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A THEOREM ON ABSTRACT SEGAL ALGEBRAS OVER SOME COMMUTATIVE BANACH ALGEBRAS

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Let B be a commutative, semi-simple, regular, Tauberian Banach algebra with noncompact maximal ideal space $\Delta(B)$. Suppose B has the property that there is a constant C such that for every compact subset K of $\Delta(B)$ there exists a $f \in B$ with $\hat{f} = 1$ on K, $\|f\|_B \leq C$ and $\hat{f}$ has compact support. We prove that if A is a proper abstract Segal algebra over B then for every positive integer n there exists $f \in B$ such that $f^n \notin A$ but $f^{n+1} \in A$. As a consequence of this result we prove that if G is a nondiscrete locally compact abelian group, μ a positive unbounded Radon measure on Γ (the dual group of G), $1 \leq p < q < \infty$ and $A_p(G, \mu) = \{f \in L_1(G) : \hat{f} \in L_p(\Gamma, \mu)\}$, then $A_p(G, \mu) \subsetneq A_q(G, \mu)$.

1. Introduction

Let G be a nondiscrete locally compact abelian group. Normed ideals in $L_1(G)$ were introduced by Cigler [2]. A normed ideal in $L_1(G)$ is a dense ideal A of $L_1(G)$ which is a Banach algebra with a norm of its own such that $\|f\|_A \geq \|f\|_1$ for every $f \in A$ and there is a constant M such that $\|f * g\|_A \leq M\|f\|_1\|g\|_A$ for $f \in L_1(G)$ and $g \in A$. Graham [3] proved

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that if A is a proper normed ideal in $L_1(G)$ and n is a positive integer then there exists a $f \in L_1(G)$ such that $f^n \notin A$ but $f^{n+1} \in A$. In this paper, using essentially the ideas of Graham, we prove a more general result, where $L_1(G)$ is replaced by a commutative, semi-simple, regular, Tauberian Banach algebra B with noncompact maximal ideal space $\Delta(B)$ such that there is a constant C with the property that for every compact subset K of $\Delta(B)$ there is a $f \in B$ with $\hat{f} = 1$ on K, $\|f\|_B \leq C$ and $\hat{f}$ has compact support, and A is replaced by a proper abstract, Segal algebra (for definition, see §2) over B.

As an application of our theorem for the case $B = L_1(G)$, we deduce that if $1 \leq p < q < \infty$ then $A_p(G, \mu) \subsetneq A_q(G, \mu)$, where G is a non-discrete locally compact abelian group, μ a positive unbounded Radon measure on Γ (the dual group of G) and for $1 \leq p < \infty$,

$$A_p(G, \mu) = \{f \in L_1(G) : \hat{f} \in L_p(\Gamma, \mu)\} .$$

$A_p(G, \mu)$ is a Banach algebra with convolution as multiplication and the norm given by

$$\|f\|_{A_p(G,\mu)} = \|f\|_1 + \|\hat{f}\|_{L_p(\Gamma,\mu)} , \text{ for } f \in A_p(G, \mu) .$$

The algebras $A_p(G, \mu)$ first appeared in [*10*, p. 25, Example (vi)] as examples of Segal algebras and their importance was indicated there. When μ is the Haar measure on Γ, $A_p(G, \mu)$ is denoted by $A_p(G)$. The algebras $A_p(G)$ were introduced by Larsen, Liu and Wang [7] in 1964 and have been studied by several authors. In [5] and [6], Larsen stated without proof that the algebras $A_p(G)$ are distinct for distinct p; however a proof, which is nontrivial, was given by Tewari and Gupta in [*11*]. The proof in [*11*] depends, among other things, on the structure theorem of locally compact abelian groups and does not seem to generalise to give a corresponding result for the algebras $A_p(G, \mu)$. Our proof for this fact based on our main theorem is very simple and elegant. The essential difference between the proof of our main theorem and that of Graham for the case $B = L_1(G)$ consists in the following. Whereas Graham needs to construct two sequences $\{f_j\}$ and $\{\phi_j\}$ with certain properties we need to construct only one to carry the proof through.

2. Main results

Let B be a commutative Banach algebra and let $\Delta(B)$ be its maximal ideal space. We say that B has *property* (P) if there is a constant C such that for every compact subset K of $\Delta(B)$ there exists $x \in B$ with $\hat{x} = 1$ on K, $\hat{x}$ has compact support and $\|x\|_B \leq C$.

The group algebra $L_1(G)$ of any locally compact abelian group has property (P) (see [9], Chapter 5). If X is a locally compact noncompact Hausdorff space, then the Banach algebra $C_0(X)$, consisting of all continuous functions vanishing at infinity, also has property (P). It is well known that $C_0(X)$ is a commutative, semisimple, regular, Tauberian Banach algebra with noncompact maximal ideal space.

