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THRESHOLD OF GLOBAL EXISTENCE FOR THE CRITICAL NONLINEAR GROSS–PITAEVSKII EQUATION

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Abstract This paper is concerned with the critical nonlinear Gross–Pitaevskii equation, which describes the attractive Bose–Einstein condensate under a magnetic trap. We derive a sharp threshold between the global existence and the blowing-up of the system. Furthermore, we answer the question: how small are the initial data, such that the system has global solutions for the nonlinear critical power $p = 1 + (4/N)$?

Keywords: threshold; global existence; blow-up; critical; Gross–Pitaevskii equation

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1. Introduction

In this paper we consider the nonlinear Gross–Pitaevskii equation

$$\mathrm{i}\varphi_t = -\Delta\varphi + |x|^2\varphi - |\varphi|^{p-1}\varphi, \quad \varphi(x, 0) = \varphi_0. \tag{1.1}$$

Here $\varphi : \mathbb{R}^N \times [0, \infty) \to \mathbb{C}$ is a complex-valued function, $1 < p < \infty$ when $N = 1, 2$ and $1 < p < (N + 2)/(N - 2)$ when $N \geqslant 3$. Equation (1.1) models the Bose–Einstein condensate with attractive interparticle interactions under a magnetic trap [**2**, **8**, **13**, **19**].

Meanwhile, as a class of nonlinear Schrödinger equation with potentials, equation (1.1) has a special mathematical significance. For equation (1.1) with a general real-valued potential function $V(x)$, when $|D^\alpha V(x)|$ is bounded for all $\alpha \geqslant 2$, Fujiwara [**9**] proved that the smoothness of the time 0 of Schrödinger kernel for potentials of quadratic growth. And Yajima [**21**] showed that, for superquadratic potentials, the Schrödinger kernel is nowhere C^1. It is shown in [**16**] that quadratic potentials are the highest order of potential for local well-posedness of the equation. Then $V(x) = |x|^2$ is the critical potential for the local existence of the Cauchy problem.

In the case of equation (1.1), Oh [**16**] and Cazenave [**5**] established the local existence of the Cauchy problem in the natural energy space. Zhang [**24**] proved that, when $p < 1 + (4/N)$, global solutions of the Cauchy problem (1.1) exist for any initial data in the energy space. On the other hand, when $p \geqslant 1 + (4/N)$, Cazenave [**5**], Carles [**3**, **4**] and

Zhang [**23**] showed that the solutions of the Cauchy problem (1.1) blow up in a finite time for some initial data, especially for a class of sufficiently large data; but the solutions of the Cauchy problem (1.1) exist globally for other initial data, especially for a class of sufficiently small data [**3**, **4**, **24**]. So the problem of finding the sharp threshold between the global solutions and the blowing-up solutions for the Cauchy problem (1.1) arises for $p \geqslant 1 + (4/N)$. Owing to the fact that equation (1.1) plays an important role in Bose–Einstein condensates, this problem has also been pursued strongly from a physics point of view [**2**, **6**, **8**, **10**, **12**–**14**, **17**–**19**, **22**].

Chen and Zhang [**7**] obtained a sharp threshold between the global solutions and the blowing-up solutions of the Cauchy problem (1.1), but this result holds only for the case of supercritical power, $p > 1 + (4/N)$. On the other hand, for the critical power, $p = 1 + (4/N)$, Zhang [**23**] obtained a sharp condition of global existence for the Cauchy problem (1.1) that was similar to that in Weinstein [**20**]. However, the result in [**23**] does not solve the problem of finding the sharp threshold between the global solutions and the blowing-up solutions, because Zhang's result shows that when the initial values φ_0 are less than N_{c}, only global solutions appear; however, when the initial values φ_0 are greater than or equal to N_{c}, there exist not only blowing-up solutions, but also global solutions such as large standing-wave solutions. Here N_{c} is a number that depends only on the dimension. Therefore, for the critical power, $p = 1 + (4/N)$, the problem of finding the sharp threshold between the global solutions and the blowing-up solutions for the Cauchy problem (1.1) is still open. In this paper, we shall solve this problem.

