H. Fujimoto
Nagoya Math. J.
Vol. 46 (1972), 49–61

ON HOLOMORPHIC MAPS INTO A TAUT COMPLEX SPACE

HIROTAKA FUJIMOTO

Introduction. The purpose of this paper is to study the extension problem of holomorphic maps of a complex manifold into a taut complex space, which is defined by analogy with a taut complex manifold given by H. Wu ([11]).

Let D be a domain in a complex manifold M and f be a holomorphic map of D into a taut complex space. We can construct the existence domain of f as in the case of holomorphic functions. We shall first prove the following theorem, which is essentially due to the Docquier-Grauert's theorem ([2]).

THEOREM A. *If D is an (unramified) Riemann domain over a Stein manifold M, the existence domain of f is a Stein manifold.*

Using Theorem A, we can easily prove that, for domains D and D' ($D \subset D'$) in a Stein manifold, if every holomorphic function on D has a holomorphic extension to D', then every holomorphic map of D into a taut complex space X can be extended to a holomorphic map of D' into X.

For holomorphic maps defined on a complex manifold minus an analytic set of codimension one, we have the following improvement of Theorem 5 in [9], p. 18.

THEOREM B. *Let S be an irreducible analytic subset of codimension one in a domain D in C^n and f be a holomorphic map of $D - S$ into a taut complex space X. If f has a cluster value in X at some regular point of S, then f can be extended to a holomorphic map of D into X.*

In connection with Theorem B, we give the following generalization of the big Picard theorem, whose proof is essentially due to [3].

Received May 7, 1971.

THEOREM C. *If S is a regular thin analytic subset of a domain D in $\boldsymbol{C}^n$, then every holomorphic map of $D-S$ into the N-dimensional complex projective space $P_N(\boldsymbol{C})$ minus $2N+1$ hyperplanes in general position can be extended to a holomorphic map of D into $P_N(\boldsymbol{C})$.*

In case of holomorphic maps defined on a complex space, the analogous extension theorem is not valid in general. We shall construct a normal complex space Y of dimension $\geqq 2$ such that there exists a holomorphic map of Y minus one point into a taut complex space which has no holomorphic extension to Y.

§1. **Local extension of holomorphic maps.** In this paper, a complex space means a reduced complex space and all complex spaces and manifolds are assumed to be σ-compact and connected unless stated to the contrary.

For complex spaces M and X, we denote the space of all holomorphic maps of M into X endowed with the compact-open topology by Hol(M, X). A sequence $\{f_\nu\}$ in Hol(M, X) is said to be compactly divergent if, for any compact sets K in M and L in X, there is some ν_0 such that $f_\nu(K) \cap L = \phi$ for any $\nu \geqq \nu_0$.

DEFINITION 1.1. A complex space X is said to be *taut* if Hol(M, X) is normal for any complex manifold M, i.e., any sequence in Hol(M, X) has a subsequence which is convergent in Hol(M, X) or compactly divergent.

For example, a complete hyperbolic complex manifold in the sense of S. Kobayashi [8] is taut. In particular, a Riemann surface which is hyperbolic in the classical sense is a taut complex manifold. Moreover, if a relatively compact subdomain D of a Stein space X can be written $D=\{x\in X;\ u(x)<c\}$ with a plurisubharmonic function $u(x)$ on X and a constant c, then D is a taut complex space (W. Kaup [6], Satz 1.2, p. 306 and Satz 4.5, p. 318).

Now, we give the following definition for the convenience of description.

DEFINITION 1.2. Let D be a domain in a complex manifold M and x_0 be a boundary point of D. We shall say that D saitisfies *the condition* (C) *at* x_0 if there is a sequence of maps $\{\varphi_\nu;\ \nu=1,2,\cdots\}$ of $B:=\{\zeta\in \boldsymbol{C};\ |\zeta|<1\}$ into M such that (1) $\{\varphi_\nu\}$ converges to φ in Hol(B, M), (2) $\varphi_\nu(\zeta)\in D$ for any

ν and $\zeta \in B$, (3) $\varphi(0) = x_0$ and $\varphi(\zeta) \in D$ for any ζ with $0 < |\zeta| < 1$.

LEMMA 1.3. *Assume that a domain D in a complex manifold M satisfies the condition (C) at a boundary point x_0. Then, every holomorphic map f of D into a taut complex space X has a possibly many-valued holomorphic extension to a neighborhood U of x_0, i.e., a holomorphic map $g: U \to X$ such that $f(x) = g(x)$ in some non-empty open subset of $U \cap D$.*

