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A SUFFICIENT CONDITION FOR A GRAPH TO BE A FRACTIONAL (f, n)-CRITICAL GRAPH

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Abstract. Let a, b and n be non-negative integers such that $1 \le a \le b$, and let G be a graph of order p with $p \ge \frac{(a+b-1)(a+b-2)+bn-2}{a}$ and f be an integer-valued function defined on $V(G)$ such that $a \le f(x) \le b$ for all $x \in V(G)$. Let $h : E(G) \to [0, 1]$ be a function. If $\sum_{e \ni x} h(e) = f(x)$ holds for any $x \in V(G)$, then we call $G[F_h]$ a fractional f-factor of G with indicator function h, where $F_h = \{e \in E(G) : h(e) > 0\}$. A graph G is called a fractional (f, n)-critical graph if after deleting any n vertices of G the remaining graph of G has a fractional f-factor. In this paper, it is proved that G is a fractional (f, n)-critical graph if $|N_G(X)| > \frac{(b-1)p+|X|+bn-1}{a+b-1}$ for every non-empty independent subset X of $V(G)$, and $\delta(G) > \frac{(b-1)p+a+b+bn-2}{a+b-1}$. Furthermore, it is shown that the result in this paper is best possible in some sense.

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1. Introduction. Many physical structures can conveniently be modelled by networks. Examples include a communication network with the nodes and links modelling cities and communication channels, respectively, or a railroad network with nodes and links representing railroad stations and railways between two stations, respectively. Factors and factorisations in networks are very useful in combinatorial design, network design, circuit layout and so on. It is well known that a network can be represented by a graph. Vertices and edges of the graph correspond to nodes and links between the nodes, respectively. Henceforth, we use the term 'graph' instead of 'network'.

We investigate the fractional factor problem in graphs, which can be considered as relaxations of the well-known cardinality matching problem. The fractional factor problem has wide-range applications in areas such as network design, scheduling and combinatorial polyhedra. For instance, in a communication network if we allow several large data packets to be sent to various destinations through several channels, the efficiency of the network will be improved if we allow the large data packets to be partitioned into small parcels. The feasible assignment of data packets can be seen as a fractional flow problem and it becomes a fractional matching problem when the destinations and sources of a network are disjoint (i.e. the underlying graph is bipartite).

Graphs considered here are finite undirected graphs without loops or multiple edges. For notation and terminology not defined here we refer the reader to [**1**, **6**].

Let G be a graph. We denote by $V(G)$ the set of vertices of G and by $E(G)$ the set of edges. For any $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G and by $N_G(x)$

the set of vertices adjacent to x in G. The minimum degree of G is denoted by $\delta(G)$. For $S \subseteq V(G)$, we write $N_G(S) = \bigcup_{x\in S} N_G(x)$ and $d_G(S) = \sum_{x\in S} d_G(x)$. For disjoint subsets S and T of $V(G)$, we denote by $e_G(S, T)$ the number of edges from S to T, by $G[S]$ the subgraph of G induced by S and by $G - S$ the subgraph obtained from G by deleting all the vertices in S together with the edges incident to vertices in S. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges.

Let f be a non-negative integer-valued function defined on $V(G)$. Then a spanning subgraph F of G is called an f-factor if $d_F(x) = f(x)$ for each $x \in V(G)$. If $f(x) = k$ for all $x \in V(G)$, then an f-factor is simply called a k-factor. Let $h : E(G) \to [0, 1]$ be a function. If $\sum_{e\ni x} h(e) = f(x)$ holds for any $x \in V(G)$, then we call $G[F_h]$ a fractional f-factor of G with indicator function h, where $F_h = \{e \in E(G) : h(e) > 0\}$. If $f(x) = k$ for each $x \in V(G)$, then a fractional f-factor is a fractional k-factor. A graph G is called a fractional (f, n)-critical graph if after deleting any n vertices of G the remaining graph of G has a fractional f-factor. If G is a fractional (f, n)-critical graph, then we also say that G is fractional (f, n)-critical. If $f(x) = k$ for each $x \in V(G)$, then a fractional (f, n)-critical graph is a fractional (k, n)-critical graph. A fractional (k, n)-critical graph is also called a fractional n-critical graph if $k = 1$.

Many authors have investigated factors [**2**, **3**, **5**, **8**, **9**, **12**] and fractional factor [**4**, **7**, **10**]. The following results on fractional k-factors and fractional (f, n)-critical graphs are known.

