{\mathrm{st}}$ which guarantees that $\mathbf{P}_{\mathrm{aug}}$ is aspherical. Specifically, suppose there is a non-negative weight function θ on $\mathbf{P}^{\mathrm{st}}$ satisfying the following conditions:

(i) The weight of every admissible cyclically reduced path in $\mathbf{P}^{\mathrm{st}}$ is at least 2.

(ii) If U is an element of $\mathbf{u}$ as in (3.9). then

$$\sum_{\lambda=1}^{m}(2-\theta(u_\lambda\dots u_m u_1\dots u_{\lambda-1},1)-\theta(u_\lambda\dots u_m u_1\dots u_{\lambda-1},-1))\geqq 2.$$

Then $\mathbf{P}_{\mathrm{aug}}$ is aspherical.

Example 3. Let

$$\mathbf{P}=\langle A,B,C,D,E;\ a_1b_1e_1d_1b_2c_1,\ (a_2c_2)^k,\ (d_2e_2)^l\rangle$$

where $a_1,a_2\in A-\{1\}$, b_1, $b_2\in B-\{1\}$ and so on, and where k, l are positive integers. Let p_A denote the length of a shortest non-empty freely reduced word in a_1 and a_2 which is equal to 1 in A (if no such word exists, put $p_A=\infty$). Define p_B, p_C etc. similarly. Then $\mathbf{P}$ satisfies (3.7) provided not both a_2,c_2 have order 2, and not both d_2, e_2 have order 2. Also, using the weight test, we have that $\mathbf{P}$ satisfies (3.8) if

$$\frac{1}{p_A}+\frac{1}{p_C}+\frac{1}{k}\leqq 1$$

$$\frac{1}{p_D}+\frac{1}{p_E}+\frac{1}{l}\leqq 1.$$

Appropriate weights on $\mathbf{P}^{\mathrm{st}}$ are given below.

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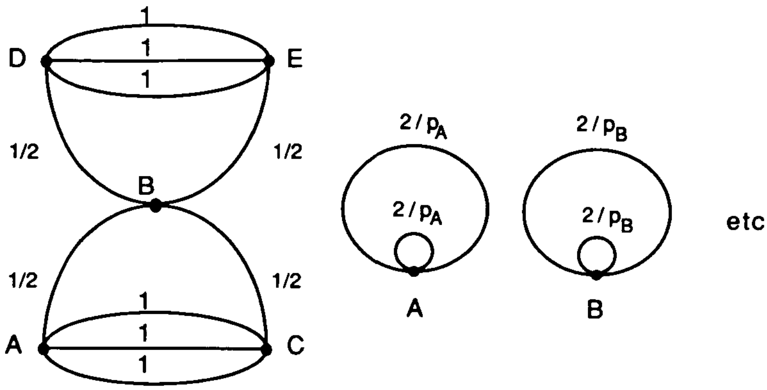

FIGURE 15

3.6. *LOG-presentations*

Consider a group presentation $\mathbf{L}$ in which each relator has the form

$$ilt^{-1}l^{-1} \tag{3.12}$$

where i, t, l are generators. Associated to $\mathbf{L}$ are three geometric graphs, each with vertex set equal to the set of generators, and having one (geometric) edge for each relator. For the first of these, denoted LOG($\mathbf{L}$), the edge corresponding to the relator (3.12) has initial vertex i, terminal vertex t, and is labelled by the vertex (i.e. generator) l. The graph LOG($\mathbf{L}$) is a *labelled oriented graph*, and $\mathbf{L}$ is an *LOG-presentation* (see **[1, 10]**). The other two graphs are $I(\mathbf{L})$ and $T(\mathbf{L})$; these are unlabelled and undirected. Corresponding to the relator (3.12), there is an edge with endpoints i and l in $I(\mathbf{L})$, and there is an edge in $T(\mathbf{L})$ with endpoints t and l.