Proof. Let $\varphi(\zeta)$ and $\varphi_\nu(\zeta)$ be holomorphic maps with the properties (1), (2) and (3) in Definition 1.2. Then $\{f \cdot \varphi_\nu\}$ is a sequence in Hol(B, X). And, since $\lim_{\nu \to \infty} f \cdot \varphi_\nu(\zeta_0) = f \cdot \varphi(\zeta_0) \in X$ for an arbitrarily fixed ζ_0 with $0 < |\zeta_0| < 1$, $\{f \cdot \varphi_\nu\}$ cannot have a compactly divergent subsequence and so has a convergent subsequence by Definition 1.1. There is no harm in assuming that $\{f \cdot \varphi_\nu\}$ itself has a limit h in Hol(B, X). Obviously, $h(\zeta) = f \cdot \varphi(\zeta)$ if $0 < |\zeta| < 1$. Put $q_0 = h(0)$ and take a Stein neighborhood V of q_0. Then, for a sufficiently small $\rho > 0$ and a sufficiently large ν_0, we see $(f \cdot \varphi_\nu)(\zeta) \in V$ and $h(\zeta) \in V$ if $\nu \geqq \nu_0$ and $|\zeta| \leqq \rho$. Consider the open set $D' := D \cap f^{-1}(V)$. Obviously, $\varphi(\zeta) \in D'$ if $0 < |\zeta| \leqq \rho$ and $\varphi_\nu(\zeta) \in D'$ if $\nu \geqq \nu_0$ and $|\zeta| \leqq \rho$. So, the set

$$K := \{\varphi(\zeta);\ |\zeta| = \rho\} \cup \cup_{\nu=\nu_0}^{\infty}\{\varphi_\nu(\zeta);\ |\zeta| = \rho\})$$

is compact in D'. As is easily seen, by the maximum principle, the set

$$\hat{K} := \{x \in D';\ |h(x)| \leqq \sup |h(K)| \text{ for any holomorphic function } h \text{ on } D'\}$$

includes $\cup_{\nu=\nu_0}^{\infty}\{\varphi_\nu(\zeta);\ |\zeta| \leqq \rho\}$, whose closure contains x_0. In this situation, by the well-known argument, we can find a neighborhood U of x_0 such that every holomorphic function on D' has a possibly many-valued holomorphic extension to U. We consider the restriction $f|D'$ of f to D' which has the image in a Stein space V. Then, as is well-known, $f|D'$ has a possibly many-valued holomorphic extension $g: U \to V$, which is also considered as an extension of f. This concludes the proof of Lemma 1.3.

LEMMA 1.4. *Let D be a domain in the $(z_1, z_2, \cdots, z_n)$-space and $x_0 := (a_1, \cdots, a_n)$ be a boundary point of it. Consider hyperspheres B with the center (a_1, a_2) and S whose boundary contains (a_1, a_2) in the (z_1, z_2)-space. If D includes the set of all points $(z_1, z_2, a_3, \cdots, a_n)$ such that (z_1, z_2) is contained in the interior of B and in the exterior of S, then D satisfies the condition (C) at x_0.*

Proof. Let (b_1, b_2) be the center of S. We may assume that $a_1 \neq b_1$ and write

$$S : |z_1 - a_1|^2 + |z_2 - a_2|^2 + 2 \times \mathrm{Re}\,((z_1 - a_1)(\bar{a}_1 - \bar{b}_1) + (z_2 - a_2)(\bar{a}_2 - \bar{b}_2)) \leqq 0.$$

For our purpose, it suffices to take the maps defined as follows;

$$\varphi_\nu(\zeta) : z_1 = a_1 + \frac{1}{\bar{a}_1 - \bar{b}_1}\Big(\frac{1}{\nu + \nu_0} - \rho\cdot\zeta(\bar{a}_2 - \bar{b}_2)\Big),\ z_2 = a_2 + \rho\cdot\zeta,\ z_i = a_i\ (3 \leqq i \leqq n),$$

and

$$\varphi(\zeta) : z_1 = a_1 - \rho\cdot\zeta\frac{\bar{a}_2 - \bar{b}_2}{\bar{a}_1 - \bar{b}_1},\ z_2 = a_2 + \rho\cdot\zeta,\ z_i = a_i\ (3 \leqq i \leqq n)$$

for a sufficiently large ν_0 and a sufficiently small $\rho > 0$, which have the properties (1), (2) and (3) in Definition 1.2.

The following proposition on local extensions of holomorphic maps is used later to give a general global extension theorem.

PROPOSITION 1.5. *In C^n, consider the domains*

$$D := \{\delta < |z_1| < \rho_1,\ |z_2| < \rho_2, \cdots, |z_n| < \rho_n\}$$
$$\cup \{|z_1| < \rho_1,\ |z_2| < \rho_2', \cdots, |z_n| < \rho_n'\}$$

and

$$D' := \{|z_1| < \rho_1, \cdots, |z_n| < \rho_n\},$$

where $0 < \delta < \rho_1$, $0 < \rho_i' \leqq \rho_i$ $(2 \leqq i \leqq n)$. Then, every holomorphic map f of D into a taut complex space X can be extended to a holomorphic map of D' into X.