THEOREM 1. [**10**] *Let k be a positive integer and G a graph of order p with $p \geq 4k - 6$. Then*
(*a*) *if k is even,*

$$|N_G(X)| \geq \frac{(k-1)p + |X| - 1}{2k-1}$$

for every non-empty independent subset X of $V(G)$ and

$$\delta(G) \geq \frac{k-1}{2k-1}(p+2),$$

then G has a fractional k-factor; and
(*b*) *if k is odd,*

$$|N_G(X)| > \frac{(k-1)p + |X| - 1}{2k-1}$$

for every non-empty independent subset X of $V(G)$ and

$$\delta(G) > \frac{k-1}{2k-1}(p+2),$$

then G has a fractional k-factor.

THEOREM 2. [**4**] *Let $k \geq 2$ be an integer. A graph G of order p with $p \geq (k+1)$ has a fractional k-factor if $t(G) \geq k - \frac{1}{k}$.*

THEOREM 3. [**11**] *Let $n \geq 1$ be an integer, and G be a graph. If vertex-connectivity of G $\kappa(G) \geq n - 1$, and $bind(G) \geq n$, then G is fractional n-critical.*

THEOREM 4. [**13**] *Let G be a graph of order p, a, b and n be non-negative integers such that $2 \leq a \leq b$ and f be an integer-valued function defined on $V(G)$ such that*

$a \le f(x) \le b$ for each $x \in V(G)$. If $bind(G) > \frac{(a+b-1)(p-1)}{ap-(a+b)-bn+2}$ and $p \ge \frac{(a+b)(a+b-3)}{a} + \frac{bn}{a-1}$, then G is fractional (f, n)-critical.

In this paper, we prove the following result, which is an extension of Theorem 1. We extend Theorem 1 to fractional (f, n)-critical graphs.

THEOREM 5. *Let a, b and n be non-negative integers such that $1 \le a \le b$, and G be a graph of order p with $p \ge \frac{(a+b-1)(a+b-2)+bn-2}{a}$ and f be an integer-valued function defined on $V(G)$ such that $a \le f(x) \le b$ for all $x \in V(G)$. Suppose*

$$|N_G(X)| > \frac{(b-1)p + |X| + bn - 1}{a+b-1}$$

for every non-empty independent subset X of $V(G)$, and

$$\delta(G) > \frac{(b-1)p + a + b + bn - 2}{a+b-1},$$

then G is a fractional (f, n)-critical graph.

In Theorem 5, if $n = 0$, then we get the following corollary.

COROLLARY 1. *Let a and b be non-negative integers such that $1 \le a \le b$, G be a graph of order p with $p \ge \frac{(a+b-1)(a+b-2)-2}{a}$ and f be an integer-valued function defined on $V(G)$ such that $a \le f(x) \le b$ for all $x \in V(G)$. Suppose*

$$|N_G(X)| > \frac{(b-1)p + |X| - 1}{a+b-1}$$

for every non-empty independent subset X of $V(G)$, and

$$\delta(G) > \frac{(b-1)p + a + b - 2}{a+b-1},$$

then G has a fractional f-factor.

In Theorem 5, if $a = b = k$, then we obtain the following corollary.

COROLLARY 2. *Let k and n be non-negative integers such that $k \ge 1$ and G be a graph of order p with $p \ge 4k - 6 + n$. Suppose*

$$|N_G(X)| > \frac{(k-1)p + |X| + kn - 1}{2k-1}$$

for every non-empty independent subset X of $V(G)$, and

$$\delta(G) > \frac{(k-1)p + 2k + kn - 2}{2k-1},$$

then G is a fractional (k, n)-critical graph.

In Corollary 2, if $n = 0$, then we have the following corollary.

Corollary 3. *Let k be a non-negative integer such that $k \geq 1$ and G be a graph of order p with $p \geq 4k - 6$. Suppose*

$$|N_G(X)| > \frac{(k-1)p + |X| - 1}{2k-1}$$

for every non-empty independent subset X of $V(G)$, and

$$\delta(G) > \frac{(k-1)p + 2k - 2}{2k-1},$$

then G has a fractional k-factor.