An LOG-presentation $\mathbf{L}$ is said to be the *join* of LOG-presentations $\mathbf{L}'$, $\mathbf{L}''$ if $\mathbf{L}=\mathbf{L}'\cup\mathbf{L}''$, where $\mathbf{L}'$ and $\mathbf{L}''$ intersect in a single generator. A *shelling* of an LOG-presentation $\mathbf{L}$ is a filtration

$$\mathbf{K}=\mathbf{L}_0\subseteq\mathbf{L}_1\ldots\subseteq\mathbf{L}_n=\mathbf{L}$$

where $\mathbf{L}_0, \mathbf{L}_1, \ldots, \mathbf{L}_n$ are LOG-presentations, and for $m=1,\ldots,n$, *either*: (i) $\mathbf{L}_m$ is the join of $\mathbf{L}_{m-1}$ and an LOG-presentation whose I-graph or T-graph is a forest; *or* (ii) $\mathbf{L}_{m-1}$ is a subpresentation of $\mathbf{L}_m$, and each edge of LOG($\mathbf{L}_m$)$-$LOG($\mathbf{L}_{m-1}$) is extremal in LOG($\mathbf{L}_m$). We call $\mathbf{K}$ the *core* of the shelling.

If $\mathbf{K}$ is a subpresentation of an (arbitrary) presentation $\mathbf{L}$, then there is a relative presentation $\mathbf{L}//\mathbf{K}$, which defines the same group as $\mathbf{L}$. One simply removes the relators of $\mathbf{K}$, replaces the generators of $\mathbf{K}$ by the group presented by $\mathbf{K}$, and views the occurrences of generators of $\mathbf{K}$ in relators of $\mathbf{L}-\mathbf{K}$ as coefficients.

The proof of the main result of **[1]** demonstrates the following

Theorem 3.10. *If* $\mathbf{K}$ *is the core of a shelling of an LOG-presentation* $\mathbf{L}$, *then* $\mathbf{L}//\mathbf{K}$ *is aspherical.*

Since $\mathbf{L}//\mathbf{K}$ is orientable, the results of § 1.7 apply.

Briefly, the proof of Theorem 3.10 uses covering space topology to detect dipoles in spherical pictures over $\mathbf{L}//\mathbf{K}$.

See [1] for explicit examples.

4. Topological aspects

In this section we present an independent topological treatment of the theory of aspherical orientable relative presentations. The combinatorial discussions of §§ 1.5, 1.6 are made topological; Theorems 1.1 and 1.2 are recast in homotopy-theoretic terms. The main results of § 1.7 (see (0.1)–(0.4) of the Introduction) follow from a theorem of Howie (Theorem 4.2) in [12]) and standard arguments. We also discuss generalizations which arise naturally from the topological viewpoint, including the generalized presentations of § 3.3.

Let $\mathbf{P}=\langle H,\mathbf{x};\mathbf{r}\rangle$ be an aspherical orientable relative presentation for a group G. Select a $K(H,1)$-space $\mathbf{K}$ and let $\mathbf{K}_1=\mathbf{K}\vee\bigvee_{x\in\mathbf{x}}\mathbf{S}^1_x$. For each $R\in\mathbf{r}$, let $\phi_R\colon\mathbf{S}^1_R\to\mathbf{K}_1$ be a (based) loop in $\mathbf{K}_1^{(1)}$ representing the root $\mathring{R}$ in $\pi_1\mathbf{K}_1\cong H*\text{free}(\mathbf{x})$. Also, let $\mathbf{D}_{p(R)}$ be a $K(\mathbb{Z}/p(R)\mathbb{Z},1)$-space with two-skeleton modelled on the presentation $\langle\rho;\,\rho^{p(R)}\rangle$; thus, $\mathbf{D}^{(1)}_{p(R)}=\mathbf{S}^1_R$. Let $\mathbf{M}$ be the adjunction space as in the pushout diagram

$$\begin{array}{ccc} \bigvee_{R\in\mathbf{r}}\mathbf{S}^1_R & \longrightarrow & \bigvee_{R\in\mathbf{R}}\mathbf{D}_{p(R)} \\ {\scriptstyle\langle\phi_R\rangle}\downarrow & p.o. & \downarrow{\scriptstyle\Phi} \\ \mathbf{K}_1 & \longrightarrow & \mathbf{M} \end{array} \tag{4.1}$$

