BULL. AUSTRAL. MATH. SOC. 46C50, 46B20
VOL. 65 (2002) [93-104]

CHARACTERISATIONS OF ORTHOGONALITY IN CERTAIN BANACH SPACES

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In this paper we adopt the notion of orthogonality introduced by the author in a previous article. We establish a characterisation for orthogonality in the spaces $l_S^p(\mathbf{C})$, $1 \leqslant p < \infty$, where S is a set of positive integers and $\mathbf{C}$ is the field of complex numbers. Generalisations of the usual characterisation of orthogonality in the Hilbert spaces $l_S^2(\mathbf{C})$, via inner products, are obtained.

1. INTRODUCTION

Throughout this paper $\mathbf{K}$ is the field of real or complex numbers, p is a real number in $[1,\infty)$, S is a finite or infinite set of positive integers, and $l_S^p(\mathbf{K})$ is the usual Banach space consisting of all sequences $(x_n) := (x_n)_{n\in S}$ in $\mathbf{K}$ satisfying $\left\|(x_n)_{n\in S}\right\|_p < \infty$, where

$$\left\|(x_n)_{n\in S}\right\|_p := \left(\sum_{n\in S} |x_n|^p\right)^{1/p}.$$

If N is a positive integer and $S = \{1,2,\dots,N\}\big(= \{1,2,\dots\}\big)$ then

$$l_S^p(\mathbf{K}) := l_N^p(\mathbf{K})\big(:= l^p(\mathbf{K})\big).$$

Usually, the notion of orthogonality is associated with inner product spaces. Many extensions to Banach spaces have been introduced through the decades by various authors, for example, Birkoff [**1**], Roberts [**4**], James [**2**], Singer [**6**], Khalil [**3**] and, more recently, in [**5**].

Each of the previously introduced extensions shares a number of important features with Hilbert space orthogonality, but lacks certain other attributes. Clearly, one cannot expect all the Hilbert space features to remain valid in general Banach spaces. Nevertheless, one would like to obtain as rich of a structure as possible, without requiring too much at the expense of the applicability and the usefulness of the concepts. Some extensions that are indeed rich in structure were introduced in [**3**] and, more recently, in [**5**]. The drawback in the extension of [**3**] is that the orthogonality of a sequence $(x_n)_{n\in S}$ in a Banach space E is dependent on a (non-unique) choice of a sequence of

Received 23rd May, 2001

functionals in E^* which is defined in terms of the sequence $(x_n)_{n\in S}$. Therefore we adopt here the more straightforward and simpler extension of orthogonality introduced in [**5**].

DEFINITION 1: A finite or infinite sequence $(x_n)_{n\in S}$ in a Banach space E is said to be orthogonal if

$$\Big\|\sum_{n\in S} a_n x_n\Big\| = \Big\|\sum_{n\in S} |a_n|\, x_n\Big\|, \quad \text{for each} \quad \sum_{n\in S} a_n x_n \in E,$$

where the a_n's are scalars. If, in addition, $\|x_n\| = 1$ for all $n \in S$, then $(x_n)_{n\in S}$ is said to be orthonormal. We write $x \perp y$ if x is orthogonal to y.

Note that Definition 1 gives an extension of the usual notion of orthogonality since, in an inner product space H, $x \perp y$ in our sense if and only if $\langle x, y\rangle = 0$, where $\langle .,.\rangle$ denotes the inner product in H, [**5**].

In [**3**], it was stated that $\left(1/2^{1/p}, 1/2^{1/p}\right)$ and $\left(1/2^{1/p}, -1/2^{1/p}\right)$ are orthogonal in $l_2^p(\mathbf{K})$. We point out that this is true only in the case where $\mathbf{K} = \mathbf{R}$ but not when $\mathbf{K} = \mathbf{C}$, where $\mathbf{R}$ is the set of real numbers and $\mathbf{C}$ is the set of complex numbers. This lead us to the following problem.

PROBLEM 1. Under what conditions are two vectors in $l_S^p(\mathbf{C})$ orthogonal?

It is well known that in the Hilbert spaces $l_S^2(\mathbf{C})$, $(a_j)_{j\in S} \perp (b_j)_{j\in S}$ if and only if

$$\sum_{j\in S} a_j \overline{b_j} = 0.$$

Therefore, our aim in this paper is to establish generalisations of this characterisation of orthogonality in the spaces $l_S^p(\mathbf{C})$, $1 \leqslant p < \infty$. In particular, in the cases where $p = 2$, we get back the usual characterisation of orthogonality in $l_S^2(\mathbf{C})$. Finally, an open problem is presented.