Proof. By virtue of Lemma 1.4, Proposition 1.5 can be proved by the argument as in the proof of the equivalence of two different types of definitions for pseudoconcave sets in Tadokoro [10]. On account of the possiblility of multivalence of the extended function, we need some careful checks. To prove Proposition 1.5, we may assume that $\rho_i = \rho_i'$ $(3 \leqq i \leqq n)$. Indeed, if it is proved in this case, the proof for the general case is easily given by mathematical induction. Take an arbitrary point $z' := (z_1', a_2', \cdots, a_n')$ with $|z_1'| = \dfrac{\delta + \rho_1}{2}$, $|a_2'| < \rho_2, \cdots, |a_n'| < \rho_n$ and consider the straight line $L : z_1 = t\cdot z_1',\ z_2 = a_2', \cdots, z_n = a_n'$ $(0 \leqq t \leqq 1)$. If f is analytically continuable along L just before $a' = (a_1', \cdots, a_n') \in L$ and not continuable to a' itself, then we call the point a' an α-point with respect to z'.

Let E be the set of all α-points in D'. For our purpose, it suffices to show $E = \phi$. Assume the contrary. Then, by the same manner as in [10], pp. 284~285, it can be proved that, for a suitable a_0 in E and local coordinates $w_1, \cdots, w_n$ in a neighborhood of a_0 (let $a_0 = (\alpha_1, \cdots, \alpha_n)$), in the (w_1, w_2)-space we can find a sufficiently small hypersphere B with the center (α_1, α_2) and a hypersphere S whose boundary contains (α_1, α_2) such that the set of all points $(w_1, w_2, \alpha_3, \cdots, \alpha_n)$ with the property that (w_1, w_2) is contained in the interior of B and in the exterior of S does not intersect E. By the definition of E, f can be extended to a single-valued holomorphic map of a neighborhood of E into X. In this situation, f has a possibly many-valued extension g to a neighborhood of $E \cap U$. This contradicts the definition of α-points.

§ 2. **Existence domains of holomorphic maps.** Let M be an (unramified) Riemann domain over a complex manifold N with projection map $\pi: M \to N$ and X be an arbitrary complex space. By $\mathcal{O}^X$ we denote the sheaf of germs of holomorphic maps defined on open subsets of N into X. The set $\mathcal{O}^X$ has a canonically defined structure of complex manifold and the projection $\tilde{\pi}: \mathcal{O}^X \to N$ is locally biholomorphic. Then, by putting $v(f_x) = f(x)$ for each $f_x \in \mathcal{O}^X_x (x \in N)$, we can define a continuous map $v: \mathcal{O}^X \to X$.

Now, let f be a holomorphic map of M into X. We consider the map $\sigma: M \to \mathcal{O}^X$ which assigns the germ of $f \cdot (\pi | U)^{-1}$ at $\pi(x)$ to each point x in M, where U is a neighborhood of x such that $\pi | U: U \to \pi(U)$ is biholomorphic. Obviously, $v \cdot \sigma = f$, $\tilde{\pi}\sigma = \pi$ and σ is continuous. So, $\sigma(M)$ is connected. By $H^f(M)$ we denote the connected component of $\mathcal{O}^X$ which includes $\sigma(M)$. Then, the map $\tilde{\pi} | H^f(M): H^f(M) \to N$ may be considered to define a Riemann domain over N which includes M as a Riemann domain. Moreover, the map $f' := v | H^f(M): H^f(M) \to X$ is a holomorphic extension of f to $H^f(M)$ because $f' \cdot \sigma = f$. As is easily seen, $H^f(M)$ is the largest one among Riemann domains over N which includes M as a Riemann domain and to which f can be holomorphically extended. Modeling after the case of functions, we give

DEFINITION 2.1. For a holomorphic map $f: M \to X$, we call the Riemann domain $H^f(M)$ over N constructed as the above *the existence domain* of f.

THEOREM 2.2. *Let M be a Riemann domain over a Stein manifold and f be a holomorphic map of M into a taut complex space. Then, the existence domain of f is a Stein manifold.*

For the proof we use the Docquier-Grauert's result on the Levi problem for Riemann domains over a Stein manifold. In their paper [2], many equivalent definitions of convexity for a Riemann domain were given. Among them, we shall use here particularly the notion of "p_7-convexity". For a Riemann domain M over N with the projection map $\pi : M \to N$, we denote the set of all (accessible) boundary points of M by ∂M and put $\check{M} = M \cup \partial M$. The set $\check{M}$ has a canonically defined Hausdorff topology and π has a continuous extension $\check{\pi} : \check{M} \to N$. The definition of p_7-convexity is given as follows:

Let $\mathscr{D} := \{(z_1, \cdots, z_n); |z_1| \leqq 1, |z_k| < 1, 2 \leqq k \leqq n\}$, $\delta\mathscr{D} := \{z \in \mathscr{D}; |z_1| = 1\}$, $\overset{0}{\mathscr{D}} := \{z \in \mathscr{D}; |z_1| < 1\}$ and $\bar{\mathscr{D}} := \{|z_k| \leqq 1, 1 \leqq k \leqq n\}$ in $\boldsymbol{C}^n$. A Riemann domain M of dimension n over a Stein manifold N with projection map $\pi : M \to N$ is said to be p_7-convex if and only if there is no continuous map $\varphi : \bar{\mathscr{D}} \to \check{M}$ with the property that 1) $\varphi(\delta\mathscr{D}) \Subset M$, $\varphi(\overset{0}{\mathscr{D}}) \subset M$, 2) $\varphi(\bar{\mathscr{D}}) \cap \partial M \neq \phi$ and 3) $\check{\pi}\varphi$ is the restriction to $\bar{\mathscr{D}}$ of a biholomorphic map of a neighborhood of $\bar{\mathscr{D}}$ onto an open subset of N.