of CW complexes. Then $\mathbf{K}\cup\mathbf{M}^{(2)}=\mathbf{K}_1\cup\bigcup_{R\in\mathbf{r}}c^2_R$ where c^2_R is a two-cell attached to $\mathbf{K}_1$ along a (based) loop representing $R\in H*\text{free}(\mathbf{x})\cong\pi_1\mathbf{K}_1$; in particular, $\pi_1\mathbf{M}\cong\pi_1(\mathbf{K}\cup\mathbf{M}^{(2)})\cong G$. Comparing with § 1.6, the choice of $\mathbf{K}$ is analogous to the choice of the ordinary presentation $\mathbf{Q}$ for H, while the choice of the maps ϕ_R parallels the lifting of $\mathbf{r}$ to words. The two-skeleton $\mathbf{M}^{(2)}$ models the lifted ordinary presentation $\mathbf{P}$ for G. The following is analogous to Theorem 1.1.

Theorem 4.1. $\pi_2(\mathbf{M},\mathbf{K})=0$.

Proof. As in Proposition 2 of [15], any map $\eta\colon(\mathbf{B}^2,\mathbf{S}^1)\to(\mathbf{M},\mathbf{K})$ can be represented by a picture $\mathbb{P}$ over a lifted presentation $\mathbf{P}$ for $G\cong\pi_1\mathbf{M}$. The pictures $\mathbb{P}$ determines the pictures $\mathbb{P}$ determines the homotopy class of η. Since $\eta(\mathbf{S}^1)\subseteq\mathbf{K}$, there are no $\mathbf{x}$-arcs meeting $\partial\mathbb{P}$. By Lemma 1.5, either $\mathbb{P}$ has no $\mathbf{r}$-discs, in which case η represents an element of $\pi_2(\mathbf{K}_1,\mathbf{K})=0$, or else $\mathbb{P}$ contains an $\mathbf{r}$-dipole. In the presence of the three-cells homotopic relative $\mathbf{S}^1$ to a map $(\mathbf{B}^2, \mathbf{S}^1) \to (\mathbf{M}, \mathbf{K})$ with a representative picture having two fewer $\mathbf{r}$-discs than $\mathbb{P}$. Induction completes the proof.

Using the long exact homotopy sequence for $(\mathbf{M}, \mathbf{K})$, the result (0.1) of the Introduction follows immediately, as does the fact that $\pi_2\mathbf{M} = 0$. Since the three-cell of $\mathbf{D}_{p(R)}$ is attached along a homologically trivial spherical map, Theorem 4.1 further implies that the composite

$$\pi_2(\mathbf{K} \cup \mathbf{M}^{(2)}) \to \pi_2(\mathbf{K} \cup \mathbf{M}^{(2)}, \mathbf{K}_1) \to H_2(\mathbf{K} \cup \mathbf{M}^{(2)}, \mathbf{K}_1)$$

is trivial (a restatement of Corollary 3 to Theorem 1.1). The result (0.2) follows using the argument of Huebschmann [**14**], as in the proof of Corollary 4 to Theorem 1.1.

To prove an analogue of Theorem 1.2, we employ a result of Howie.

Theorem ([**12**, Theorem 4.2]). *Let the CW complex* $\mathbf{X}$ *be a pushout as in the diagram*

$$\begin{tikzcd} \mathbf{X}_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "p.o."] & \mathbf{X}_1 \arrow[d] \\ \mathbf{X}_2 \arrow[r] & \mathbf{X} \end{tikzcd}$$

of aspherical CW complexes and cellular maps. Suppose $n \geqq 2$ *and that for each choice of basepoint*

(i) $hd(\ker(\pi_1\mathbf{X}_0 \to \pi_1\mathbf{X})) \leqq n-1$,

(ii) $hd(\ker(\pi_1\mathbf{X}_i \to \pi_1\mathbf{X})) \leqq n\,(i=1,2)$, *and*

(iii) $\pi_j\mathbf{X} = 0, 2 \leqq j \leqq n$.

Then, $\mathbf{X}$ *is aspherical.*

We remark that Howie states this theorem for the case where $\mathbf{X}$ is the union of aspherical subcomplexes $\mathbf{X}_1$ and $\mathbf{X}_2$ with aspherical intersection $\mathbf{X}_0$. This generalizes to pushouts using the mapping cylinder construction.