We note that orthogonality in the sense of [**3**] implies orthogonality in our sense, [**5**].

2. CHARACTERISATION OF ORTHOGONALITY IN $l_S^p(\mathbf{C})$

The support of a finite or infinite sequence $(a_j)_{j\in S}$ in $\mathbf{C}$ is given by

$$\operatorname{supp}(a_j) := \{j \in S : a_j \neq 0\}.$$

For any two sequences $(a_j)_{j\in S}$ and $(b_j)_{j\in S}$ in $\mathbf{C}$, we set

$$J := \operatorname{supp}(a_j) \cap \operatorname{supp}(b_j) := \{j \in S : a_j b_j \neq 0\},$$
$$A := \left\{\left|\frac{b_j}{a_j}\right| : j \in J\right\}$$

and, for each $r > 0$,

$$J_r := \left\{ j \in J : \left| \frac{b_j}{a_j} \right| = r \right\},$$

$$J_r^+ := \left\{ j \in J : \left| \frac{b_j}{a_j} \right| > r \right\}$$

and

$$J_r^- := \left\{ j \in J : \left| \frac{b_j}{a_j} \right| < r \right\}$$

The cardinality of J is denoted by $|J|$. Note that for each $r > 0$,

$$J = J_r^- \cup J_r \cup J_r^+$$

and

$$A = \left\{ \left| \frac{b_j}{a_j} \right| : j \in J_r^- \right\} \cup \{r\} \cup \left\{ \left| \frac{b_j}{a_j} \right| : j \in J_r^+ \right\},$$

where the unions are disjoint. Also, note that J is the disjoint union

$$J = \bigcup_{r \in A} J_r.$$

We start with the following direct consequences of Definition 1:

REMARK 1. Let $(a_j)_{j\in S}$ and $(b_j)_{j\in S}$ be two elements in $l_S^p(\mathbf{C})$.

(i) If $J = \emptyset$, then $(a_j)_{j\in S} \perp (b_j)_{j\in S}$.

(ii) If $J \neq \emptyset$, then $(a_j)_{j\in S} \perp (b_j)_{j\in S}$ if and only if $(a_j)_{j\in J} \perp (b_j)_{j\in J}$.

(iii) If J is a finite or infinite union of nonempty and disjoint subsets, $J = \bigcup_{k\in I} J^k$, and if $(a_j)_{j\in J^k} \perp (b_j)_{j\in J^k}$ for each $k \in I$, then $(a_j)_{j\in J} \perp (b_j)_{j\in J}$.

In view of Remark 1, we assume for the remainder of this paper that $J \neq \emptyset$.

For $r, \theta \in \mathbf{R}$ and $i := \sqrt{-1}$, we set

$$f(r,\theta) := \sum_{j\in J} \left| b_j + re^{i\theta} a_j \right|^p.$$

The following observation actually holds for orthogonality in any Banach space.

LEMMA 1. *Two elements $a, b \in l_S^p(\mathbf{C})$ are orthogonal if and only if, for each real number $r > 0$ satisfying $r, (1/r) \notin A$, $f(r,\theta)$ is independent of $\theta \in \mathbf{R}$.*

PROOF: From Remark 1. (ii), it follows that $a \perp b$ in $l_S^p(\mathbf{C})$ if and only if $a \perp b$ in $l_J^p(\mathbf{C})$. Hence $a \perp b$ in $l_S^p(\mathbf{C})$ if and only if

$$\sum_{j\in J} |\lambda b_j + \mu a_j|^p = \sum_{j\in J} \left| |\lambda| b_j + |\mu| a_j \right|^p \text{ for all } \lambda, \mu \in \mathbf{C},\ \lambda \neq 0.$$

Dividing by $|\lambda|$, we get that this is equivalent to

$$\sum_{j\in J}\left|b_j+\frac{\mu}{\lambda}a_j\right|^p=\sum_{j\in J}\left|b_j+\left|\frac{\mu}{\lambda}\right|a_j\right|^p \text{ for all } \frac{\mu}{\lambda}\in\mathbf{C}.$$

In other words we have, setting $\mu/\lambda := re^{i\theta}$ $(r>0)$,

$$\sum_{j\in J}\left|b_j+re^{i\theta}a_j\right|^p=\sum_{j\in J}|b_j+ra_j|^p \text{ for all } r>0 \text{ and } \theta\in\mathbf{R}.$$

The lemma now follows from the continuity of the norm and the fact that A is countable, hence $\{r>0 : r, (1/r)\notin A\}$ is dense in $(0,\infty)$. ∎

We note that any of the sets J_r, J_r^+ and J_r^- may be empty. Hence, for the remainder of this paper, we assume that any summation over an empty index-set is equal to zero.