The Docquier-Grauert's result which we need here is the following

THEOREM ([2], Satz 10, p. 113). *Any p_7-convex Riemann domain over a Stein manifold is Stein.*

Proof of Theorem 2.2. Without loss of generality, we may assume $H^f(M) = M$. It suffices to show that M is p_7-convex. Assume that there is a continuous map $\varphi : \bar{\mathscr{D}} \to M$ satisfying the above conditions 1), 2) and 3). By the condition 3), $\check{\pi}\varphi$ is extended to a biholomorphic map ϕ of a neighborhood U of $\bar{\mathscr{D}}$ onto an open set W in N. If we put $G := \varphi^{-1}(M) \cap U$, the map $g := f \cdot \varphi | G : G \to X$ is holomorphic because $\varphi | G$ is a biholomorphic map onto an open set in M. Moreover, since $\delta\mathscr{D} \Subset G$ and $\overset{0}{\mathscr{D}} \subset G$ by the condition 1), we can find real numbers δ, ρ_1, ρ_1' and ρ_i $(2 \leqq i \leqq n)$ with $0 < \delta < 1 < \rho_1$, $0 < \rho_i' < 1 < \rho_i (2 \leqq i \leqq n)$ such that

$$D := \{\delta < |z_1| < \rho_1, \ |z_i| < \rho_i \ (2 \leqq i \leqq n)\} \cup \{|z_1| < \rho_1, \ |z_i| < \rho_i'(2 \leqq i \leqq n)\}$$

is included in G. In this situation, Proposition 1.5 implies that the map $g|D : D \to X$ has a holomorphic extension $h : D' := \{|z_i| < \rho_i \ (1 \leqq i \leqq n)\} \to X$. Here, we may assume $U = D$. On the other hand, by the condition 2), there exists a point $z \in \bar{\mathscr{D}}$ such that $\varphi(z) \in \partial M$. If we take connected neighborhoods $U'(\Subset U)$ of z and $W'(\Subset W)$ of $\check{\pi} \cdot \varphi(z)$, then a suitable connected component W_i' of $\check{\pi}^{-1}(W') \cap M$ gives a neighborhood of $\varphi(z)$ in $\check{M}$

https://doi.org/10.1017/S0027763000014768 Published online by Cambridge University Press

by the definition of the topology of $\check{M}$ and $\pi|W_i'$ is a biholomorphic map onto an open subset $\tilde{W}'$ of W'. The holomorphic map $f\cdot\pi^{-1}|\tilde{W}' : \tilde{W}' \to X$ has a holomorphic extension $h\cdot\psi^{-1}|W' : W' \to X$. It follows from Definition 2.1 that $\pi|W_i' : W_i' \to W'$ is a biholomorphic map. This contradicts the fact $\varphi(z) \in \partial M$ and completes the proof.

As a result of Theorem 2.2, we have the following global extension theorem.

THEOREM 2.3. *Let M be a Riemann domain over a Stein manifold and $H(M)$ be its envelope of holomorphy. Then, for any taut complex space X, every holomorphic map $f : M \to X$ can be extended to a holomorphic map $f : H(M) \to X$.*

Proof. By the definition of the envelope of holomorphy, we have a holomorphic map $\tau : M \to H(M)$ with $\pi = \tilde{\pi}\tau$, where π and $\tilde{\pi}$ are projection maps of Riemann domains M and $H(M)$ respectively. On the other hand, the existence domain $H^f(M)$ is Stein by Theorem 2.1. By the well-known argument the map $\sigma : M \to H^f(M)$ can be extended to a holomorphic map $\tilde{\sigma} : H(M) \to H^f(M)$ with $\tilde{\sigma}\tau = \sigma$. The map $f'\cdot\tilde{\sigma} : H(M) \to X$ is a holomorphic extension of f to $H(M)$, where f' is a holomorphic extension of f to $H^f(M)$.

COROLLARY 2.4 (c.f. H. Wu, [11], p. 211). *If a Riemann domain M over a Stein manifold is taut, then M is Stein.*

Proof. The identity map $\mathrm{id}_M : M \to M$ has a holomorphic extension $h : H(M) \to M$ with $h\tau = \mathrm{id}_M$, where $\tau : M \to H(M)$ is a canonical holomorphic map with $\tilde{\pi}\tau = \pi$. Obviously, τh is the identity map of $H(M)$. This shows that M is biholomorphic with $H(M)$, whence M is Stein.

