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RESEARCH ARTICLE

Generic Beauville's Conjecture

Izzet Coskun [1], Eric Larson [2] and Isabel Vogt [3]

[1]Department of Mathematics, Statistics, and CS, University of Illinois at Chicago, 851 S. Morgan Street, Chicago, IL 60607, United States; E-mail: icoskun@uic.edu.
[2]Department of Mathematics, Brown University, 151 Thayer Street, Providence, RI 02912, United States; E-mail: elarson3@gmail.com.
[3]Department of Mathematics, Brown University, 151 Thayer Street, Providence, RI 02912, United States; E-mail: ivogt.math@gmail.com (corresponding author).

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Abstract
Let $\alpha\colon X \to Y$ be a finite cover of smooth curves. Beauville conjectured that the pushforward of a general vector bundle under α is semistable if the genus of Y is at least 1 and stable if the genus of Y is at least 2. We prove this conjecture if the map α is general in any component of the Hurwitz space of covers of an arbitrary smooth curve Y.

Contents

1. Introduction

Motivated by the study of the theta linear series on the moduli spaces of vector bundles on curves, Beauville in [B00] (see also [B06, Conjecture 6.5]) made the following celebrated conjecture:

Conjecture 1.1 (Beauville). *Let $\alpha\colon X \to Y$ be a finite morphism between smooth irreducible projective curves, and let V be a general vector bundle on X. Then α_*V is stable if the genus of Y is at least* 2 *and semistable if the genus of Y is* 1.

We prove Beauville's conjecture when Y is an *arbitrary* smooth irreducible projective curve and X is a general member of *any* component of the Hurwitz space of genus g degree r covers of Y.

Statement of results

Let $\alpha\colon X \to Y$ be a finite map of degree r from a smooth irreducible projective curve X of genus g to a smooth irreducible projective curve Y of genus h. We always work over an algebraically closed field of characteristic 0 or greater than r.

For a vector bundle V on a curve X, the **slope** $\mu(V)$ is defined by $\mu(V) = \frac{\deg(V)}{\mathrm{rk}(V)}$. The bundle V is called **(semi)stable** if, for every proper subbundle W, we have $\mu(W) \underset{(-)}{<} \mu(V)$. Semistable bundles satisfy nice cohomological and metric properties and form projective moduli spaces. Consequently, determining the stability of naturally defined bundles is an important and fundamental problem.

Let $\mathscr{H}_{r,g}(Y)$ denote the Hurwitz space parameterizing smooth connected degree r genus g covers of Y. In general, $\mathscr{H}_{r,g}(Y)$ is reducible, and when $g > r(h-1)+1$, the irreducible components correspond to subgroups of the (étale) fundamental group $\pi_1(Y)$ of index diving r. With this notation, our main theorem is the following.

Theorem 1.2. *Let Y be any smooth irreducible projective curve of genus h. Let $\alpha\colon X \to Y$ be a general morphism in any component of $\mathscr{H}_{r,g}(Y)$. Let V be a general vector bundle of any degree and rank on X.*

1. *If $h = 1$, then α_*V is semistable.*
2. *If $h \geq 2$, then α_*V is stable.*

Remark 1.3. It may happen that, for special V, the bundle α_*V is not semistable. For example, $\alpha_*\mathcal{O}_X$ has $\mathcal{O}_Y$ as a direct summand. When the map α is ramified, $\mathcal{O}_Y$ destabilizes $\alpha_*\mathcal{O}_X$ (see [CLV22]).

History of the problem

Beauville made Conjecture 1.1 in an unpublished note dating to 2000 [B00]. In the same note, Beauville proved the conjecture if

1. α is étale [B00, Propostion 4.1], or
2. $r < g(\sqrt{3}+1) - 1$ [B00, Corollary 3.4], or
3. V is a line bundle with $|\chi(V)| \leq g + \frac{g^2}{r}$ [B00, Proposition 3.2].

It is an elementary observation that, when $h = 0$, the pushforward of a general vector bundle splits as $\bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(a_i)$, where $|a_i - a_j| \leq 1$ for every i, j (see [B00, §1]).