Before we continue, we introduce the following notations, where k,l are two non-negative integers, r is a positive real number and $(a_j)_{j\in S}$, $(b_j)_{j\in S}$ are two elements in $l_S^p(\mathbf{C})$:

$$c_p(0):=1 \quad\text{and}\quad c_p(k):=\frac{\prod_{t=0}^{k-1}((p/2)-t)}{k!}=\frac{\prod_{t=0}^{k-1}(p-2t)}{2^k k!} \text{ if } k\geqslant 1,$$

$$B_r^+(k,l):=c_p(k)c_p(l)\sum_{j\in J_r^+}|b_j|^p\left(\frac{a_j}{b_j}\right)^k\left(\frac{\overline{a_j}}{\overline{b_j}}\right)^l,$$

$$B_r^-(k,l):=c_p(k)c_p(l)\sum_{j\in J_r^-}|a_j|^p\left(\frac{b_j}{a_j}\right)^k\left(\frac{\overline{b_j}}{\overline{a_j}}\right)^l.$$

Lemma 2. *We have $(a_j)_{j\in S}\perp(b_j)_{j\in S}$ in $l_S^p(\mathbf{C})$ if and only if, for every integer $m\geqslant 1$,*

$$\sum_{k=0}^{\infty}B_r^+(k+m,k)r^{2k+m}+\sum_{k=0}^{\infty}B_r^-(k,k+m)r^{p-(2k+m)}=0,$$

for all real numbers $r>0$.

Proof: By Lemma 1, $(a_j)_{j\in S}\perp(b_j)_{j\in S}$ if and only if $f(r,\theta)$ is independent of θ for each $r>0$ satisfying r, $(1/r)\notin A$. So let r be a positive real number satisfying r, $(1/r)\notin A$.

If α, $\beta\in\mathbf{C}$ and $\alpha\neq 0$, then we have

$$(\alpha+\beta)^{p/2}=\sum_{k=0}^{\infty}c_p(k)\alpha^{p/2}\left(\frac{\beta}{\alpha}\right)^k \text{ if } \left|\frac{\beta}{\alpha}\right|<1,$$

where $(\alpha+\beta)^{p/2}$ represents the principal branch of $e^{(p/2)\operatorname{Log}(\alpha+\beta)}$ and where the series is absolutely convergent for $|\beta/\alpha|<1$. This implies, since $\left|(re^{i\theta}a_j)/b_j\right|<1$ if $j\in J_r^+$, that

$$\begin{aligned}\sum_{j\in J_r^+}\left|b_j+re^{i\theta}a_j\right|^p &= \sum_{j\in J_r^+}\left(b_j+re^{i\theta}a_j\right)^{p/2}\left(\overline{b_j}+re^{-i\theta}\overline{a_j}\right)^{p/2}\\ &= \sum_{j\in J_r^+}\left(\sum_{k=0}^{\infty}c_p(k)b_j^{p/2}\left(\frac{a_j}{b_j}\right)^k r^k e^{ik\theta}\right)\left(\sum_{l=0}^{\infty}c_p(l)\left(\overline{b_j}\right)^{p/2}\left(\frac{\overline{a_j}}{\overline{b_j}}\right)^l r^l e^{-il\theta}\right).\end{aligned}$$

Since all series converge absolutely, we may interchange the order of the summations to get

$$\begin{aligned}\sum_{j\in J_r^+}\left|b_j+re^{i\theta}a_j\right|^p &= \sum_{k=0}^{\infty}\left(\sum_{l=0}^{\infty}\left(c_p(k)c_p(l)\sum_{j\in J_r^+}|b_j|^p\left(\frac{a_j}{b_j}\right)^k\left(\frac{\overline{a_j}}{\overline{b_j}}\right)^l\right)r^{k+l}\right)e^{i(k-l)\theta}\\ &= \sum_{k=0}^{\infty}\left(\sum_{l=0}^{\infty}B_r^+(k,l)r^{k+l}\right)e^{i(k-l)\theta}.\end{aligned}$$