In this paper we fix $p = 1 + (4/N)$. In the next section, we give some preliminaries. In § 3, we construct some proper functionals, and pose a constrained variational problem, which we then solve. In § 4, by combining the variational character with the invariant properties of the local semi-flows of the evolution system, we get a sharp threshold between the global existence and blowing-up. These arguments originate in [**1**, **25**] and in Levine's concavity method [**15**]. Furthermore, we answer the question: how small are the initial data, such that the Cauchy problem (1.1) has global solutions for $p = 1 + (4/N)$?

2. Preliminaries

Firstly, we naturally set

$$H := \left\{ u \in H^1(\mathbb{R}^N) : \int |x|^2 |u|^2 \,\mathrm{d}x < \infty \right\}, \tag{2.1}$$

where $H^1(\mathbb{R}^N) = \{u : u \in L^2(\mathbb{R}^N) \text{ and } \partial_{x_i} u \in L^2(\mathbb{R}^N),\ i = 1, 2, \dots, N\}$. Henceforth, for simplicity, we denote $\int_{\mathbb{R}^N} \cdot \,\mathrm{d}x$ by $\int \cdot \,\mathrm{d}x$. H becomes a Hilbert space, continuously embedded in $H^1(\mathbb{R}^N)$, when endowed with the inner product

$$\langle \varphi, \psi \rangle_H = \int [\nabla\varphi \nabla\bar{\psi} + \varphi\bar{\psi} + |x|^2 \varphi\bar{\psi}] \,\mathrm{d}x, \tag{2.2}$$

whose associated norm we denote by $\|\cdot\|_H$. In addition, we use $\|\cdot\|_p$ to denote the norm of $L^p(\mathbb{R}^N)$.

Lemma 2.1 (Zhang [24]). *Let* $1 \leqslant p < \infty$. *Then the embedding* $H \hookrightarrow L^{p+1}(\mathbb{R}^N)$ *is compact.*

Proposition 2.2 (Cazenave [5], Glassey [11], Oh [16]). *Assume that* $1 < p < \infty$ *when* $N = 1, 2$, $1 < p < (N+2)/(N-2)$ *when* $N \geqslant 3$ *and* $\varphi_0 \in H$. *There then exists a unique solution* φ *of the Cauchy problem (1.1) in* $C([0,T); H)$ *for some* $T \in [0, \infty)$ *(maximal existence time), and* φ *satisfies the following two mass- and energy-conservation laws:*

$$M(\varphi) := \int |\varphi|^2 \, \mathrm{d}x = M(\varphi_0), \tag{2.3}$$

$$E(\varphi) := \int \left[|\nabla \varphi|^2 + |x|^2 |\varphi|^2 - \frac{2}{p+1} |\varphi|^{p+1} \right] \mathrm{d}x = E(\varphi_0), \tag{2.4}$$

for all $t \in [0, T)$. *Furthermore, we have the following alternatives:* $T = \infty$ *or else* $T < \infty$ *and* $\lim_{t \to T} \|\varphi\|_H = \infty$ *(blow-up).*

Proposition 2.3 (Cazenave [5]). *Let* $\varphi_0 \in H$. *Then for* $1 < p < 1 + (4/N)$, *the Cauchy problem (1.1) has a unique bounded global solution* φ *on* $t \in [0, \infty)$ *in* H. *For* $p \geqslant 1 + (4/N)$, *when* $\|\varphi_0\|_H$ *is sufficiently small, the Cauchy problem (1.1) has a unique bounded global solution in* H*; when* $\|\varphi_0\|_H$ *is sufficiently large, the Cauchy problem (1.1) has a unique solution blowing up in a finite time in* H.

Proposition 2.4 (Cazenave [5]). *For* $p \geqslant 1 + (4/N)$, *when* $E(\varphi_0) < 0$, *the solution* φ *of the Cauchy problem (1.1) blows up in a finite time in* H.

