LATTICE PATHS WITH DIAGONAL STEPS

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1. Introduction. The André-Poincaré "problème du scrutin" [9] can be stated as follows: In an election between two candidates A polls m votes, B polls n, $m > n$. If the votes are counted one by one what is the probability that A leads B throughout the counting? Many derivations and interpretations of the solution $\frac{m-n}{m+n}$ have been given and a convenient summary of methods till 1956 can be found in Feller [1]. So numerous are the generalizations of ballot problems and their applications since this date that we do not even attempt an enumeration here. However, it does not seem to be realized that many results on "ballot theorems" are directly applicable to other enumeration problems. In this paper we derive various results on lattice paths with diagonal steps by a uniform treatment based on Feller [1] and Narayana [6, 7]. The formulae obtained by this unified method represent both a simplification and generalization of recently published formulae, eg. by Rohatgi [10] and Stocks [11]. Indeed as will be evident from our approach, many other results on lattice paths with diagonal steps can be derived with great ease using the well known techniques for ballot theorems.

In section 2 we indicate essentially two different proofs of a "ballot problem with ties" and apply it to certain enumeration problems for lattice paths with diagonal steps in the plane. Without appealing de nouveau to the reflection principle or to specialized results on convolutions we show in the next section that many of these results can be extended directly to three or more dimensions. We conclude with a statement of the "duality principle" (Feller [1], Narayana [5]) for lattice paths with diagonal steps, and briefly indicate how many classical results could be generalized in a similar fashion. Such generalizations provide further combinatorial results on lattice paths with diagonal steps.

2. The ballot problem with ties. Consider the following ballot problem, which for $r = 0$ reduces to the classical "problème du scrutin": Let candidate A poll $m - r$ votes, candidate B poll $n - r$ votes and let a further r votes be counted for both A and B. These r votes represent ties, and so the total number of votes for A, B are m, n respectively, where of course $m > n$ and $0 \leq r \leq n$. What is the probability that A leads B throughout the counting?

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We give two proofs that the required probability is $\frac{m-n}{m+n-r}$ if $0 \leq r \leq n$. Our first proof follows almost verbatim the classical proof of Poincaré [9], Feller [1], while the second utilizes the simple idea of "placing balls into cells" used in occupancy problems. It will be seen in sections 3 and 4 that the methods used in our second proof are particularly useful in solving certain lattice path enumeration problems in three or more dimensions and indeed in generalizing many classical results involving enumeration of lattice paths. Following the second proof we illustrate more explicitly the ways in which A can hold a lead over B throughout the count by applying occupancy techniques introduced by Narayana [6]. A few remarks and a corollary conclude the section.

We now state formally as Theorem 1 the ballot problem with ties.

THEOREM 1. The probability that A leads B throughout the counting when A has m votes, B has n votes, $m > n$ and $0 \leq r \leq n$ votes are ties is $\frac{m-n}{m+n-r}$.

Proof 1. We prove the theorem for $0 \leq r < n$, noting that the case $r = n$ is trivial and can be easily established. Let a vote for A be represented by a unit horizontal step, a vote for B by a unit vertical step and a vote for both A and B by a diagonal step. Surely the number of distinct paths from $(0, 0)$ to (m, n) with $m - r$ horizontal steps, $n - r$ vertical steps and r diagonal steps is $(m, n, r) = \frac{(m+n-r)!}{(m-r)!\,(n-r)!\,r!}$. Define four mutually exclusive and exhaustive types of paths from $(0, 0)$ to (m, n) as follows:

(1) An α path starts with a unit vertical step.

(2) A β path starts with a unit horizontal step and then at some other step touches the line $y = x$.

(3) A γ path starts with a diagonal step.

(4) A δ path starts with a horizontal step and remains below the line $y = x$.

The number of α paths is equivalent to the number of paths from $(0, 1)$ to (m, n) with $(n - r - 1)$ vertical steps, $(m - r)$ horizontal steps and r diagonal steps which is $\frac{(m+n-r-1)!}{(m-r)!\,(n-r-1)!\,r!}$. Using the reflection principle which the probability literature attributes to D. André (1887) we see that the β paths are in 1:1 correspondence with the α paths and hence are the same in number. The number of γ paths is equivalent to the number of paths from $(1, 1)$ to (m, n) with $(m - r)$ horizontal steps, $(n - r)$ vertical steps and $r - 1$ diagonal steps which

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is $\dfrac{(m+n-r-1)!}{(m-r)!\,(n-r)!\,(r-1)!}$. Thus the number of δ paths is

$$(1) \qquad \frac{(m+n-r)!}{(m-r)!\,(n-r)!\,r!} - \frac{2(m+n-r-1)!}{(m-r)!\,(n-r-1)!\,r!} - \frac{(m+n-r-1)!}{(m-r)!\,(n-r)!\,(r-1)!} \;;$$

on dividing (1) by the total number of paths (m, n, r) we see that the required probability is $\dfrac{m-n}{m+n-r}$. Our first proof is complete.