Setting $l=k-m$ then interchanging the order of the summations we obtain, since

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}=\sum_{k=0}^{\infty}\sum_{-\infty}^{m=k}=\sum_{-\infty}^{m=-1}\sum_{k=0}^{\infty}+\sum_{m=0}^{\infty}\sum_{k=m}^{\infty},$$

that

$$\begin{aligned}\sum_{j\in J_r^+}\left|b_j+re^{i\theta}a_j\right|^p = \sum_{-\infty}^{m=-1}&\left(\sum_{k=0}^{\infty}B_r^+(k,k-m)r^{2k-m}\right)e^{im\theta}\\ &+\sum_{m=0}^{\infty}\left(\sum_{k=m}^{\infty}B_r^+(k,k-m)r^{2k-m}\right)e^{im\theta}.\end{aligned}$$

Replacing m by $(-m)$ in the first summation and replacing k by $(k+m)$ in the fourth summation we obtain

$$\begin{aligned}\text{(2.1)}\quad \sum_{j\in J_r^+}\left|b_j+re^{i\theta}a_j\right|^p = \sum_{m=1}^{\infty}&\left(\sum_{k=0}^{\infty}B_r^+(k,k+m)r^{2k+m}\right)e^{-im\theta}\\ &+\sum_{m=0}^{\infty}\left(\sum_{k=0}^{\infty}B_r^+(k+m,k)r^{2k+m}\right)e^{im\theta}.\end{aligned}$$

Similarly, since $\left|b_j/\left(re^{i\theta}a_j\right)\right| < 1$ if $j \in J_r^-$, we obtain

$$\begin{aligned}\sum_{j\in J_r^-}\left|b_j + re^{i\theta}a_j\right|^p &= \sum_{k=0}^{\infty}\left(\sum_{l=0}^{\infty}\left(c_p(k)c_p(l)\sum_{j\in J_r^-}|a_j|^p\left(\frac{b_j}{a_j}\right)^k\left(\frac{\overline{b_j}}{\overline{a_j}}\right)^l\right)r^{p-(k+l)}\right)e^{-i(k-l)\theta}\\ &= \sum_{k=0}^{\infty}\left(\sum_{l=0}^{\infty}B_r^-(k,l)r^{k+l}\right)e^{-i(k-l)\theta}.\end{aligned}$$

Setting $l = k+m$ then interchanging the order of the summations we obtain, since

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} = \sum_{k=0}^{\infty}\sum_{m=-k}^{\infty} = \sum_{-\infty}^{m=-1}\sum_{k=-m}^{\infty} + \sum_{m=0}^{\infty}\sum_{k=0}^{\infty},$$

that

$$\begin{aligned}\sum_{j\in J_r^-}\left|b_j + re^{i\theta}a_j\right|^p = \sum_{-\infty}^{m=-1}&\left(\sum_{k=-m}^{\infty}B_r^-(k,k+m)r^{p-(2k+m)}\right)e^{im\theta}\\ &+\sum_{m=0}^{\infty}\left(\sum_{k=0}^{\infty}B_r^-(k,k+m)r^{p-(2k+m)}\right)e^{im\theta}.\end{aligned}$$

Replacing m by $(-m)$ in the first summation then replacing k by $(k+m)$ in the second summation, we obtain

$$\begin{aligned}(2.2)\quad \sum_{j\in J_r^-}\left|b_j + re^{i\theta}a_j\right|^p = \sum_{m=1}^{\infty}&\left(\sum_{k=0}^{\infty}B_r^-(k+m,k)r^{p-(2k+m)}\right)e^{-im\theta}\\ &+\sum_{m=0}^{\infty}\left(\sum_{k=0}^{\infty}B_r^-(k,k+m)r^{p-(2k+m)}\right)e^{im\theta}.\end{aligned}$$