Beauville, Narasimhan and Ramanan [BNR89] earlier proved that a general vector bundle V of degree d and rank r on Y arises as the pushforward of a line bundle from some cover of degree r. Hence, there exists covers of Y for which Beauville's conjecture is true.

Mehta and Pauly [MP07] proved that if α is the Frobenius morphism, $h \geq 2$, and V is semistable, then α_*V is semistable.

Strategy

Recall that $\alpha\colon X \to Y$ is called **primitive** if the map $\alpha_*\colon \pi_1(X) \to \pi_1(Y)$ induced on (étale) fundamental groups is surjective. Every degree r cover $\alpha\colon X \to Y$ factors into a primitive map $\alpha^{\mathrm{pr}}\colon X \to Y'$ followed by an étale map $\alpha^{\text{ét}}\colon Y' \to Y$, where Y' is the étale cover associated with the subgroup $\alpha_*\pi_1(X) \subset \pi_1(Y)$.

We prove Theorem 1.2 by specializing X to a nodal curve. Let $\alpha\colon X \to Y$ be a general element in any component of $\mathscr{H}_{r,g}(Y)$. Let $\alpha = \alpha^{\mathrm{pr}} \circ \alpha^{\text{ét}}$ be the primitive-étale factorization of α. Let $\alpha^{\text{ét}}$ and α^{pr} have degrees r' and $s = \frac{r}{r'}$, respectively. Let $\beta_0\colon X_0 \to Y'$ be a degree s cyclic étale cover of Y'. The resulting map $\alpha_0\colon X_0 \to Y$ is étale and Conjecture 1.1 holds for the map $X_0 \to Y$ by [B00, Proposition 4.1].

Let p_j and p'_j be points on X_0 contained in a fiber of β_0 such that the cyclic action takes p_j to p'_j. We identify the appropriate number of pairs p_j and p'_j on X_0 to form a nodal curve X_1 of genus g. Let $\nu\colon X_0 \to X_1$ be the normalization map. The induced map $\beta_1\colon X_1 \to Y'$ is primitive (see Proposition 3.1), and so $\alpha_1 = \alpha^{\text{ét}} \circ \beta_1\colon X_1 \to Y$ is in the same irreducible component of $\mathscr{H}_{r,g}(Y)$ as X (see Lemma 2.1). For a general bundle V on X_0, the pushforward $\alpha_{0*}V = \alpha_{1*}(\nu_*V)$ is stable if $h \geq 2$ and semistable if $h = 1$. Finally, we observe that ν_*V is a limit of vector bundles on nearby deformations of X_1 (see Proposition 3.2). Together with the openness of (semi)stability, this proves Theorem 1.2.

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2. Preliminaries

2.1. Basic facts

Let $\alpha\colon X \to Y$ be a finite map of degree r from a smooth irreducible projective curve X of genus g to a smooth irreducible projective curve Y of genus h. Since the characteristic is 0 or greater than r, the map α is separable. By the Riemann–Hurwitz formula

$$2g - 2 = r(2h - 2) + b,$$

where b is the degree of the ramification divisor. In particular, $g \geq r(h-1)+1$ with equality if and only if α is étale.

If V is a vector bundle of rank s and degree d on X, then $\alpha_*(V)$ is a vector bundle of rank rs on Y. Using the fact that $\chi(V) = \chi(\alpha_* V)$ and the Riemann–Roch formula, we compute the degree d' of $\alpha_*(V)$ via

$$d + s(1-g) = \chi(V) = \chi(\alpha_* V) = d' + rs(1-h).$$

We conclude that $d' = d + s(1-g) - rs(1-h)$.

2.2. The primitive-étale factorization

Let $\mathscr{H}_{r,g}(Y)$ denote the Hurwitz space parameterizing smooth connected degree r genus g covers of Y. If $g < r(h-1)+1$, then $\mathscr{H}_{r,g}(Y)$ is empty. If $g = r(h-1)+1$, then the degree r covers of genus g are étale and there are finitely many. In general, the Hurwitz space $\mathscr{H}_{r,g}(Y)$ is not irreducible. The following lemma characterizes the irreducible components.

