BULL. AUSTRAL. MATH. SOC. 47C05, 46B20
VOL. 34 (1986) 271-274

CONTRACTIVE PROJECTIONS ON BANACH SPACE

MARK SPIVACK

In this note we prove the uniqueness of a projection onto a given subspace with strictly contractive complement. We also show that, if one completely contractive projection is invariant under another, then the two commute.

There are many interesting problems concerning the existence and uniqueness of contractive projections on a Banach space (see [2] for example). Despite their geometrical importance within a space (for example [4]) their behaviour is poorly understood. In this note we prove the uniqueness of a projection, onto a given subspace, whose complement is strictly contractive. We also show that, if one completely contractive projection is invariant under another, then the two commute.

Recall that a *projection* p is a linear operator such that $p = p^2$. p is *contractive* if $||p|| = 1$, and *strictly contractive* if, in addition, $||p\xi|| < ||\xi||$ whenever $p\xi \neq \xi$. $p^\perp$ denotes the projection $1 - p$. We call p *completely contractive* if both p and $p^\perp$ are strictly contractive.

Note that in a strictly convex Banach space E a contractive projection p is strictly contractive: Suppose $||p\xi|| = ||\xi|| = 1$, where

---

Received 22 November 1985.

---

https://doi.org/10.1017/S0004972700010133 Published online by Cambridge University Press

$p\xi \neq \xi$. Then the line-segment joining ξ and $p\xi$ lies on the unit sphere of E.

LEMMA 1. *Let p and q be projections such that $p^{\perp}$ is strictly contractive and q is contractive. Then for any ξ in E, $pq\xi = 0$ if and only if $qpq\xi = 0$. In particular, $qp = 0$ implies thet $pq = 0$.*

Proof. Let $qp = 0$. Suppose $pq\xi \neq 0$ for some ξ . We can assume that $\xi = q\xi$. Then $p^{\perp}\xi \neq \xi$. Therefore $||\xi|| > ||p^{\perp}\xi|| \geq ||qp^{\perp}\xi||$ $= ||\xi||$, since $qp^{\perp}\xi = q(1-p)\xi = \xi$. This gives a contradiction, and so $pq = 0$.

In Hilbert space the contractive projections are exactly those which are self-adjoint. Hilbert space proofs of these results would normally use the inner product, and those here are of some interest in providing a geometrical generalisation.

THEOREM 2. *Let p and q be projections onto the same subspace, such that $p^{\perp}$ is contractive and $q^{\perp}$ is strictly contractive. Then $p = q$.*

Proof. $pq = q$ so that $p^{\perp}q = 0$. We can now apply Lemma 1 to show that $qp^{\perp} = p^{\perp}q = 0$. Thus p and q commute, and the result follows.

It follows, for example, that any L_{∞} projection on an $L_p(\Omega)$ space is the only projection onto its range having a contractive complement. (Here $1 < p < \infty$, Ω is a positive measure space, and the L_{∞} functions are considered as multiplication operators). This holds because L_p is strictly convex.

Note that this result no longer holds if we replace 'strictly contractive' by 'contractive'. As a counter-example consider the real plane with the unit ball represented by a regular hexagon. Then each of the three lines through opposite vertices is parallel to a pair of opposite sides. Let any two correspond to the null-spaces of p and q respectively and let the third be their common range. Then p , q , and their complements are all contractive.

Now let p and q be completely contractive projections on a Banach space E. Suppose that q is invariant under p, that is $q^{\perp}pq = 0$. The first lemma can be stated without proof:

LEMMA 3. *$q^{\perp}p = q^{\perp}pq^{\perp}$, $pq = qpq$, and $q^{\perp}p$, pq, and $pq^{\perp}p$ are all strictly contractive projections.*

LEMMA 4. *$qpq^{\perp}p = 0$, and qp is a strictly contractive projection.*

Proof. Suppose that $qpq^{\perp}p\xi \neq 0$ for some ξ. Then, since $pq^{\perp}p$ is a contractive projection, we can apply Lemma 1 to show that $(pq^{\perp}p)qpq^{\perp}p\xi \neq 0$. However this contradicts $q^{\perp}pq = 0$, and so $qpq^{\perp}p = 0$ as claimed. The second assertion is equivalent, and can be seen by expanding $qpq^{\perp}p$.

LEMMA 5. *$q^{\perp}p^{\perp}$ is a strictly contractive projection.*

Proof. This follows easily by expanding $(q^{\perp}p^{\perp})^2 - q^{\perp}p^{\perp}$ and applying the previous results.

THEOREM 6. *Let q be invariant under p. Then p and q commute.*

Proof. We show first that $pq^{\perp}p^{\perp} = 0$: Suppose not. Then, by Lemmas 1 and 5, $q^{\perp}p^{\perp}pq^{\perp}p^{\perp}$ is non-zero, giving a contradiction. Then $qpq^{\perp} = qpq^{\perp}p + qpq^{\perp}p^{\perp} = 0$, by Lemma 4. A routine computation shows that $qp = pq$, as required.

REMARKS. This result enables us to consider reflexivity of sets of projections. A set P of self-adjoint projections on (or subspaces of) a Hilbert space is *reflexive* if $P = \mathrm{lat}(\mathrm{alg}(P))$. Arveson [1] and Davidson [3] showed that complete commutative lattices on Hilbert space are reflexive. (This can be viewed as a non-self-adjoint analogue of the Double Commutant Theorem). If we define completeness for lattices of projections on Banach space as in [5], and denote by $\mathrm{lat}(A)$ those completely contractive projections which are invariant under an algebra A, we can conjecture that complete commutative lattices of completely contractive projections are reflexive in the obvious sense. From Theorem 6, since P is contained in $\mathrm{alg}(P)$, we have immediately:

https://doi.org/10.1017/S0004972700010133 Published online by Cambridge University Press

COROLLARY 7. *Complete maximal commutative sets of completely contractive projections are reflexive.*

The equivalent conjecture for subspaces fails. As a counterexample consider the set S of subspaces generated by the natural basis of l_∞. The subspace c_0 is then invariant under all operators in $\mathrm{alg}(S)$.

References

[1] W.B. Arveson, "Operator algebras and invariant subspaces", *Ann. Math.* 100 (1974) 433-532.

[2] E.W. Cheney and K.M. Price, "Minimal Projections", in *Approximation Theory*, Ed. A. Talbot, (Academic Press, New York, 1970).

[3] K.R. Davidson, "Commutative subspace lattices", *Indiana Univ. Math. J.* 27 (1978) 479-490.

[4] S. Kakutani and G.W. Mackey, "Ring and lattice characterisations of complex Hilbert space", *Bull. Amer. Math. Soc.* 52 (1946) 727-733.

[5] M. Spivack, "Derivations and nest algebras on Banach space", *Israel J. Math.* 50 (1985) 193-200.

Department of Applied Mathematics
and Theoretical Physics
University of Cambridge
Silver Street
Cambridge CB3 9EW
United Kingdom