It follows from equations (2.1) and (2.2) that

$$\begin{aligned}f(r,\theta) &= \sum_{j\in J}\left|b_j + re^{i\theta}a_j\right|^p = \sum_{j\in J_r^-}\left|b_j + re^{i\theta}a_j\right|^p + \sum_{j\in J_r^+}\left|b_j + re^{i\theta}a_j\right|^p\\ &= \sum_{m=1}^{\infty}\left(\sum_{k=0}^{\infty}B_r^+(k,k+m)r^{2k+m} + \sum_{k=0}^{\infty}B_r^-(k+m,k)r^{p-(2k+m)}\right)e^{-im\theta}\\ &\qquad + \sum_{m=0}^{\infty}\left(\sum_{k=0}^{\infty}B_r^+(k+m,k)r^{2k+m} + \sum_{k=0}^{\infty}B_r^-(k,k+m)r^{p-(2k+m)}\right)e^{im\theta}\\ &:= \sum_{m=1}^{\infty}E_m(r)e^{-im\theta} + \sum_{m=0}^{\infty}D_m(r)e^{im\theta}.\end{aligned}$$

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Noting that

$$B_r^+(l,k) = \overline{B_r^+(k,l)}$$

and

$$B_r^-(l,k) = \overline{B_r^-(k,l)},$$

we obtain that $E_m(r) = \overline{D_m(r)}$ for all $m \geqslant 1$. Since $\{e^{im\theta} : -\infty < m < \infty\}$ is a linearly independent set of functions of θ, it follows that $f(r,\theta)$ is independent of θ if and only if, for each $r > 0$ satisfying $r, (1/r) \notin A$ and for each integer $m \geqslant 1$,

$$D_m(r) := \sum_{k=0}^{\infty} B_r^+(k+m,k) r^{2k+m} + \sum_{k=0}^{\infty} B_r^-(k,k+m) r^{p-(2k+m)} = 0. \tag{2.3}$$

From the definitions of J_r^+ and J_r^-, we get

$$\begin{aligned} \left|B_r^+(k+m,k) r^{2k+m}\right| &\leqslant c_p(k+m)c_p(k)\Big(\sum_{j\in J_r^+} |b_j|^p\Big) \\ &\leqslant c_p(k+m)c_p(k)\Big(\sum_{j\in J} |b_j|^p\Big) \end{aligned}$$

and

$$\begin{aligned} \left|B_r^-(k,k+m) r^{p-(2k+m)}\right| &\leqslant c_p(k)c_p(k+m)\Big(\sum_{j\in J_r^-} |a_j|^p\Big) r^p \\ &\leqslant c_p(k)c_p(k+m)\Big(\sum_{j\in J} |a_j|^p\Big) r^p. \end{aligned}$$

Therefore, since $\sum_{k=0}^{\infty} c_p(k)c_p(k+m)$ converges, we obtain that the two series in equation (2.3) are both uniformly convergent in r on every compact subset of $(0,\infty)$. Hence, since $\{r > 0 : r, (1/r) \notin A\}$ is dense in $(0,\infty)$, we obtain by continuity that equation (2.3) holds for all $r \in (0,\infty)$, which completes the proof. Note that the convergence of $\sum_{k=0}^{\infty} c_p(k)c_p(k+m)$ follows from the convergence of $\sum_{k=0}^{\infty} c_p(k)$ which, in turn, follows from the fact that, for $k \geqslant 1$,

$$\left|\frac{c_p(k+1)}{c_p(k)}\right| = 1 - \frac{1+p/2}{k+1} \leqslant 1 - \frac{3/2}{k+1} \leqslant \Big(1 - \frac{1}{k+1}\Big)^{3/2} = \frac{1/(k+1)^{3/2}}{1/k^{3/2}}$$

and the fact that $\sum_{k=1}^{\infty} 1/k^{3/2}$ converges. ∎

We are now in a position to state our main characterisation theorems for orthogonality in $l_S^p(\mathbf{C})$. First we start with the case where p is not an even integer. We have the following.

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THEOREM 1. *If $p \in [1, \infty)$ is not an even integer, then $(a_j)_{j\in S} \perp (b_j)_{j\in S}$ in $l^p_S(\mathbf{C})$ if and only if, for every real number $r > 0$,*

$$\sum_{j\in J_r} |b_j|^p \left(\frac{a_j}{b_j}\right)^m = 0,$$

for all integers $m \geqslant 1$. Equivalently, if and only if, for every real number $r > 0$,

$$\sum_{j\in J_r} |a_j|^p \left(\frac{b_j}{a_j}\right)^m = 0,$$

for all integers $m \geqslant 1$.