Lemma 2.1. *Let Y be a smooth and irreducible curve of genus h defined over an algebraically closed field of characteristic* 0 *or larger than r. Let $g > r(h-1)+1$. Then the components of $\mathscr{H}_{r,g}(Y)$ are in bijection with subgroups of $\pi_1(Y)$ of index dividing r.*

Proof. Let $\mathscr{H}$ be an irreducible component of the Hurwitz scheme $\mathscr{H}_{r,g}(Y)$. Given $\alpha\colon X \to Y$ in $\mathscr{H}$, the subgroup $\alpha_*\pi_1(X)$ of $\pi_1(Y)$ has index dividing r. Since this is a discrete invariant and is constant in irreducible families, $\alpha_*\pi_1(X)$ is an invariant of $\mathscr{H}$.

Conversely, given a subgroup $G \subset \pi_1(Y)$ of index $r^{\text{ét}}$ dividing r, up to isomorphism there is a unique étale cover $\delta\colon Y' \to Y$ corresponding to G of degree $r^{\text{ét}}$ and genus $h' = r^{\text{ét}}(h-1)+1$. Let $r^{\text{pr}} = r/r^{\text{ét}}$. Given the inequality

$$g > r(h-1)+1 = r^{\text{pr}}(h'-1)+1,$$

there exists a genus g primitive cover $\gamma\colon X \to Y'$ of degree r^{pr}. For any such cover, we obtain an element of $\mathscr{H}_{r,g}(Y)$ by taking $\alpha = \delta \circ \gamma$. Furthermore, $\alpha_*\pi_1(X) = G$. On the other hand, if $\gamma\colon X \to Y'$ is not primitive but $\gamma_*\pi_1(X)$ has index s in $\pi_1(Y')$, then $\alpha_*\pi_1(X)$ has index $sr^{\text{ét}}$ in $\pi_1(Y)$ and cannot be G. We conclude that if $\alpha_*\pi_1(X) = G$, then α must factor as the composition of δ and a primitive cover of Y'. By results of Clebsch [C1872], Fulton [F69], and Gabai–Kazez [GK90] (see [CLV22, Proposition 2.2]), the Hurwitz scheme parameterizing genus g degree r^{pr} primitive covers of Y' is irreducible. We conclude that there is a bijection between irreducible components of $\mathscr{H}_{r,g}$ and subgroups of $\pi_1(Y)$ of index dividing r. □

2.3. The étale case

We briefly recall Beauville's proof of Conjecture 1.1 [B00, Proposition 4.1] in the étale case (see also the proof of [CLV22, Proposition 1.3]).

First, we show that it suffices to consider the case of line bundles. Given a vector bundle V on X of degree d and rank s, let $\delta\colon Z \to X$ be an étale cover of degree s. If L is a line bundle of degree d on Z,

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then $\delta_* L$ is a vector bundle of rank s and degree d on X. Hence, if we prove Conjecture 1.1 for (étale) maps in the case of line bundles, it follows for (étale) maps in higher rank as well.

Let $\alpha\colon X \to Y$ be étale, and let $\rho\colon Z \to Y$ be the Galois cover associated to α with Galois group G. Let Σ be the set of Y-morphisms $\sigma\colon Z \to X$. Then

$$W := \rho^*\alpha_* L \cong \bigoplus_{\sigma\in\Sigma} \sigma^* L.$$

The pullback by ρ of any destabilizing subbundle of $\alpha_* L$ would destabilize W. Hence, $\alpha_* L$ is semistable for *every* line bundle L on X.

If $\alpha_* L$ has a proper subbundle F of the same slope as $\alpha_* L$, then $\rho^* F$ is a G-invariant subbundle of W. Since the category of semistable bundles of a fixed slope is abelian with simple objects stable bundles, $\rho^* F \cong \oplus_{\sigma\in\Sigma'} \sigma^* L$ for some $\Sigma' \subset \Sigma$. Since G acts transitively on Σ, it suffices to show that, if $h > 1$ and L is general, the line bundles $\sigma^* L$ are pairwise nonisomorphic as σ varies in Σ.