§ 3. **A counter example.** Theorem 2.2 shows that the simultaneous continuability of holomorphic functions leads to the simultaneous continuability of holomorphic maps defined on subdomains in a Stein manifold into a taut complex space. On the contrary, the analogous extension theorem for holomorphic maps defined on a complex space is not valid in general. In this section, we shall give a counter example.

Let P be a projective algebraic manifold of dimension $\geqq 1$. Moreover, assume that P is hyperbolic in the sense of S. Kobayashi ([8], p. 465 and [9], p. 10). For example, a closed Riemann surface of genus at least two satisfies these conditions. According to H. Grauert [4], taking a suitable line bundle F over P, we can find a relatively compact strongly pseudoconvex neighborhood X of the zero section Z of F. Then, there are a normal complex space M and a holomorphic map $\tau : X \to M$ such that, for a special point x_0 in M, $\tau^{-1}(x_0) = Z$ and $\tau | X - Z : X - Z \to M - \{x_0\}$ is a biholomorphic map (c.f. [4], Satz 5, p. 340). Shrinking a neighborhood X of Z we may assume that M is complete hyperbolic. Then we want to prove that the space X is taut.

By d_P, d_M and d_X we denote the so-called Kobayashi pseudo-distance on P, M and X respectively (c.f. [8], p. 462 and [9], Definition, p. 10). We shall first show that X is hyperbolic. Assume the contrary. Then, there are two points $x, y \in X \, (x \neq y)$ such that $d_X(x, y) = 0$. By the distance-decreasing property of holomorphic maps with respect to the Kobayashi pseudo-distance, we have $d_M(\tau(x), \tau(y)) \leqq d_X(x, y) = 0$. This implies that $\tau(x) = \tau(y)$ from the hyperbolicity of M. So, $\tau(x) = \tau(y) = x_0$, i.e., $x, y \in Z$, because τ is injective on $X - Z$. On the other hand, since F is a line bundle over P, the canonical projection map $\pi : F \to P$ gives a biholomorphic map $\pi | Z : Z \to P$. Again using the distance-decreasing property, we see $d_P(\pi(x), \pi(y)) \leqq d_X(x, y) = 0$. By the hyperbolicity of P, we have $\pi(x) = \pi(y)$ and hence $x = y$, which is a contradiction. Therefore, X is hyperbolic. Now, it is easily proved that X is complete with respect to the distance d_X because M is complete and $\tau : X \to M$ is proper. Accordingly, we come to the conclusion that X is taut.

Consider the map $f := (\tau | X - Z)^{-1}$. It is a holomorphic map of $M - \{x_0\}$ into a taut complex space X which obviously cannot be extended to a holomorphic map of M into X. This gives a desired counter example.

§ 4. Extensions across an analytic subset of codimension one.

Let S be a thin analytic subset of a domain D in $\boldsymbol{C}^n$ and f be a holomorphic map of $D - S$ into a taut complex space X. If $\dim S \leqq n - 2$, f is continuable to a holomorphic map of D into X by virtue of Theorem 2.3. In case of $\dim S = n - 1$, we can prove

THEOREM 4.1. *Assume that, for each irreducible component S_i of S, there is a sequence $\{a_\nu\}$ in $D - S$ which converges to some regular point of S_i such that $\{f(a_\nu)\}$ has a limit point in X. Then f can be extended to a holomorphic map of D into X.*

For the proof, we need the following result of M. H. Kwack in [9].

THEOREM. *Let X be a hyperbolic complex space and f be a holomorphic map of the domain $B^* := \{z \in \boldsymbol{C};\ 0 < |z| < 1\}$ into X. If, for a suitable sequence $\{\alpha_\nu\}$ in B^* with $\lim_{\nu\to\infty} \alpha_\nu = 0$, $\{f(\alpha_\nu)\}$ converges to a point in X, then f can be extended to a holomorphic map of $B := \{|z| < 1\}$ into X.*

Remark. Though the above theorem is slightly modified from the original, we can prove it by the same argument as in the proof of Theorem 3 in [9], p. 14.

Proof of Theorem 4.1. Without loss of generality, we may assume that S is irreducible. As is well-known, the set of all singularities of S is an analytic set of dimension $\leqq n-2$ in D. There is no harm in assuming that S is regular. Moreover, it may be assumed that $D := \{|z_1| < 1, \cdots, |z_n| < 1\}$ and $S := \{z = (z_1, \cdots, z_n) \in D;\ z_n = 0\}$. Indeed, if Theorem 4.1 is proved in this case, the set of all $x \in S$ such that f is continuable to a neighborhood of x is an open and closed subset of S and hence coincides with the whole set S.