PROOF: First note that if $r \notin A$ then $J_r = \emptyset$ and the result follows by our assumption that $\sum\limits_{\emptyset} := 0$. Since p is not an even integer, it follows that, for each $m \geqslant 1$, the set of functions $\left\{g_k(r) = r^{2k+m} : k \geqslant 0\right\} \cup \left\{h_k(r) = r^{p-(2k+m)} : k \geqslant 0\right\}$ is linearly independent on $(0, \infty)$. Hence we obtain from Lemma 2 that, for each $m \geqslant 1$ and each $k \geqslant 0$,

$$B_r^+(k+m, k) = B_r^-(k, k+m) = 0 \text{ for all } r > 0.$$

Let $r_1 > 0$ be fixed. Then, for all $r \in (0, r_1)$, we have

$$B_r^+(k+m, k) - B_{r_1}^+(k+m, k) = 0.$$

Since p is not an even integer we get that $c_p(k+m)c_p(k) \neq 0$ for all $m \geqslant 1$ and all $k \geqslant 0$. Hence we obtain, for each $m \geqslant 1$ and each $k \geqslant 0$,

$$\sum_{j\in J_r^+ \setminus J_{r_1}^+} |b_j|^p \left(\frac{a_j}{b_j}\right)^{k+m} \left(\frac{\overline{a_j}}{\overline{b_j}}\right)^k = 0 \text{ for all } r \in (0, \infty).$$

Note that

$$(J_r^+ \setminus J_{r_1}^+) \searrow J_{r_1} \quad \text{as } r \nearrow r_1.$$

Therefore, since the series $\sum\limits_{j\in J} |b_j|^p (a_j/b_j)^{k+m} \left(\overline{a_j}/\overline{b_j}\right)^k$ is absolutely convergent, we obtain by taking the limit as $r \nearrow r_1$ that

$$\sum_{j\in J_{r_1}} |b_j|^p \left(\frac{a_j}{b_j}\right)^{k+m} \left(\frac{\overline{a_j}}{\overline{b_j}}\right)^k = \sum_{j\in J_{r_1}} |b_j|^p \left|\frac{a_j}{b_j}\right|^{2k} \left(\frac{a_j}{b_j}\right)^m = 0\,.$$

But, $|a_j/b_j| = r_1$, for all $j \in J_{r_1}$. Therefore, dividing by r_1^{2k}, we obtain

$$\sum_{j\in J_{r_1}} |b_j|^p (a_j/b_j)^m = 0\,. \tag{2.4}$$

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Multiplying by $r_1^{p-2m} = \left(|a_j|^p \,/\, |b_j|^p\right)\left(\overline{b_j}/\overline{a_j}\right)^m (b_j/a_j)^m$ and then taking the conjugate we obtain

$$\sum_{j\in J_{r_1}} |a_j|^p \left(\frac{b_j}{a_j}\right)^m = 0\,. \tag{2.5}$$

Conversely, if Equation (2.4) or Equation (2.5) holds then both hold true. Multiplying by $c_p\,(k+m)\,c_p\,(k)\,r_1^{2k}$ and by $c_p\,(k)\,c_p\,(k+m)\,r_1^{-2k}$ respectively and noting that $r_1 = |a_j/b_j|$, we get that, for each $r_1 > 0$,

$$B_{r_1}^{+}(k+m,k) = B_{r_1}^{-}(k,k+m) = 0\,,$$

for all $k \geqslant 0$ and all $m \geqslant 1$. Hence, by Lemma 2, $(a_j)_{j\in S} \perp (b_j)_{j\in S}$ in $l_S^p(\mathbf{C})$. This completes the proof. ∎

The following equivalent version for Theorem 1 follows immediately.

Corollary 1. *If $p \in [1,\infty)$ is not an even integer, then $(a_j)_{j\in S} \perp (b_j)_{j\in S}$ in $l_S^p(\mathbf{C})$ if and only if, for every real number $r > 0$,*

$$\sum_{j\in J_r} |b_j|^p\, e^{im\theta_j} = 0\,,$$

for all integers $m \geqslant 1$. Equivalently, if and only if, for every real number $r > 0$,

$$\sum_{j\in J_r} |a_j|^p\, e^{-im\theta_j} = 0\,,$$

for all integers $m \geqslant 1$, where $|a_j/b_j|\, e^{i\theta_j} := a_j/b_j$, $j \in J$.