For any fixed $\sigma \in \Sigma'$, let $H \subset G$ be the subgroup fixing σ. The subvariety $\sigma^* \operatorname{Pic}^d X \subset \operatorname{Pic} Z$ has dimension g, contains $\sigma^* L$ and is invariant under H. On the other hand, if $\sigma^* \operatorname{Pic}^d X$ is invariant under a subgroup H' with $H \subsetneq H' \subset G$, then it would be pulled back from $\operatorname{Pic}(Z/H')$. By the Riemann–Hurwitz formula, the genus of Z/H' is strictly smaller than g. Hence, by dimension reasons, this containment is impossible and the $\sigma^* L$ are pairwise distinct. This shows the stability of $\alpha_* L$.

3. Proof of Theorem 1.2

Let Y be a curve of genus h, and let g be an integer such that $g > r(h-1)+1$. Fix an étale cover $\alpha^{\text{ét}}\colon Y' \to Y$ of degree $r^{\text{ét}} \mid r$. We first explain how to construct a nodal cover $\alpha_1\colon X_1 \to Y$ of arithmetic genus g, whose primitive-étale factorization is

$$X_1 \to Y' \xrightarrow{\alpha^{\text{ét}}} Y.$$

The first step of our construction is to fix a cyclic étale cover $\beta_0\colon X_0 \to Y'$ of degree $r^{\text{pr}} = r/r^{\text{ét}}$. Such covers correspond to points of order r^{pr} in $\operatorname{Jac}(Y')$, which always exist. Write $\tau\colon X_0 \to X_0$ for the automorphism corresponding to the generator of $\mathbb{Z}/r^{\text{pr}}\mathbb{Z}$. Let $n := g - r(h-1) - 1$. We then pick general points $p_1, p_2, \dots, p_n \in X_0$, and let $p_i' = \tau(p_i)$. Let X_1 be the curve obtained from X_0 by gluing every p_i to p_i' for $1 \le i \le n$, and denote the normalization $\nu\colon X_0 \to X_1$. Let $\beta_1\colon X_1 \to Y'$ be the induced morphism, and let $\alpha_1 := \alpha^{\text{ét}} \circ \beta_1$.

Proposition 3.1. *The cover $\beta_1\colon X_1 \to Y'$ is primitive.*

Proof. We must show that the pushforward $\pi_1(X_1) \to \pi_1(Y')$ is surjective. By [S09, Proposition 5.5.4(2)], this is equivalent to the assertion that for all finite étale connected covers $Y'' \to Y'$, the fiber product $X_1 \times_{Y'} Y''$ is connected.

Consider the dominant map $\epsilon\colon X_0 \times_{Y'} Y'' \to X_1 \times_{Y'} Y''$. By construction, $\mathbb{Z}/r^{\text{pr}}\mathbb{Z}$ acts transitively on the components of $X_0 \times_{Y'} Y''$. Therefore, it suffices to show that for any component $Z \subset X_0 \times_{Y'} Y''$, the components $\epsilon(Z)$ and $\epsilon(\tau(Z))$ intersect. Since $Z \to X_0$ is surjective, Z contains a point of the form (p_1, y'') for some $y'' \in Y''$ and so $\tau(Z)$ contains (p_1', y''). Since $\epsilon((p_1, y'')) = \epsilon((p_1', y''))$, the components $\epsilon(Z)$ and $\epsilon(\tau(Z))$ intersect as desired. □

By the theory of formal patching (see [Li03, Lemma 5.6]), the map β_1 can be smoothed to a map $\beta\colon X \to Y'$ with a smooth domain X. Since being primitive is a deformation invariant, the resulting smoothing β is also primitive by Proposition 3.1.