Proposition 2.5 (Cazenave [5]). *Let* $\varphi_0 \in H$ *and let* φ *be a solution of the Cauchy problem (1.1) on* $[0, T)$. *Set*

$$J(t) = \int |x|^2 |\varphi|^2 \, \mathrm{d}x.$$

Then one has

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} J(t) = 8 \int \left[|\nabla \varphi|^2 - |x|^2 |\varphi|^2 - \frac{N(p-1)}{2(p+1)} |\varphi|^{p+1} \right] \mathrm{d}x. \tag{2.5}$$

Lemma 2.6 (Weinstein [20]). *Let* $\varphi \in H$. *Then we have*

$$\int |\varphi|^2 \, \mathrm{d}x \leqslant \frac{2}{N} \left(\int |\nabla \varphi|^2 \, \mathrm{d}x \right)^{1/2} \left(\int |x|^2 |\varphi|^2 \, \mathrm{d}x \right)^{1/2}. \tag{2.6}$$

3. A variational problem

We define two functionals on H as follows

$$S(u) = \frac{1}{2} \int [|\nabla u|^2 + |x|^2 |u|^2] \, \mathrm{d}x \tag{3.1}$$

and

$$R(u) = \frac{1}{2} \int \left[|u|^2 - \frac{2}{p+1} |u|^{p+1} \right] \mathrm{d}x. \tag{3.2}$$

Then set

$$\left.\begin{aligned} d &= \inf_{u\in\Sigma} S(u), \\ \Sigma &= \{u \in H \setminus \{0\} : R(u) = 0\}. \end{aligned}\right\} \tag{3.3}$$

Theorem 3.1. *Let $p = 1 + (4/N)$. There then exists a $u \in \Sigma$ such that*

$$d = \min_{u\in\Sigma} S(u). \tag{3.4}$$

Furthermore, $d > 0$.

Proof. Choose a minimizing sequence $\{u_n\}_{n\in\mathbb{N}}$, so we have $u_n \in \Sigma$ and

$$\lim_{n\to\infty} \frac{1}{2}\int [|\nabla u_n|^2 + |x|^2|u_n|^2]\,\mathrm{d}x = d. \tag{3.5}$$

By the Gagliardo–Nirenberg inequality and $u_n \in \Sigma$, we have

$$0 < \int |u_n|^2\,\mathrm{d}x = \frac{2}{p+1}\int |u_n|^{p+1}\,\mathrm{d}x \leqslant C\|\nabla u_n\|_2^{p-1}\|u_n\|_2^2. \tag{3.6}$$

Henceforth, for simplicity, we use C to denote various positive constants. Therefore, there exists a positive constant C such that

$$\int |\nabla u_n|^2\,\mathrm{d}x \geqslant C > 0,$$

which implies that

$$\frac{1}{2}\int [|\nabla u_n|^2 + |x|^2|u_n|^2]\,\mathrm{d}x \geqslant C > 0. \tag{3.7}$$

Hence, $d > 0$.

In addition, it follows from (3.5), Lemma 2.6 and the Cauchy inequality that

$$\begin{aligned} \int |u_n|^2\,\mathrm{d}x &\leqslant \frac{2}{N}\biggl(\int |\nabla u_n|^2\,\mathrm{d}x\biggr)^{1/2}\biggl(\int |x|^2|u_n|^2\,\mathrm{d}x\biggr)^{1/2} \\ &\leqslant \frac{1}{N}\biggl(\int |\nabla u_n|^2\,\mathrm{d}x + \int |x|^2|u_n|^2\,\mathrm{d}x\biggr) \\ &< C. \end{aligned} \tag{3.8}$$

By (3.5) and (3.8), one finds that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in H. Therefore, there exists $u \in H$ such that a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ (which we still denote by $\{u_n\}_{n\in\mathbb{N}}$) satisfies

$$u_n \rightharpoonup u \quad \text{weakly in } H. \tag{3.9}$$

By Lemma 2.1 we have

$$\left.\begin{aligned} u_n &\to u \quad \text{in } L^2(\mathbb{R}^N), \\ u_n &\to u \quad \text{in } L^{p+1}(\mathbb{R}^N). \end{aligned}\right\} \tag{3.10}$$