Put $a_\nu = (a_1^{(\nu)}, \cdots, a_{n-1}^{(\nu)}, a_n^{(\nu)})$ and $\lim_{\nu\to\infty} a_\nu = (a_1, \cdots, a_{n-1}, 0)$. The condition $a_\nu \in D - S$ implies $0 < |a_n^{(\nu)}| < 1$. For each ν, we take an integer k_ν such that $2^{-(k_\nu+2)} \leqq |a_n^{(\nu)}| < 2^{-(k_\nu+1)}$. Choosing a subsequence and changing indices if necessary, we may assume that $1 \leqq k_1 < k_2 < \cdots$. Moreover, $\{2^{k_\nu} a_n^{(\nu)}\}$ may be assumed to converge to a point b with $\frac{1}{4} \leqq |b| \leqq \frac{1}{2}$. Now, we define the holomorphic maps $f_\nu(z_1, \cdots, z_{n-1}, z_n) := f\left(z_1, \cdots, z_{n-1}, \frac{z_n}{2^{k_\nu}}\right)$ of $D - S$ into X. Then, for $a_\nu = (a_1^{(\nu)}, \cdots, a_{n-1}^{(\nu)}, 2^{k_\nu} a_n^{(\nu)})$, $\{f_\nu(a_\nu)\}$ converges to a point in X and $\lim_{\nu\to\infty} a_\nu = (a_1, \cdots, a_{n-1}, b) \in D - S$. The sequence $\{f_\nu\}$ in $\mathrm{Hol}(D-S, X)$ cannot have a compactly divergent subsequence and hence it has a convergent subsequence because $\mathrm{Hol}(D-S, X)$ is normal. We may assume that $\lim_{\nu\to\infty} f_\nu = g$ exists in $\mathrm{Hol}(D-S, X)$. Take an arbitrary point z' in $D' := \{|z_1| < 1, \cdots, |z_{n-1}| < 1\}$ and put $h(z_n) := f(z', z_n)$, which is considered as a map of $B^* := \{0 < |z_n| < 1\}$ into X. Then for $\alpha_\nu := \frac{1}{2^{k_\nu+1}} (\nu = 1, 2, \cdots)$ $\lim_{\nu\to\infty} \alpha_\nu = 0$ and $\lim_{\nu\to\infty} h(\alpha_\nu) = \lim_{\nu\to\infty} f_\nu\left(z', \frac{1}{2}\right) = g\left(z', \frac{1}{2}\right) \in X$ exists. On the other hand, it was shown by P. Kiernan ([7]) that a taut complex manifold is always hyperbolic and this assertion can be easily generalized to the case of complex spaces. So, X is hyperbolic. According to the above M. H. Kwack's result, h is extended to a map of $B := \{|z_n| < 1\}$ into X.

https://doi.org/10.1017/S0027763000014768 Published online by Cambridge University Press

Thus, we obtain a map $f(z_1, \cdots, z_n): D \to X$, which is an extension of f and holomorphic in z_n for each fixed $z' = (z_1, \cdots, z_{n-1})$ in D'. To complete the proof of Theorem 4.1, it suffices to show that f is continuous on D. To this end, take an arbitrary sequence $z'_\nu = (z_1^{(\nu)}, \cdots, z_{n-1}^{(\nu)})$ in D' with $\lim_{\nu\to\infty} z'_\nu = z'_0 =: (z_1^{(0)}, \cdots, z_{n-1}^{(0)}) \in \dot{D}'$ and consider the maps $h_\nu(z_n) := f(z'_\nu, z_n)$ in $\mathrm{Hol}(B, X)$. Since $\lim_{\nu\to\infty} h_\nu(z_n) = f(z'_0, z_n) \in X$ for any fixed z_n with $0 < |z_n| < 1$, $\{h_\nu\}$ has no compactly divergent subsequence. By the normality of $\mathrm{Hol}(B, X)$, $\{h_\nu\}$ has a sequence which converges in $\mathrm{Hol}(B, X)$ and whose limit is necessarily equal to $h_0(z_n) := f(z_1^{(0)}, \cdots, z_{n-1}^{(0)}, z_n)$. Accordingly, $\{h_\nu\}$ itself converges to h_0. This shows that f is continuous and completes the proof of Theorem 4.1.

Remark. In Theorem 4.1, we cannot omit the assumption that the limit of $\{a_\nu\}$ is a regular point of S_i. For example, consider the taut complex space X defined as the Riemann sphere minus $\{0, 1, \infty\}$ and the analytic subset $S := \{z_1 = 0\} \cup \{z_2 = 0\} \cup \{z_1 = z_2\}$ in $\boldsymbol{C}^2$. Putting $f(z_1, z_2) = \frac{z_2}{z_1}$ on $\boldsymbol{C}^2 - S$, we have the holomorphic map of $\boldsymbol{C}^2 - S$ into X which cannot be extended to a holomorphic map of $\boldsymbol{C}^2$ into X but has an arbitrary point in X as a cluster value at the origin.

As direct consequences of Theorem 4.1, we have

COROLLARY 4.2 (M.H. Kwack). *Let S be a thin analytic subset of a domain D in $\boldsymbol{C}^n$. Then every holomorphic map of $D - S$ into a compact taut complex space X can be extended to a holomorphic map of D into X.*

COROLLARY 4.3. *Under the same assumption as in Corollary 4.2, if a holomorphic map of $D - S$ into a taut complex space X can be extended to an open subset of D which intersects with each irrreducible component of S, it can be extended to the whole set D.*

Remark. Corollary 4.3 is also a consequence of Theorem 2.3. Because, as is well-known, if a domain D' with $D - S \subset D' \subset D$ intersects with each irreducible component of S, then the envelope of holomorphy of D' includes D.