Corollary 2. *If $J \neq \emptyset$ and if $p \in [1,\infty)$ is not an even integer, then $(a_j)_{j\in S} \perp (b_j)_{j\in S}$ in $l_S^p(\mathbf{C})$ if and only if, for each real number $r \in A$, $(a_j)_{j\in J_r} \perp (b_j)_{j\in J_r}$ in $l_{J_r}^p(\mathbf{C})$.*

For the cases where p is an even integer, we have the following.

Theorem 2. *If p is a positive even integer, then $(a_j)_{j\in S} \perp (b_j)_{j\in S}$ in $l_S^p(\mathbf{C})$ if and only if*

$$\sum_{j\in J} |b_j|^p \left|\frac{a_j}{b_j}\right|^{2k} \left(\frac{a_j}{b_j}\right)^m = 0\,,$$

for all integers m, k, $1 \leqslant m \leqslant p/2$, $0 \leqslant k \leqslant (p/2) - m$. Equivalently, if and only if

$$\sum_{j\in J} |a_j|^p \left|\frac{b_j}{a_j}\right|^{2k} \left(\frac{b_j}{a_j}\right)^m = 0\,,$$

for all integers m, k, $1 \leqslant m \leqslant p/2$, $0 \leqslant k \leqslant (p/2) - m$.

PROOF: Let $p = 2n$, $n \geqslant 1$. The binomial formula gives

$$\begin{aligned}
f(r,\theta) &= \sum_{j \in J} \left(b_j + re^{i\theta} a_j\right)^n \left(\overline{b_j} + re^{-i\theta}\overline{a_j}\right)^n \\
&= \sum_{j \in J} \left(\sum_{k=0}^{n} \binom{n}{k} b_j^{n-k} a_j^k r^k e^{ik\theta}\right) \left(\sum_{l=0}^{n} \binom{n}{l} \left(\overline{b_j}\right)^{n-l} \left(\overline{a_j}\right)^l r^l e^{-il\theta}\right) \\
&= \sum_{k=0}^{n} \sum_{l=0}^{n} \binom{n}{k} \binom{n}{l} \left(\sum_{j \in J} a_j^k \left(\overline{a_j}\right)^l b_j^{n-k} \left(\overline{b_j}\right)^{n-l}\right) r^{k+l} e^{i(k-l)\theta}.
\end{aligned}$$

Setting $l = k - m$ then interchanging the order of the summations we obtain, since

$$\sum_{k=0}^{n} \sum_{l=0}^{n} = \sum_{k=0}^{n} \sum_{m=k-n}^{k} = \sum_{m=-n}^{-1} \sum_{k=0}^{n+m} + \sum_{m=0}^{n} \sum_{k=m}^{n},$$

that

$$\begin{aligned}
f(r,\theta) = &\sum_{m=-n}^{-1} \sum_{k=0}^{n+m} \binom{n}{k} \binom{n}{k-m} \left(\sum_{j \in J} a_j^k \left(\overline{a_j}\right)^{k-m} b_j^{n-k} \left(\overline{b_j}\right)^{n-k+m}\right) r^{2k-m} e^{im\theta} \\
&+ \sum_{m=0}^{n} \sum_{k=m}^{n} \binom{n}{k} \binom{n}{k-m} \left(\sum_{j \in J} a_j^k \left(\overline{a_j}\right)^{k-m} b_j^{n-k} \left(\overline{b_j}\right)^{n-k+m}\right) r^{2k-m} e^{im\theta}.
\end{aligned}$$

Replacing m by $(-m)$ in the first summation then replacing k by $(k+m)$ in the fourth summation, we obtain

$$\begin{aligned}
f(r,\theta) = &\sum_{m=1}^{n} \sum_{k=0}^{n-m} \binom{n}{k} \binom{n}{k+m} \left(\sum_{j \in J} a_j^k \left(\overline{a_j}\right)^{k+m} b_j^{n-k} \left(\overline{b_j}\right)^{n-k-m}\right) r^{2k+m} e^{-im\theta} \\
&+ \sum_{m=0}^{n} \sum_{k=0}^{n-m} \binom{n}{k+m} \binom{n}{k} \left(\sum_{j \in J} a_j^{k+m} \left(\overline{a_j}\right)^{k} b_j^{n-k-m} \left(\overline{b_j}\right)^{n-k}\right) r^{2k+m} e^{im\theta}.
\end{aligned}$$