The notion of abstract Segal algebras was introduced by Burnham in [1]. Let B be a commutative Banach algebra. An abstract Segal algebra A over B is a dense ideal of B which is a Banach algebra with a norm of its own and such that the following conditions hold:

(i) there is a constant M such that

$$\|f\|_B \leq M\|f\|_A \text{, for every } f \in A ;$$

(ii) there is a constant N such that

$$\|fg\|_A \leq N\|f\|_B\|g\|_A \text{, for } f \in B \text{ and } g \in A .$$

Burnham defined abstract Segal algebras over Banach algebras which are not necessarily commutative. He also observed that an abstract Segal algebra over $L_1(G)$ is simply a normed ideal (see Cigler [2]), of $L_1(G)$. We are now ready to state and prove our generalisation of Graham's main result in [3].

THEOREM 1. *Let B be a commutative, semi-simple, regular, Tauberian Banach algebra having property* (P) *with non-compact maximal ideal space. Suppose A is a proper abstract Segal algebra over B. Then for each*

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positive integer n *, there exists* $f \in B$ *such that* $f^n \notin A$ *and* $f^{n+1} \in A$.

Proof. Let $F = \{f \in B : \hat{f}$ has compact support$\}$. Since A is a dense ideal in B , A has empty cospectrum. It follows from Theorem 25E, p. 85 and the lemma on p. 86 of [8] that $F \subseteq A$. We shall first construct a sequence $\{f_j\}$ in F such that

(i) $f_j \cdot f_k = 0$ if $j \neq k$, and

(ii) $\|f_j\|_B \leq 1$ and $\left\|f_j^{n+1}\right\|_A \geq j$.

To this end, let us first observe that $\sup\left\{\|g^{n+1}\|_A : \|g\|_B \leq 1,\ g \in F\right\}$ is infinite. Suppose not; then there exists a constant L such that $g \in F$ and $\|g\|_B \leq 1$ implies that $\|g^{n+1}\|_A \leq L$. Let $g \in F$; choose $h \in F$ such that $\hat{h} = 1$ on support of g and $\|h\|_B \leq C$. Then, $h^{n+1} \cdot g = g$ and

$$(1) \qquad \|g\|_A = \|h^{n+1}g\|_A \leq N\|h^{n+1}\|_A\|g\|_B \leq LNC^{n+1}\|g\|_B ,$$

by the definition of the abstract Segal algebra and the property of the constant L . It follows that the A and B norms on F are equivalent. Since B is Tauberian, F is dense in B . It now follows that $B \subseteq A$, contradicting that A is proper. Therefore,

$$\sup\left\{\|g^{n+1}\|_A : \|g\|_B \leq 1,\ g \in F\right\}$$

is infinite. Hence there is a sequence $\{g_j\}$ in F such that $\|g_j\|_B \leq 1$ and $\left\|g_j^{n+1}\right\|_A \geq j$, for every j .

Let $f_1 = g_1$ and suppose $f_1, \ldots, f_m$ belonging to F have been chosen such that $\|f_j\|_B \leq 1$, $\left\|f_j^{n+1}\right\|_A \geq j$ and $\hat{f}_j$ have mutually disjoint support, $j = 1, \ldots, m$. We observe that if $\hat{f}_j$ and $\hat{f}_k$ have disjoint supports then $f_j \cdot f_k = 0$. For choosing f_{m+1} we proceed as follows.

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Let K be a compact subset of $\Delta(B)$ such that $\bigcup_{1}^{m} \operatorname{supp} \hat{f}_j$ is contained in the interior of K. Let $\phi \in F$ be such that $\hat{\phi} = 1$ on K, and $\|\phi\|_B \leq C$. For any positive integer j, we have

$$j \leq \left\|g_j^{n+1}\right\|_A \leq \sum_0^n N\binom{n+1}{k}\left\|(g_j-\phi g_j)^k\right\|_B\left\|(\phi\cdot g_j)^{n+1-k}\right\|_A + \left\|(g_j-\phi g_j)^{n+1}\right\|_A$$
$$\leq \sum_0^n \binom{n+1}{k}(C+1)^k N^2\|\phi\|_A^{n+1-k} + \left\|(g_j-\phi g_j)^{n+1}\right\|_A .$$

The first term on the right of the above inequality is independent of j and hence we can choose j to be so large that

$$\left\|(g_j-\phi g_j)^{n+1}\right\|_A \geq (C+1)^{n+1}(m+1) .$$

Let

$$f_{m+1} = \frac{g_j-\phi g_j}{\|g_j-\phi g_j\|_B} .$$

Then $f_{m+1} \in F$, $\|f_{m+1}\|_B = 1$ and $\left\|f_{m+1}^{n+1}\right\|_A \geq m+1$. Furthermore, since $\hat{\phi} = 1$ on K, $\hat{f}_{m+1} = 0$ on K and therefore support of $\hat{f}_{m+1}$ is disjoint from support of $\hat{f}_j$, $j = 1, \ldots, m$. Now, the construction of the sequence $\{f_j\}$ in F satisfying (i) and (ii) is completed by induction.