COROLLARY 4.4. *Let X be a taut complex space and assume that there exists a negative real-valued continuous function $u(x)$ on X such that $u(x)$ is plurisubharmonic on $X - K$ for a suitable compact subset K of X and $\{x \in X;\ u(x) < c\}$ is relatively compact in X for any $c(< 0)$. If S is a thin analytic subset of a domain D in $\boldsymbol{C}^n$,* *then every holomorphic map f of $D-S$ into X has a holomorphic extension to D.*

Proof. We may assume that S is regular connected and of dimension $n-1$. Suppose that f cannot be extended to S. By virtue of Theorem 4.1, $\{f(a_\nu)\}$ has no accumulation point in X for any sequence $\{a_\nu\}$ converging to a point in S. Therefore, an arbitrary point a_0 in S has a neighborhood U such that $f(U-S) \cap K = \phi$. Then, $v := u \cdot f$ is a non-positive plurisubharmonic function on $U-S \cap U$. Moreover, $\lim_{z \to a_0} v(z) = 0$. Putting $v(z) = 0$ for any $z \in S \cap U$, we have a plurisubharmonic function v on U (c.f. Grauert-Remmert [5]). This contradicts the maximum principle for plurisubharmonic functions. Thus we have Corollary 4.4.

§5. **A generalization of the big Picard theorem.** The extension problem of holomorphic maps into a taut complex space is closely related to the classical big Picard theorem. Here, we shall study holomorphic maps into the N-dimensional complex projective space $P_N(\boldsymbol{C})$ minus some hyperplanes.

In [3], J. Dufresnoy gave the following profound theorem.

THEOREM ([3], p. 18). *Let D be a domain in the complex plane and $\mathscr{F}$ be a family of holomorphic maps of D into the complement of arbitrarily given $2N+1$ hyperplanes in general position in $P_N(\boldsymbol{C})$. Then $\mathscr{F}$ is relatively compact in* Hol $(D, P_N(\boldsymbol{C}))$.

As a consequence of this result, we have the following theorem, which gives an answer to the conjecture of H. Wu (e.g., [13], p. 216).

THEOREM 5.1. *For arbitrarily given $2N+1$ hyperplanes $H_1, H_2, \cdots, H_{2N+1}$ in general position in $P_N(\boldsymbol{C})$, the space $X := P_N(\boldsymbol{C}) - (\cup_{k=1}^{2N+1} H_k)$ is a taut complex space.*

Proof. Owing to the result of T.J. Barth [1], we have only to show that Hol (B, X) is normal for the special domain $B := \{|z| < 1\}$ in $\boldsymbol{C}$. Take a sequence $\{f_\nu\}$ in Hol (B, X). By the above theorem a suitable subsequence $\{f_{\nu_k}\}$ converges to a map g in Hol $(B, P_N(\boldsymbol{C}))$. For our purpose, it suffices to ascertain that $g(B) \subset X$ if $g(B) \cap X \neq \phi$. For each $i (1 \leqq i \leqq 2N+1)$ we consider the set $E_i := \{z \in B;\ g(z) \in H_i\}$. Obviously, E_i is closed in B. On the other hand, for any $z_0 \in E_i$, we can choose neighborhoods U of z_0 and V of $g(z_0)$ such that $g(U) \subset V$ and $f_{\nu_k}(U) \subset V$ for almost all k and

$H_i \cap V$ is the set of all zeros of a non-zero linear form l on V. Since $l \cdot f_{\nu_k} \neq 0$ on V and $l \cdot g(z_0) = 0$, $\lim_{k\to\infty} l \cdot f_{\nu_k} = l \cdot g$ vanishes identically on V. This shows that E_i is open in B. Eventually, $E_i = D$ if $E_i \neq \phi$. Therefore, we see $g(D) \cap X = \phi$ if $E_i = D$ for some i and $g(D) \subset X$ if $E_i = \phi$ for any i. This completes the proof.

Now, we shall prove the following generalization of the big Picard theorem.

THEOREM 5.2. *Let S be a regular thin analytic subset of a domain D in $\boldsymbol{C}^n$. Then every holomorphic map of $D - S$ into the complement X of $2N+1$ hyperplanes $H_1, H_2, \cdots, H_{2N+1}$ in general position in $P_N(\boldsymbol{C})$ can be extended to a holomorphic map of D into $P_N(\boldsymbol{C})$.*