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Hence,

$$R(u) = \frac{1}{2}\int\left[|u|^2 - \frac{2}{p+1}|u|^{p+1}\right] dx = 0, \tag{3.11}$$

and $u \in \Sigma$, so we have $S(u) \geqslant d$. On the other hand, since $S(u_n)$ is coercive and convex, one has

$$S(u) \leqslant \liminf_{n\to\infty} S(u_n). \tag{3.12}$$

Therefore, from (3.5) and (3.12), we have

$$d \leqslant S(u) \leqslant \liminf_{n\to\infty} S(u_n) \leqslant \lim_{n\to\infty} S(u_n) = d, \tag{3.13}$$

which implies that $S(u) = d$. Therefore, Theorem 3.1 is true. □

4. Threshold of global existence

In this section, we shall give the main results and the proofs. First, we define another functional in H as

$$I(u) = \frac{1}{2}\int\left[|\nabla u|^2 + |u|^2 + |x|^2|u|^2 - \frac{2}{p+1}|u|^{p+1}\right] dx. \tag{4.1}$$

Hence, we can obtain the invariant properties of the local semi-flows of the Cauchy problem (1.1).

Proposition 4.1. *Let $p = 1 + (4/N)$ and*

$$\left.\begin{aligned} K_+ &= \{u \in H : R(u) > 0,\ I(u) < d\},\\ K_- &= \{u \in H : R(u) < 0,\ I(u) < d\}.\end{aligned}\right\} \tag{4.2}$$

Then K_+ and K_- are invariant under the local semi-flow generated by the Cauchy problem (1.1).

Proof. Let $\varphi_0 \in K_+$ and let φ be the unique solution of (1.1) with the initial datum φ_0. Therefore, it follows easily from Proposition 2.2 that $I(\varphi) = \frac{1}{2}[M(\varphi)+E(\varphi)] = \frac{1}{2}[M(\varphi_0)+E(\varphi_0)] = I(\varphi_0)$, $t \in [0,T)$. Thus, from $I(\varphi_0) < d$, it follows that

$$I(\varphi) < d, \quad t \in [0,T). \tag{4.3}$$

To check that $\varphi \in K_+$, we need to prove that

$$R(\varphi) > 0, \quad t \in [0,T). \tag{4.4}$$

If it were not, because of $R(\varphi_0) > 0$, there would exist, by continuity, a $t_1 \in (0,T)$ such that $R(\varphi(t_1)) = 0$. Then $\varphi(t_1) \in \Sigma$. Hence, $S(\varphi(t_1)) \geqslant d$. From (3.1), (3.2) and (4.1), $I(\varphi(t_1)) = S(\varphi(t_1)) + R(\varphi(t_1))$. It follows from (4.3) that $S(\varphi(t_1)) < d$, violating Theorem 3.1. Therefore, (4.4) holds. Hence, K_+ is invariant under the local semi-flow generated by the Cauchy problem (1.1).

By the same argument as that above, we can show that K_- is invariant. □

Theorem 4.2. *For $p = 1 + (4/N)$ and $N \geqslant 2$, let $\varphi_0 \in H$. We then have the following.*

(i) *If $\varphi_0 \in K_+$, the solution φ of the Cauchy problem (1.1) exists globally on $t \in [0, \infty)$ in H.*

(ii) *If $\varphi_0 \in K_-$, the solution φ of the Cauchy problem (1.1) blows up at a finite time in H.*

Proof. (i) Let $\varphi_0 \in K_+$ and let φ be the solution of (1.1) with the initial datum φ_0. It follows from Proposition 4.1 that $\varphi \in K_+$, i.e.

$$\left.\begin{aligned} I(\varphi) &= \frac{1}{2}\int\left[|\nabla\varphi|^2 + |\varphi|^2 + |x|^2|\varphi|^2 - \frac{2}{p+1}|\varphi|^{p+1}\right]\mathrm{d}x < d, \\ R(\varphi) &= \frac{1}{2}\int\left[|\varphi|^2 - \frac{2}{p+1}|\varphi|^{p+1}\right]\mathrm{d}x > 0, \end{aligned}\right\} \tag{4.5}$$

which implies that

$$\frac{1}{2}\int[|\nabla\varphi|^2 + |x|^2|\varphi|^2]\,\mathrm{d}x$$

is bounded. Furthermore, since

$$\int|\varphi|^2\,\mathrm{d}x = \int|\varphi_0|^2\,\mathrm{d}x,$$

we then find that $\|\varphi\|_H$ is bounded. Therefore, it follows from Proposition 2.2 that the solution φ of the Cauchy problem (1.1) exists globally on $t \in [0, \infty)$ in H.