Having constructed the sequence $\{f_j\}$ in $F \subseteq A$ as above, we complete the proof of the theorem as follows. Choose a subsequence $\{j_k\}$ of positive integers such that

$$\sum_{k=1}^{\infty} k^{1+(1/n)}\left\|f_{j_k}^{n+1}\right\|_A^{-1/n} < \infty .$$

Let

$$\alpha_k = k^{1/n}\left\|f_{j_k}^{n+1}\right\|_A^{-1/n} ,$$

so that

$$\sum_{k=1}^{\infty} k\alpha_k < \infty .$$

Let

$$f = \sum_{k=1}^{\infty} \alpha_k f_{j_k} .$$

Since $\|f_{j_k}\|_B \leq 1$ for each k , $f \in B$. Now

$$f^n = \sum_{k=1}^{\infty} \alpha_k^n f_{j_k}^n .$$

For any k , we have $f_{j_k} \cdot f^n = \alpha_k^n f_{j_k}^{n+1}$. Thus

$$N\|f^n\|_A \geq \left\|f_{j_k} \cdot f^n\right\|_A = \alpha_k^n \left\|f_{j_k}^{n+1}\right\|_A = k .$$

Since this is true for each k , we conclude that $f^n \notin A$. Now we shall show that $f^{n+1} \in A$. To this end, we have

$$f^{n+1} = \sum_{k=1}^{\infty} \alpha_k^{n+1} f_{j_k}^{n+1}$$

and

$$\sum_{k=1}^{\infty} \alpha_k^{n+1} \left\|f_{j_k}^{n+1}\right\|_A = \sum_{k=1}^{\infty} k\alpha_k < \infty$$

and this clearly implies that $f^{n+1} \in A$. This completes the proof of our theorem.

The following theorem of Graham [3] is a particular case of our theorem.

THEOREM 2. *Let G be a nondiscrete locally compact abelian group. Suppose A is a proper normed ideal of $L_1(G)$. Then for each positive integer n , there exists $f \in L_1(G)$ such that $f^n \notin A$ and $f^{n+1} \in A$.*

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Proof. We simply remark that $L_1(G)$ is a commutative, semi-simple, regular, Tauberian Banach algebra having property (P) with noncompact maximal ideal space and that a normed ideal of $L_1(G)$ is an abstract Segal algebra over $L_1(G)$.

The following theorem is a nontrivial generalisation of Theorem 2 of Tewari and Gupta [11].

THEOREM 3. *Let* G *be a nondiscrete locally compact abelian group and let* μ *be a positive unbounded Radon measure on* Γ *, the dual group of* G *. Then* $A_p(G, \mu)$ *is a proper subspace of* $A_q(G, \mu)$ *where* $1 \le p < q < \infty$ *. Moreover,* $\bigcup_{p<q} A_p(G, \mu)$ *is of first category in* $A_q(G, \mu)$ *.*

Proof. It is obvious from the definition that if $p < q$ then $A_p(G, \mu) \subseteq A_q(G, \mu)$. It is also well known [10] that $A_p(G, \mu)$ is Segal algebra and hence it is a normed ideal of $L_1(G)$. Since $p < q$, there exists a positive integer n such that $(n+1)p < nq$. For this n there is a $f \in L_1(G)$ such that $f^n \notin A_q(G, \mu)$ but $f^{n+1} \in A_q(G, \mu)$. Then $f^{n+1} \notin A_p(G, \mu)$. Thus $A_p(G, \mu) \subsetneq A_q(G, \mu)$. Since the inclusion map of $A_p(G, \mu)$ to $A_q(G, \mu)$ is continuous and both are Banach spaces, it follows from the open mapping theorem (11.4, [4]) that $A_p(G, \mu)$ is a set of first category in $A_q(G, \mu)$. Let $\{p_n\}$ be a sequence of real numbers such that $1 \le p_n < q$ and $p_n \to q$. Then

$$\bigcup_{1 \le p < q} A_p(G, \mu) = \bigcup_{n=1}^{\infty} A_{p_n}(G, \mu) ,$$

and it follows that $\bigcup_{1 \le p < q} A_p(G, \mu)$ is a set of first category in $A_q(G, \mu)$. This completes the proof.

REMARK. The condition that B has property (P) in Theorem 1 cannot be relaxed. For example, if $q > 1$, then $A_q(G, \mu)$ is a commutative semisimple, regular, Tauberian Banach algebra with noncompact maximal ideal space provided G is nondiscrete. Let p be such that $1 \leq p < q$. Choose a positive integer n such that $np > q$. Then $f^n \in A_p(G, q)$ for every $f \in A_q(G, \mu)$. It is obvious that $A_p(G, \mu)$ is an abstract Segal algebra over $A_q(G, \mu)$.

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