A NEW KIND OF FREE EXTENSION FOR PROJECTIVE PLANES

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1. Introduction. Marshall Hall [1] shows how projective planes of very general structure may be constructed and at the same time exhibits an extensive class which are non-Desarguesian. Here we shall indicate how his method of free extension can be generalized to yield a class of planes which seem to be distinct from those which he obtains.

A partial plane is a system consisting of two distinct sets of elements, a set of "points" P,Q,... and a set of "lines" ℓ, m,..., and a relation between these two sets, called "incidence", such that for any two distinct points, there is at most one line incident with both (or, equivalently, for any two distinct lines, there is at most one point incident with both). A partial plane is complete if every two distinct points are joined by a line and every two distinct lines intersect in a point.

Given an arbitrary partial plane π_1, we can define a sequence $\{\pi_n\}$ of partial planes as follows: let π_1 be the original partial plane and then, assuming π_r has already been defined, let π_{r+1} consist of the points, lines, and incidence relation of π_r plus (1) new points - all the pairs of lines of π_r which do not intersect in π_r, and (2) new lines - all the pairs of points of π_r which are not joined in π_r. The new points and lines are said to be incident with the elements which are used to define them. The above sequence can now be used to define a complete plane π by stipulating that an element

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belongs to π if and only if it belongs to some π_i, $i = 1, 2, \ldots$, and P and $\mathscr{L}$ are incident in π if and only if they are incident in some π_i. The plane π so defined is called the free extension of π_1 and we shall denote it by $F(\pi_1)$.

More generally, π_1 can be extended to a complete plane by taking as new lines sets of (two or more) points of π_1 such that every pair of points of π_1 not joined in π_1 is in precisely one set and such that no two points in any set are joined by a line of π_1, then carrying out a similar operation on the lines of π_1 to form new points, and so on a denumerable number of times.

2. The New Free Extension. The method of extension now to be considered differs from Marshall Hall's in that some of the new points and lines are to be sets consisting of merely one element. Thus, if P is a point of a partial plane π_1, we can extend π_1 by taking the singleton $\{P\}$ (the set consisting of the single element P) to be a line through P and through no other point already present. (In fact, we could "draw" more than one line through P by introducing indexed sets whereby, for example, the lines $\{P\}_1$ and $\{P\}_2$ would be considered as distinct. This more general method of extension will not be considered here.)

For any partial plane π_1 a sequence of partial planes π_n, $n = 0, 1, 2, \ldots$, can be defined. Let π_0 be the null set, π_1 the original system, and, assuming π_r, $r \geq 1$, has already been defined, let π_{r+1} be obtained from π_r as follows:

(1) For any two points P, Q that are not joined in π_r add a line, the pair $\{P, Q\}$;

(2) For any two lines $\mathscr{L}, m$ that do not intersect in π_r add a point, the pair $\{\mathscr{L}, m\}$;

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(3) For any point P in π_r but not in π_{r-1} add a line, the singleton $\{P\}$;

(4) For any line $\mathscr{L}$ in π_r but not in π_{r-1} add a point, the singleton $\{\mathscr{L}\}$.

As before, this sequence determines a plane which we shall denote by $F^*(\pi_1)$.

We shall say that an element κ (point or line) is of rank r, or rank $(\kappa) = r$, if κ is in π_r but not in π_{r-1}.

It should be observed that in contrast to the method of free extension, as long as π_1 is not the void plane, $F^*(\pi_1)$ always contains π_1 properly (whether π_1 is complete or not) and is nondegenerate, i.e. contains four points no three collinear. Moreover, $F^*(\pi_1)$ is always non-Desarguesian since it contains the free extension of a free 4-point (the partial plane consisting of two points on a line and two points not on it) which, as is pointed out by Marshall Hall, is itself non-Desarguesian.

A question which naturally arises at this point is whether or not the planes generated in this manner are essentially distinct from those generated purely by free extension. This question is partially answered in the following theorem.

THEOREM. If π_1 and ψ_1 are partial planes, where π_1 is finite but not empty, then $F(\pi_1)$ and $F^*(\psi_1)$ are not isomorphic.

Proof. Assume $F(\pi_1)$ and $F^*(\psi_1)$ are isomorphic. Then if $\pi_1 \rightleftarrows \pi_1'$ in the isomorphism, since π_1 is finite, π_1' is finite and hence $\{\text{rank}(\kappa) : \kappa \in \pi_1'\}$ has a greatest member, N. Let ψ_N be the system consisting of all elements of $F^*(\psi_1)$ of rank r, $r \leq N$. Then $\pi_1' \subset \psi_N$. Also, there is at least one element λ' of π_1' of rank N.

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Suppose λ' is a point P'. Then there is a point P in π_1 such that $P \rightleftarrows P'$. Let $\mathcal{L}$ be any line on P and let $\mathcal{L} \rightleftarrows \mathcal{L}'$. If $\mathcal{L}'$ is in ψ_N then $\mathcal{L}' \neq \{P'\}$ since rank $(\{P'\}) = N + 1$. If $\mathcal{L}'$ is not in ψ_N then $\mathcal{L}$ is not in π_1 and is therefore constructible from π_1 by free extension. Then, under the isomorphism, the construction for $\mathcal{L}$ induces a construction for $\mathcal{L}'$ where each element obtained is either in ψ_N or is obtainable from ψ_N by free extension. It follows that $\mathcal{L}'$ can be constructed from ψ_N by free extension and hence $\mathcal{L}' \neq \{P'\}$. Thus $\{P'\}$ is a line on P' such that if $\mathcal{L} \rightleftarrows \{P'\}$, then $\mathcal{L}$ is not on P. This contradicts the existence of an isomorphism.

Similarly, if λ' is a line a contradiction is obtained. This proves the theorem.

It seems very likely that the above theorem is also true when π_1 is infinite and (of course) incomplete.

The results of this paper are contained in an M. Sc. thesis written under the supervision of Professor N. S. Mendelsohn.

REFERENCE

1. Hall, Marshall, "Projective Planes," Trans. Amer. Math. Soc., LIV (1943), 229-277.

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