Proof 2. The number of paths to $(m-r, n-r)$ lying below the line $y=x$ and having only horizontal and vertical steps is by the classical ballot problem

$$(2) \qquad \frac{m-n}{m+n-2r}\binom{m+n-2r}{m-r}.$$

Since each path to $(m-r, n-r)$ passes through $m+n-2r+1$ lattice points, which for convenience, we refer to as cells, we see that there are $m+n-2r$ cells in which the r diagonal steps, or alternatively balls, can be placed. The excluded cell is of course the origin, since starting with a diagonal step violates the conditions of the ballot problem. The number of ways of placing r diagonal steps into $(m+n-2r)$ cells is

$$(3) \qquad \binom{m+n-r-1}{r}.$$

Hence the total number of paths to (m, n) satisfying the conditions of the theorem is from (2) and (3)

$$(4) \qquad \frac{m-n}{m+n-2r}\binom{m+n-2r}{m-r}\binom{m+n-r-1}{r}.$$

Division of (4) by the total number of paths (m, n, r) yields the result $\dfrac{m-n}{m+n-r}$ as before.

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We can be more specific about the structure of the paths from (0, 0) to (m, n) using the idea of partial orders through the relation of domination introduced by Narayana [6, 7]. Ignoring diagonal steps, we define a turn as a horizontal step or steps followed by a vertical step or steps. Note that a sequence of vertical steps followed by one or more horizontal steps is not a turn. In an extension of the "probléme du scrutin", Narayana [7, p. 94], has shown that the number of paths with k turns and no diagonal steps to (m - r, n - r) such that each path lies entirely below the line y = x is

$$(5) \qquad \binom{n-r-1}{k-1} \sum_{i=2}^{m-n+1} \binom{m-r-i}{k-1} - \binom{n-r-1}{k} \sum_{i=2}^{m-n+1} \binom{m-r-i}{k-2} .$$

With this result in hand we state the following

THEOREM 2. *The number of paths from* (0, 0) *to* (m, n) *lying below the line* y = x *with* r $(0 \leq r < n)$ *diagonal steps and* $k \geq 1$ *turns is*

$$(6) \qquad \binom{m+n-r-1}{r} \left\{ \binom{n-r-1}{k-1} \binom{m-r-1}{k} - \binom{n-r-1}{k} \binom{m-r-1}{k-1} \right\}.$$

Proof. Applying the well known identity

$$\sum_{\nu=0}^{n} \binom{a-\nu}{r} = \binom{a+1}{r+1} - \binom{a-n}{r+1} ,$$

we see that (5) reduces to

$$(7) \qquad \binom{n-r-1}{k-1} \binom{m-r-1}{k} - \binom{n-r-1}{k} \binom{m-r-1}{k-1} .$$

Multiplying (7) by the number of ways of placing r balls into m + n - 2r cells we see that the required number of paths is given by (6).

Remarks. 1. Since it is a trivial matter to calculate the number of paths to (m, n) with r = n diagonal steps, we have not stated this case here.

2. Summing (6) over k from 1 to $n-r$ yields, after some simplification, $\frac{(m+n-r-1)!\,(m-n)}{r!\,(m-r)!\,(n-r)!}$ paths to (m, n) with r diagonal steps, as in (4).

3. Clearly Theorem 2 represents a considerable generalization of the ballot problem (without using the reflection principle).

As a sample of other results which follow immediately from Theorem 2 we now state a corollary with an idea of the proof briefly indicated.

COROLLARY 1. The number of paths from $(0, 0)$ to (n, n) lying entirely below the line $y=x$ (except of course for the end points) with (a) k turns $(k \geq 1)$, (b) r diagonal steps $(0 \leq r \leq n-2)$ is

$$\binom{2n-r-2}{r}\left\{\binom{n-r-2}{k-1}\binom{n-r-1}{k}-\binom{n-r-2}{k}\binom{n-r-1}{k-1}\right\}. \tag{8}$$

Observe that a path satisfying the above requirements must end with a vertical step so that we are essentially dealing with paths below the diagonal to $(n, n-1)$.

We remark that (8) clearly represents a generalization of Moser and Zayachkowski [4, p. 225, eq. 2.8] as can be seen by summing (8) over all k and then over all r. Similarly closed expressions can be obtained for $Q(m, n)$ and $Q'(m, n)$ defined by Rohatgi in [10] for all cases $m \geq n$.

3. Lattice paths in E^3 with cube diagonal steps. In this section we illustrate the simplicity with which the "balls into cells" technique can be used to solve lattice path problems of a more general nature than those discussed in the previous section. In three dimensions, a diagonal step may be taken as a cube diagonal. Each lattice path has steps of the following types:

1) x-increasing only, e.g. $[(m,n,k), (m+1, n, k)]$,

2) y-increasing only, e.g. $[(m,n,k), (m,n+1, k)]$,

3) z-increasing only, e.g. $[(m,n,k), (m,n,k+1)]$

4) Cube diagonal, e.g. $[(m,n,k), (m+1, n+1, k+1)]$.