Noting that the first double summation is the conjugate of the second, it follows, since $\{e^{im\theta} : -n \leqslant m \leqslant n\}$ is a linearly independent set of functions of θ, that $f(r,\theta)$ is independent of θ if and only if, for each m, $1 \leqslant m \leqslant n$,

$$\sum_{k=0}^{n-m} \binom{n}{k} \binom{n}{k+m} \left(\sum_{j \in J} a_j^{k+m} \left(\overline{a_j}\right)^{k} b_j^{n-k-m} \left(\overline{b_j}\right)^{n-k}\right) r^{2k+m} = 0, \ r > 0.$$

This is equivalent to, since the set of functions $\{g_k(r) = r^{2k+m} : 0 \leqslant k \leqslant n-m\}$ is linearly independent on $(0,\infty)$,

$$\sum_{j\in J} a_j^{k+m}\left(\overline{a_j}\right)^k b_j^{n-k-m}\left(\overline{b_j}\right)^{n-k} = \sum_{j\in J} |b_j|^p \left|\frac{a_j}{b_j}\right|^{2k}\left(\frac{a_j}{b_j}\right)^m = 0,$$

for all m, k, $1 \leqslant m \leqslant n$, $0 \leqslant k \leqslant n-m$. In other words, setting $k' := n-m-k$,

$$\sum_{j\in J} |b_j|^{2k'} |a_j|^{p-2m-2k'} \left(\overline{a_j} b_j\right)^m = \sum_{j\in J} |a_j|^p \left|\frac{b_j}{a_j}\right|^{2k'}\left(\frac{b_j}{a_j}\right)^m = 0,$$

for all m, k', $1 \leqslant m \leqslant n$, $0 \leqslant k' \leqslant n-m$. This completes the proof. $\square$

In particular, if $p=2$ then $n=m=1$, $k=0$ and $\sum_{j\in S} a_j\overline{b_j} = \sum_{j\in J} |b_j|^2 (a_j/b_j)$. This implies that in the cases of the Hilbert spaces $l_S^2(\mathbf{C})$ we get back the usual notion of orthogonality. Therefore we have the following.

Corollary 3. *$(a_j)_{j\in S} \perp (b_j)_{j\in S}$ in $l_S^2(\mathbf{C})$ if and only if*

$$\sum_{j\in S} a_j\overline{b_j} = 0\,.$$

Clearly, one has the following equivalent version for Theorem 2.

Corollary 4. *If p is a positive even integer, then $(a_j)_{j\in S} \perp (b_j)_{j\in S}$ in $l_S^p(\mathbf{C})$ if and only if*

$$\sum_{j\in J} |b_j|^p \left|\frac{a_j}{b_j}\right|^{2k+m} e^{im\theta_j} = 0\,,$$

for all integers m, k, $1 \leqslant m \leqslant p/2$, $0 \leqslant k \leqslant (p/2)-m$. Equivalently, if and only if

$$\sum_{j\in J} |a_j|^p \left|\frac{b_j}{a_j}\right|^{2k+m} e^{-im\theta_j} = 0\,,$$

for all integers m, k, $1 \leqslant m \leqslant p/2$, $0 \leqslant k \leqslant (p/2)-m$, where $|a_j/b_j|\, e^{i\theta_j} := a_j/b_j$, $j \in J$.

We note that in the case of $l_S^p(\mathbf{R})$ things are different. Indeed, in the case where $p=4$ for example, the binomial formula gives that

$$f(r,0) := |a_1 + b_1 r|^4 + |a_2 + b_2 r|^4 = \sum_{k=0}^{4} \binom{4}{k} \left(a_1^{4-k} b_1^k + a_2^{4-k} b_2^k\right) r^k.$$

Since $(a_1, a_2) \perp (b_1, b_2)$ in $\ell_2^4(\mathbf{R})$ if and only if $f(r,0) = f(-r,0)$ for all $r \in \mathbf{R}$. we obtain the following.

Lemma 3. *$(a_1, a_2) \perp (b_1, b_2)$ in $\ell_2^4(\mathbf{R})$ if and only if*

$$a_1^3 b_1 + a_2^3 b_2 = a_1 b_1^3 + a_2 b_2^3 = 0.$$

We finish with the following open problem.

Problem 2. Under which conditions are two vectors in $l_S^p(\mathbf{R})$ orthogonal?

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