Theorem 4.2. $\mathbf{M}$ *is aspherical:* $\pi_j\mathbf{M} = 0$ *for* $j \geqq 2$.

Proof. Apply Howie's theorem to the pushout diagram (4.1) and take $n = 2$. The conditions (i) and (ii) hold by (0.1), (0.2) and the subgroup theorem for free products. The condition (iii) follows from Theorem 4.1 as above.

The results (0.3) and (0.4) now follow from Theorem 4.2 using standard arguments and the theorem of Serre [**14**].

Consider next a generalized presentation

$$\mathbf{P} = \langle H_i\,(i \in I);\, \mathbf{u} \rangle$$

for a group G as in (3.6), and assume that (3.7) and (3.8) are satisfied. A topological model for $\mathbf{P}$ is constructed as follows. For $i \in I$, let $\mathbf{K}_i$ be a $K(H_i, 1)$-space, and let $\mathbf{W}$ be obtained from the disjoint union $\bigcup_{i \in I} \mathbf{K}_i$ by adding a disjoint zero-cell e^0, together with oriented one-cells e_i^1 $(i \in I)$, where e_i^1 has initial point e^0 and terminal point in $\mathbf{K}_i$. For $U \in \mathbf{u}$, let $\sigma_U: \mathbf{S}_U^1 \to \mathbf{W}$ be a (based) loop in $\mathbf{W}^{(1)}$ realizing the root $U \in *_{i \in I} H_i \cong \pi_1 \mathbf{W}$. Also, let $\mathbf{D}_{p(U)}$ be a $K(\mathbb{Z}/p(U)\mathbb{Z}, 1)$-space with two-skeleton modelled on $\langle v : v^{p(U)} \rangle$. Let $\mathbf{Y}$ be the adjunction space as in the pushout diagram

$$\begin{tikzcd} \bigvee_{U \in \mathbf{u}} \mathbf{S}_U^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "p.o."] & \bigvee_{U \in \mathbf{u}} \mathbf{D}_{p(U)} \arrow[d] \\ \mathbf{W} \arrow[r] & \mathbf{Y} \end{tikzcd} \tag{4.2}$$

of CW complexes. This compares with §3.3 as follows. Let $q: \mathbf{Y} \to \mathbf{M}$ be the quotient map which identifies all endpoints of the one-cells e_i^1 $(i \in I)$ with the zero-cell e^0. Set $\mathbf{K} = q(\bigcup_{i \in I} \mathbf{K}_i)$ and $\mathbf{K}_1 = q(\mathbf{W})$. Then the triple $(\mathbf{M}, \mathbf{K}_1, \mathbf{K})$ models the augmented relative presentation $\mathbf{P}_{\mathrm{aug}}$ as in (4.1). The following is analogous to Lemma 3.3.

Lemma 4.1. $\pi_2 \mathbf{Y} = 0$.

Proof. The map $q_\#: \pi_1 \mathbf{Y} \to \pi_1 \mathbf{M}$ is the inclusion of a free factor, and $0 = \pi_2 \mathbf{M} \cong \mathbb{Z}\pi_1 \mathbf{M} \otimes_{\pi_1 \mathbf{Y}} \pi_2 \mathbf{Y}$, which implies that $\pi_2 \mathbf{Y} = 0$.

Using Lemma 3.2 and Howie's theorem, we obtain:

Theorem 4.3. $\mathbf{Y}$ *is aspherical.*

From this, one may easily deduce Theorem 3.6 and Theorem 3.7.

More generally, one may use these techniques to build Eilenberg–Maclane spaces for the fundamental group of a "generalized two-complex", as defined in [13]. Specifically, let Γ be a (connected) graph of groups with trivial edge groups. For each vertex group H_v, select a $K(H_v, 1)$-space $\mathbf{K}_v$, attached to Γ at v. Given a collection $\mathbf{u}$ of elements of $\pi_1(\Gamma, \{H_i\})$, let G be the quotient of $\pi_1(\Gamma, \{H_i\})$ by the normal closure of $\mathbf{u}$. Identifying the vertices of Γ to a point, there results a topological model for a relative (augmented) presentation; if this is aspherical and orientable, then a $K(G, 1)$-space can be constructed in the manner of (4.1) and (4.2).