Without appealing to the reflection principle we now prove a theorem which generalizes the theorems in Stocks [11], and conclude this section with a few corollaries and remarks.

THEOREM 3. The number of paths from $(0, 0, 0)$ to (m, n, k), $m > n$, with r cube diagonal steps such that each component of each path lies entirely to the $(m, 0, 0)$ side of the diagonal plane $y = x$ is

$$\frac{(m-n)\,(m+n+k-2r-1)!}{(m-r)!\,(n-r)!\,(k-r)!\,r!} \quad (9)$$

Proof. The number of paths to $(m-r, n-r, 0)$ lying entirely to the $(m, 0, 0)$ side of the diagonal plane $y = x$ is by the classical ballot problem

$$\frac{m-n}{m+n-2r}\binom{m+n-2r}{m-r} . \quad (10)$$

Now $k-r$ vertical steps can be placed into $m+n-2r$ cells in

$$\binom{m+n+k-3r-1}{k-r} \quad (11)$$

ways. Hence by (10) and (11) the number of paths to $(m-r, n-r, k-r)$ without cube diagonal steps is

$$\frac{m-n}{m+n-2r}\binom{m+n-2r}{m-r}\binom{m+n+k-3r-1}{k-r} . \quad (12)$$

Placing r cube diagonal steps into the available $(m+n+k-3r)$ cells in

$$\binom{m+n+k-2r-1}{r} \quad (13)$$

ways, we see from (12) and (13) that the total number of paths to (m, n, k) satisfying the stated conditions is

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$$(14) \qquad \frac{m-n}{m+n-2r}\binom{m+n-2r}{m-r}\binom{m+n+k-3r-1}{k-r}\binom{m+n+k-2r-1}{r}.$$

A simple computation reduces (14) to (9).

As a sample of other results which follow immediately from Theorem 3, we now state two corollaries with an idea of their proofs briefly indicated.

COROLLARY 2. *The number of paths from* (0, 0, 0) *to* (n n, n) *with* r *cube diagonal steps such that, excepting of course the end points, each component of each path lies entirely to the* (n, 0, 0) *side of the diagonal plane* $y = x$ *is*

$$(15) \qquad \frac{(3n-2r-2)!}{(n-r)^2\,[(n-r-1)!]^3\,r!} \quad .$$

The proof is immediate if one observes that the required number of paths is equivalent to the number of paths to (n, n-1, n) lying entirely to the (n, 0, 0) side of the diagonal plane $y = x$.

COROLLARY 3. *The number of paths from* (0, 0, 0) *to* (m, n, k), $m \geq n$, *such that no path has a component on the* non-(m, 0, 0) *side of the diagonal plane* $y = x$ *is*

$$\frac{(m-n+1)\,(m+n+k-2r)!}{(m-r+1)!\,(n-r)!\,(k-r)!\,r!} \quad .$$

To prove Corollary 3 observe that the required number of paths is equivalent to the number of paths to (m+1, n, k) lying entirely to the (m+1, 0, 0) side of the diagonal plane.

Remarks. 1. Summing (15) over r from 0 to n-1 yields a much simpler expression than that obtained by Stocks [11, p. 656] for the number of paths to (n, n, n) with components to the (n, 0, 0) side of the diagonal plane. A little elementary algebra, however, simplifies Stocks expression to ours.

2. Setting $m = n = k$ in (16) and summing over r from 0 to n clearly simplifies another expression obtained by Stocks [11, p. 658], for the number of paths to (n, n, n) not crossing the diagonal plane. Again, it is a simple matter to show equivalence of his expression to ours.

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4. Conclusion. It is clear that by using Theorem 1 of Narayana [7], which is valid also in higher dimensions, we can obtain other results pertaining to enumeration of lattice paths. Our formulation is also consistent with the duality principle for ballot problems as in Narayana [5] and Feller [1]. Since this can be verified easily a result on first passages with ties can be formulated as follows: Let us assume that each step in a ballot problem with ties takes a unit of time. By duality we see immediately that the probability that a first passage through $m - n$ occurs at time $m + n - r$ is $\frac{m - n}{m + n - r}$ where $m > n \geq r$. We suspect that the same technique can be applied to other results in Chapter 3 of Feller [1] as well but it does not seem worthwhile to do so. Of course the balls into cells technique provides an alternative expression for the main result in [4] using, for example, the results of Grossman [2].

Perhaps essentially new results can be obtained by trying to extend this technique to further refinements of the ballot problem, for example, 'An Analogue of the Multinomial Theorem' by Narayana [8]. This appears to have been partially studied by Mohanty and Handa [3].

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