Proof. Without loss of generality, we may assume that S is of dimension $n-1$. Furthermore, it may be assumed that $D = \{|z_1| < 1, \cdots, |z_n| < 1\}$ and $S := \{(z_1, \cdots, z_n) \in D; z_n = 0\}$. As in the proof of Theorem 4.1, consider the holomorphic maps $f_\nu(z_1, \cdots, z_{n-1}, z_n) := f\left(z_1, \cdots, z_{n-1}, \frac{z_n}{2^\nu}\right)$ $(z := (z_1, \cdots, z_{n-1}, z_n) \in D - S$ and $\nu \geqq 1)$ in $\mathrm{Hol}(D-S, X)$. By the above result of Dufresnoy, a suitable subsequence $\{f_{\nu_k}\}$ converges to g in $\mathrm{Hol}(D-S, P_N(\boldsymbol{C}))$. By the same argument as in the proof of Theorem 5,1, it holds either $g(D-S) \subset H_i$ or $g(D-S) \cap H_i = \phi$ for each $i (1 \leqq i \leqq 2N+1)$. Since $H_i (1 \leqq i \leqq 2N+1)$ are located in general position, we can choose some i_0 such that $g(D-S) \cap H_{i_0} = \phi$. Let $w_0 : w_1 : \cdots : w_N$ be a system of homogeneous coordinates on $P_N(\boldsymbol{C})$ such that $\{w_0 = 0\} = H_{i_0}$. The space $P_N(\boldsymbol{C}) - H_{i_0}$ may be considered as the space $\boldsymbol{C}^N$ and $w^{(1)} = \frac{w_1}{w_0}, \cdots, w^{(N)} = \frac{w_N}{w_0}$ give the global coordinate system on $\boldsymbol{C}^N$. Put $f_k^{(i)} = w^{(i)} \cdot f_{\nu_k}$, $f^{(i)} = w^{(i)} \cdot f$ and $g^{(i)} = w^{(i)} \cdot g (1 \leqq i \leqq N)$. Since $\lim_{k\to\infty} f_k^{(i)} = g^{(i)}$ uniformly on the compact set $E := \left\{|z_1| \leqq \frac{1}{2}, \cdots, |z_{n-1}| \leqq \frac{1}{2}, \ |z_n| = \frac{1}{2}\right\}$ in $D-S$, we can find a real constant M such that $|f_k^{(i)}(z)| \leqq M (1 \leqq i \leqq N)$ for any $z \in E$ and k. Therefore, if $|z_1| \leqq \frac{1}{2}, \cdots, |z_{n-1}| \leqq \frac{1}{2}, \ |z_n| = \frac{1}{2^{\nu_k+1}}$, we have $|f^{(i)}(z_1, \cdots, z_n)| = |f_k^{(i)}(z_1, \cdots, z_{n-1}, 2^{\nu_k} z_n)| \leqq M$. Then, considering the holomorphic functions $h^{(i)}(z_n) := f^{(i)}(z_1, \cdots, z_{n-1}, z_n)$ in z_n for arbitrarily fixed $z_1, \cdots, z_{n-1}$ with $|z_i| \leqq \frac{1}{2} (1 \leqq i \leqq n-1)$ and applying the maximum principle, we obtain $|f^{(i)}(z_1, \cdots, z_n)| \leqq M$ if $\frac{1}{2^{\nu_{k+1}+1}} \leqq |z_n| \leqq \frac{1}{2^{\nu_k+1}}$ for any k. Eventually, $|f^{(i)}(z_1, \cdots, z_n)| \leqq M$ on $\left\{|z_1| \leqq \frac{1}{2}, \cdots, |z_{n-1}| \leqq \frac{1}{2}, \ 0 < |z_n| \leqq \frac{1}{2}\right\}$. In this situation, Theorem 5.2 is an immediate consequence of the classical Riemann's theorem on removable singularities of bounded holomorphic functions.

References

[1] T.J. Barth, Taut and tight complex manifolds, Proc. Amer. Math. Soc., **24** (1970), 429–431.

[2] F. Docquier-H. Grauert, Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten, Math. Ann., **140** (1960), 94–123.

[3] J. Dufresnoy, Théorie nouvelle des familles complexes normales; applications à l'étude des fonctions algébroïdes, Ann. E.N.S., (3) **61** (1944), 1–44.

[4] H. Grauert, Über Modifikationen und exzeptionelle analytische Mengen, Math. Ann., **146** (1962), 331–368.

[5] H. Grauert-R. Remmert, Plurisubharmonisches Funktionen in komplexen Räume, Math. Z., **65** (1965), 175–194.

[6] W. Kaup, Hyperbolische komplexe Räume, Ann. Inst. Fourier Grenoble, **18** (1969), 303–330.

[7] P. Kiernan, On the relations between taut, tight and hyperbolic manifolds, Bull. Amer. Math. Soc., **76** (1970), 49–51.

[8] S. Kobayashi, Invariant distances on complex manifolds and holomorphic maps, J. Math. Soc. Japan, **19** (1967), 460–480.

[9] M.H. Kwack, Generalization of the big Picard theorem, Ann. Math., **90** (1969), 9–22.

[10] M. Tadokoro, Sur les ensembles Pseudoconcaves généraux, J. Math. Soc. Japan, **17** (1965), 281–290.

[11] H. Wu, Normal families of holomorphic mappings, Acta Math., **119** (1967), 193–233.

[12] H. Wu, The equidistribution theory of holomorphic curves, Princeton Univ. Princeton, New Jersey (1970).

Nagoya University