(ii) Let $\varphi_0 \in K_-$ and let φ be the solution of (1.1) with the initial datum φ_0. It follows from Proposition 4.1 that $\varphi \in K_-$, i.e.

$$I(\varphi) = \frac{1}{2}\int\left[|\nabla\varphi|^2 + |\varphi|^2 + |x|^2|\varphi|^2 - \frac{2}{p+1}|\varphi|^{p+1}\right]\mathrm{d}x < d, \tag{4.6}$$

and

$$R(\varphi) = \frac{1}{2}\int\left[|\varphi|^2 - \frac{2}{p+1}|\varphi|^{p+1}\right]\mathrm{d}x < 0. \tag{4.7}$$

Therefore, there must exist a $\lambda \in (0, 1)$ such that $R(\lambda\varphi) = 0$, which implies that $\lambda\varphi \in \Sigma$. So, $S(\lambda\varphi) \geqslant d > I(\varphi)$, i.e.

$$\lambda^2\int[|\nabla\varphi|^2 + |x|^2|\varphi|^2]\,\mathrm{d}x > \int\left[|\nabla\varphi|^2 + |\varphi|^2 + |x|^2|\varphi|^2 - \frac{2}{p+1}|\varphi|^{p+1}\right]\mathrm{d}x. \tag{4.8}$$

When $N = 2$ and $p = 3$, it follows from $R(\lambda\varphi) = 0$ and (4.8) that

$$(\lambda^2 - 1)\int[|\nabla\varphi|^2 + |x|^2|\varphi|^2 - \tfrac{1}{2}|\varphi|^4]\,\mathrm{d}x > 0. \tag{4.9}$$

At the same time, noting that $\lambda \in (0,1)$, we have

$$E(\varphi) = \int [|\nabla\varphi|^2 + |x|^2|\varphi|^2 - \tfrac{1}{2}|\varphi|^4]\,\mathrm{d}x < 0.$$

It then follows from this result and Proposition 2.2 that $E(\varphi_0) < 0$. Thus, φ blows up, by Proposition 2.4.

When $N \geqslant 3$ and $p = 1 + (4/N)$, from Proposition 2.5, (4.8), $R(\lambda\varphi) = 0$ and (2.3), we have

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int |x|^2|\varphi|^2\,\mathrm{d}x
&= 8\int |\nabla\varphi|^2\,\mathrm{d}x - 8\int \left[|x|^2|\varphi|^2 + \frac{N(p-1)}{2(p+1)}|\varphi|^{p+1}\right]\mathrm{d}x \\
&< -8\int \left[|x|^2|\varphi|^2 + \frac{1}{1-\lambda^2}|\varphi|^2 - \frac{2}{(1-\lambda^2)(p+1)}|\varphi|^{p+1}\right]\mathrm{d}x \\
&\qquad - 8\int \left[|x|^2|\varphi|^2 + \frac{N(p-1)}{2(p+1)}|\varphi|^{p+1}\right]\mathrm{d}x \\
&< -16\int |x|^2|\varphi|^2\,\mathrm{d}x + \frac{8}{\lambda^{p-1}(1-\lambda^2)}\left[-\lambda^{p-1} + 1 - \frac{(1-\lambda^2)(p-1)N}{4}\right]\int |\varphi|^2\,\mathrm{d}x \\
&< \frac{8}{\lambda^{p-1}(1-\lambda^2)}\left[-\lambda^{p-1} + 1 - \frac{(1-\lambda^2)(p-1)N}{4}\right]\int |\varphi|^2\,\mathrm{d}x \\
&= \frac{8}{\lambda^{p-1}(1-\lambda^2)}[-\lambda^{4/N} + \lambda^2]\int |\varphi|^2\,\mathrm{d}x \\
&= \frac{8}{\lambda^{p-1}(1-\lambda^2)}[-\lambda^{4/N} + \lambda^2]\int |\varphi_0|^2\,\mathrm{d}x. \qquad (4.10)
\end{aligned}$$