Proposition 3.2. *Given a vector bundle V on X_0, the pushforward $\nu_* V$ to X_1 is a limit of vector bundles of the same rank and slope $\mu(V) + n$ on the smoothing X.*

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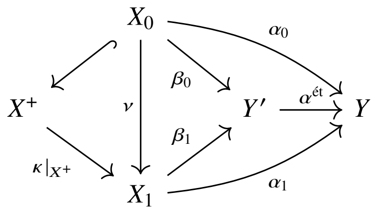
Proof. Let $\mathcal{X} \to \Delta$ denote a family of smooth curves specializing to X_1 with smooth total space. Consider the blowup at the nodes of X_1. In this family, the central fiber is the union of X_0 together with n copies of $\mathbb{P}^1$, where each $\mathbb{P}^1$ is attached at the two preimages of the corresponding node under the normalization map ν. These $\mathbb{P}^1$s appear with multiplicity 2 in the central fiber. Make a base change of order 2 and normalize the total space to obtain a family $\mathcal{X}^+ \to \Delta'$. This is a semistable family of smooth curves specializing to the union of X_0 with n copies of $\mathbb{P}^1$, where now the central fiber X^+ is reduced. Write $\kappa\colon \mathcal{X}^+ \to \mathcal{X}' := \mathcal{X} \times_\Delta \Delta'$. The following diagram illustrates the maps that exist on the central fiber.



Let V^+ denote the vector bundle on X^+ obtained by gluing the vector bundle V on X_0 to $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus \operatorname{rk} V}$ on each $\mathbb{P}^1$ component via any choice of gluing. (In fact the reader may check that any two choices result in isomorphic bundles.) Since V is locally free, $\mathcal{E}xt^i(V^+, V^+) = 0$ for all $i > 0$. Thus, by the local-to-global Ext spectral sequence, we have that $\operatorname{Ext}^2_{X^+}(V^+, V^+) = 0$. By [H, Theorem 7.3 (b)], the obstructions to extending the bundle V^+ to the whole family lie in $\operatorname{Ext}^2_{X^+}(V^+, V^+)$. Consequently, V^+ extends to a vector bundle $\mathcal{V}^+$ on $\mathcal{X}^+$. Observe that $\mathcal{V}^+$ has rank $\operatorname{rk} V$, and by the constancy of the Euler characteristic in flat families, the slope of the restriction of $\mathcal{V}^+$ to the fibers is $\mu(V^+) = \mu(V) + n$.

Now, we claim that $\kappa_* \mathcal{V}^+|_{X_1} \simeq \nu_* V$. Once we establish this claim, we obtain that $\nu_* V$ is the limit of vector bundles of the same rank and slope $\mu(V) + n$ on the smooth fibers.

Let $(\operatorname{rk}, \chi)(F)$ denote the rank and Euler characteristic of a sheaf F, and write $\mathcal{X}'_t$ and $\mathcal{X}^+_t$ for general fibers of their respective families. We first show that $(\operatorname{rk}, \chi)(\kappa_* \mathcal{V}^+|_{X_1}) = (\operatorname{rk}, \chi)(\nu_* V)$. By the constancy of the rank and the Euler characteristic in flat families and the fact that κ is an isomorphism away from the central fiber, we have

$$(\operatorname{rk}, \chi)(\kappa_* \mathcal{V}^+|_{X_1}) = (\operatorname{rk}, \chi)(\kappa_* \mathcal{V}^+|_{\mathcal{X}'_t}) = (\operatorname{rk}, \chi)(\mathcal{V}^+|_{\mathcal{X}^+_t}) = (\operatorname{rk}, \chi)(V^+).$$

Furthermore, by considering the exact sequence for restriction to X_0

$$0 \to \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus \operatorname{rk}(V)} \to V^+ \to V^+|_{X_0} = V \to 0,$$

we see that $(\operatorname{rk}, \chi)(V^+) = (\operatorname{rk}, \chi)(V)$. Finally, $(\operatorname{rk}, \chi)(V) = (\operatorname{rk}, \chi)(\nu_* V)$, which proves that $(\operatorname{rk}, \chi)(\kappa_* \mathcal{V}^+|_{X_1}) = (\operatorname{rk}, \chi)(\nu_* V)$.