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REFERENCES

1. W. A. Bogley, Local collapses for diagrammatic reducibility, in: *Topology and Combinatorial Group Theory* (Lecture Notes in Mathematics **1440** (1990), edited by P. Latiolais, Springer-Verlag).

2. S. G. Brick, Normal-convexity and equations over groups, *Invent. Math.* **94** (1988), 81–104.

3. R. Brown and J. Huebschmann, Identities among relations, in: *Low-dimensional topology* (London Mathematical Society Lecture Note Series, **48** (1982), edited by R. Brown and T. L. Thickston).

4. D. J. Collins and J. Huebschmann, Spherical diagrams and identities among relations, *Math. Ann.* **261** (1982), 155–183.

5. D. J. Collins and J. Perraud, Cohomology and finite subgroups of small cancellation quotients of free products, *Math. Proc. Cambridge Philos. Soc.* **97** (1985), 243–259.

6. E. J. Fennessey and S. J. Pride, Equivalences of two-complexes, with applications to NEC-groups, *Math. Proc. Cambridge Philos. Soc.* **106** (1989), 215–228.

7. S. M. Gersten, Reducible diagrams and equations over groups, in: *Essays in group theory* (MSRI Publications **8** (1987), edited by S. M. Gersten, Springer-Verlag).

8. S. M. Gersten, On certain equations over torsion-free groups (University of Utah), preprint.

9. P. J. Hilton and U. Stammbach, *A course in homological algebra* (Graduate Texts in Mathematics **4** (1971), Springer-Verlag).

10. J. Howie, On the asphericity of ribbon disc complements, *Trans. Amer. Math. Soc.* **289** (1985), 281–302.

11. J. Howie, How to generalize one-relator group theory, in: *Combinatorial group theory and topology* (Annals of Mathematics Studies, **111** (1987), edited by S. M. Gersten and J. R. Stallings, Princeton University Press).

12. J. Howie, The quotient of a free product of groups by a single high-powered relator. I. Pictures. Fifth and higher powers, *Proc. London Math. Soc.* (3) **59** (1989), 507–540.

13. J. Howie and S. J. Pride, A spelling theorem for staggered generalized 2-complexes, with applications, *Invent. Math.* **76** (1984), 55–74.

14. J. Huebschmann, Cohomology theory of aspherical groups and of small cancellation groups, *J. Pure Appl. Algebra* **14** (1979), 137–143.

15. J. Huebschmann, Aspherical 2-complexes and an unsettled problem of J. H. C. Whitehead, *Math. Ann.* **258** (1981), 17–37.

16. D. S. Kanevskiĭ, The structure of groups connected with automorphisms of cubic surfaces, *Math. USSR-Sb.* **32** (1977), 252–264.

17. D. S. Kanevskiĭ, On cubic planes and groups connected with cubic varieties, *J. Algebra* **80** (1983), 559–565.

18. F. Levin, Solutions of equations over groups, *Bull. Amer. Math. Soc.* **68** (1962), 603–604.

19. R. C. Lyndon and P. E. Schupp, *Combinatorial group theory* (Springer-Verlag, 1977).

20. S. J. Pride, Star-complexes and the dependence problems for hyperbolic complexes, *Glasgow Math. J.* **30** (1988), 155–170.

21. S. J. Pride, Involutary presentations, with applications to Coxeter groups, NEC-groups and groups of Kanevskiĭ, *J. Algebra* **120** (1989), 200–223.

22. S. J. Pride, The (co)homology of groups given by presentations in which each defining relator involves at most two types of generators, *J. Austral. Math. Soc.*, to appear.

23. S. J. Pride and R. Stöhr, Relation modules of groups with presentations in which each relator involves exactly two types of generators, *J. London Math. Soc.* (2) **38** (1988), 99–111.

24. J.-P. Serre, *Trees* (Springer-Verlag, 1980).

25. A. Sieradski, A coloring test for asphericity, *Quart. J. Math. Oxford* (2) **34** (1983), 97–106.

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