Set $f(\lambda) = -\lambda^{4/N} + \lambda^2$, where $\lambda \in (0,1)$. When $N \geqslant 3$, we obtain $0 < 4/N < 2$. Therefore, $\lambda^{4/N} > \lambda^2$, where $\lambda \in (0,1)$. Then $f(\lambda) < 0$. Therefore, it follows that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int |x|^2|\varphi|^2\,\mathrm{d}x < -C < 0,$$

which yields that the solution φ of the Cauchy problem (1.1) blows up in a finite time in H. □

Remark 4.3. When $N = 1$, we unfortunately cannot find the sharp threshold between the global solutions and the blowing-up solutions of the Cauchy problem (1.1) for the critical power, $p = 5$.

The following theorem answers the question: how small are the initial data, such that the Cauchy probelm (1.1) has global solutions for $p = 1 + (4/N)$?

Theorem 4.4. *Let $p = 1 + (4/N)$ and $N \geqslant 2$. If $\varphi_0 \in H$ and satisfies*

$$\tfrac{1}{2}\|\varphi_0\|_H^2 = \frac{1}{2}\int [|\nabla\varphi_0|^2 + |\varphi_0|^2 + |x|^2|\varphi_0|^2]\,\mathrm{d}x < d, \tag{4.11}$$

then the solution φ of (1.1) with the initial datum φ_0 exists globally on $t \in [0, \infty)$.

Proof. Let $\varphi_0 \neq 0$ and let it satisfy (4.11). Then obviously $I(\varphi_0) < d$. Now we show that φ_0 also satisfies $R(\varphi_0) > 0$. Firstly, we prove that

$$R(\varphi_0) = \frac{1}{2}\int \left[|\varphi_0|^2 - \frac{2}{p+1}|\varphi_0|^{p+1}\right]\mathrm{d}x \neq 0. \tag{4.12}$$

Otherwise, $\varphi_0 \in \Sigma$. It follows that $I(\varphi_0) = R(\varphi_0) + S(\varphi_0) = S(\varphi_0) < d$, which is contradictory to Theorem 3.1. Therefore, if $R(\varphi_0) > 0$ were not true, we would have

$$R(\varphi_0) = \frac{1}{2}\int \left[|\varphi_0|^2 - \frac{2}{p+1}|\varphi_0|^{p+1}\right]\mathrm{d}x < 0. \tag{4.13}$$

Thus, there exists a $\mu \in (0, 1)$ such that

$$R(\mu\varphi_0) = \frac{1}{2}\int \left[\mu^2|\varphi_0|^2 - \frac{2\mu^{p+1}}{p+1}|\varphi_0|^{p+1}\right]\mathrm{d}x = 0, \tag{4.14}$$

which means that $\mu\varphi_0 \in \Sigma$. But, by (4.11), we have

$$\begin{aligned} S(\mu\varphi_0) &= \tfrac{1}{2}\mu^2\int [|\nabla\varphi_0|^2 + |x|^2|\varphi_0|^2]\,\mathrm{d}x \\ &< \frac{1}{2}\int [|\nabla\varphi_0|^2 + |x|^2|\varphi_0|^2]\,\mathrm{d}x \\ &< \frac{1}{2}\int [|\nabla\varphi_0|^2 + |x|^2|\varphi_0|^2 + |\varphi_0|^2]\,\mathrm{d}x \\ &< d, \end{aligned} \tag{4.15}$$

which contradicts Theorem 3.1. These contradictions yield $\varphi_0 \in K_+$. By using Theorem 4.2, we find that the solution φ of (1.1) with the initial datum φ_0 exists globally on $t \in [0, \infty)$. □

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