Hence, it suffices to construct a surjective map between $\kappa_* \mathcal{V}^+|_{X_1}$ and $\nu_* V$. Consider the exact sequence on $\mathcal{X}^+$

$$0 \to \mathcal{V}^+(-X_0) \to \mathcal{V}^+ \to \mathcal{V}^+|_{X_0} \to 0.$$

Pushing forward under κ, we obtain

$$0 \to \kappa_* \mathcal{V}^+(-X_0) \to \kappa_* \mathcal{V}^+ \to \kappa_*(\mathcal{V}^+|_{X_0}) \to R^1 \kappa_* \mathcal{V}^+(-X_0) \to \cdots$$

Observe that $\kappa_*(\mathcal{V}^+|_{X_0}) \simeq \nu_* V$ and that the map $\kappa_* \mathcal{V}^+ \to \kappa_*(\mathcal{V}^+|_{X_0}) \simeq \nu_* V$ factors through $(\kappa_* \mathcal{V}^+)|_{X_1}$. Hence, it suffices to show that $R^1 \kappa_* \mathcal{V}^+(-X_0) = 0$.

Since κ is an isomorphism away from the nodes of the X_1, the sheaf $R^1 \kappa_* \mathcal{V}^+(-X_0)$ is supported on the nodes of X_1. It therefore suffices to show that its completion at every node p of X_1 vanishes. For this,

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we use the theorem on formal functions, which states that

$$R^1\kappa_*\mathcal{V}^+(-X_0)^\wedge_p \simeq \varprojlim_n H^1(\mathcal{V}^+(-X_0)|_{n\cdot\mathbb{P}^1}),$$

where $\mathbb{P}^1 = \kappa^{-1}(p)$ is a Cartier divisor on $\mathcal{X}^+$. It thus suffices to show that $H^1(\mathcal{V}^+(-X_0)|_{n\cdot\mathbb{P}^1}) = 0$ for all n. For this, we use induction on n, with base case $n = 0$, which is clear. For the inductive step, we use the exact sequence for restriction to $n \cdot \mathbb{P}^1$

$$0 \to \mathcal{V}^+(-X_0 - n\cdot\mathbb{P}^1)|_{\mathbb{P}^1} \to \mathcal{V}^+(-X_0)|_{(n+1)\cdot\mathbb{P}^1} \to \mathcal{V}^+(-X_0)|_{n\cdot\mathbb{P}^1} \to 0.$$

Since $\mathcal{V}^+(-X_0 - n\cdot\mathbb{P}^1)|_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(2n-1)^{\oplus \operatorname{rk} V}$, which has vanishing h^1, we conclude by induction that the middle term has vanishing h^1. □

Proof of Theorem 1.2. Let $\mathscr{H}$ be an irreducible component of the Hurwitz space $\mathscr{H}_{r,g}(Y)$. Assume that the corresponding covers have primitive-étale factorization

$$X \to Y' \xrightarrow{\alpha^{\text{ét}}} Y.$$

Let $\beta_0 \colon X_0 \to Y'$ be the cyclic étale cover constructed above, and let

$$\alpha_1 \colon X_1 \xrightarrow{\beta_1} Y' \xrightarrow{\alpha^{\text{ét}}} Y$$

be the cover constructed above by gluing points in the fibers of $\beta_0 \colon X_0 \to Y'$.

Let V be a general vector bundle on X_0 of arbitrary degree and rank. By [B00, Proposition 4.1] (see §2.3), the pushforward $\alpha_{0*}V$ is semistable if $h = 1$ and stable if $h \geq 2$. Since

$$\alpha_0 = \alpha^{\text{ét}} \circ \beta_0 = \alpha^{\text{ét}} \circ \beta_1 \circ \nu = \alpha_1 \circ \nu,$$

we conclude that $\alpha_{1*}\nu_*V$ is semistable if $h = 1$ and stable if $h \geq 2$.

By Proposition 3.2, the pushforward bundle ν_*V on X_1 is a limit of vector bundles on a smoothing $X \to Y' \to Y$ and these vector bundles can be chosen to have any given degree and rank. The theorem follows by the openness of (semi)stability. □

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