2. Proof of Theorem 5. Let f be an integer-valued function defined on the vertex-set $V(G)$ of a graph G. If $S \subseteq V(G)$, then we define $f(S) = \sum_{x \in S} f(x)$. If S and T are disjoint subsets of $V(G)$ define

$$\delta_G(S, T) = f(S) + d_{G-S}(T) - f(T),$$

and if $|S| \geq n$ define

$$f_n(S) = \max\{f(U) : U \subseteq S \text{ and } |U| = n\}. \tag{1}$$

Lemma 2.1. [13] *Let G be a graph, n be a non-negative integer and f be a nonnegative integer-valued function defined on $V(G)$. Then G is fractional (f, n)-critical if and only if $\delta_G(S, T) \geq f_n(S)$ for all subsets S of $V(G)$ with $|S| \geq n$, where $T = \{x \in V(G) - S : d_{G-S}(x) \leq f(x)\}$.*

Proof of Theorem 5. Suppose that G is not fractional (f, n)-critical, then, by Lemma 2.1, there exists some subset $S \subseteq V(G)$ with $|S| \geq n$ such that

$$\delta_G(S, T) \leq f_n(S) - 1, \tag{2}$$

where $T = \{x : x \in V(G) - S, d_{G-S}(x) \leq f(x)\}$. We choose such subsets S and T so that $|T|$ is as small as possible.

If $T = \emptyset$, then by (1) and (2), $f(S) - 1 \geq f_n(S) - 1 \geq \delta_G(S, T) = f(S)$, a contradiction. Hence, $T \neq \emptyset$. Define

$$h = \min\{d_{G-S}(x) : x \in T\}.$$

Obviously, we have

$$\delta(G) \leq h + |S|. \tag{3}$$

Now we prove the following claim. □

Claim 1. $d_{G-S}(x) \leq f(x) - 1 \leq b - 1$ for each $x \in T$.

Proof. If $d_{G-S}(x) \geq f(x)$ for some $x \in T$, then the subsets S and $T \setminus \{x\}$ satisfy (2). This contradicts the choice of S and T.

This completes the proof of Claim 1.

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In terms of Claim 1, we get

$$0 \leq h \leq b - 1.$$

Since $f(x) \leq b$ for each $x \in V(G)$, it follows from (1) and (2) that

$$bn - 1 \geq f_n(S) - 1 \geq \delta_G(S, T) = f(S) + d_{G-S}(T) - f(T). \tag{4}$$

In the following, we shall consider two cases according to the value of h and derive contradictions. □

Case 1. $1 \leq h \leq b - 1$.

In terms of a hypothesis of the theorem, and using (3), (4), $|S| + |T| \leq p$ and $b - h \geq 1$, we obtain

$$\begin{aligned} bn - 1 &\geq f(S) + d_{G-S}(T) - f(T) \\ &\geq a|S| + h|T| - b|T| \\ &= a|S| - (b - h)|T| \\ &\geq a|S| - (b - h)(p - |S|) \\ &= (a + b - h)|S| - (b - h)p \\ &\geq (a + b - h)(\delta(G) - h) - (b - h)p \\ &\geq (a + b - h)\left(\frac{(b-1)p + a + b + bn - 1}{a + b - 1} - h\right) - (b - h)p, \end{aligned}$$

that is,

$$0 \geq (a + b - h)\left(\frac{(b-1)p + a + b + bn - 1}{a + b - 1} - h\right) - (b - h)p - bn + 1. \tag{5}$$

Multiplying (5) by $(a + b - 1)$ and rearranging, we obtain

$$0 \geq (h - 1)(ap - (a + b - h)(a + b - 1) - bn) + (a + b - 1). \tag{6}$$

Subcase 1.1. $h = 1$.

From (6), we have

$$0 \geq (a + b - 1).$$

This is a contradiction by $1 \leq a \leq b$.

Subcase 1.2. $h = 2$.

Clearly, $b \geq 3$ since $h \leq b - 1$. Using (6) and $p \geq \frac{(a+b-1)(a+b-2)+bn-2}{a}$, we get

$$\begin{aligned} 0 &\geq (h - 1)(ap - (a + b - h)(a + b - 1) - bn) + (a + b - 1) \\ &= (ap - (a + b - 2)(a + b - 1) - bn) + (a + b - 1) \\ &\geq -2 + (a + b - 1) \geq a \geq 1, \end{aligned}$$

which is a contradiction.

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Subcase 1.3. $3 \leq h \leq b-1$.

Obviously, $b \geq 4$. According to (6) and $p \geq \frac{(a+b-1)(a+b-2)+bn-2}{a}$, we have

$$\begin{aligned} 0 &\geq (h-1)(ap-(a+b-h)(a+b-1)-bn)+(a+b-1) \\ &\geq (h-1)(ap-(a+b-3)(a+b-1)-bn)+(a+b-1) \\ &\geq (h-1)(a+b-3)+(a+b-1) \geq 2(a+b-3)+(a+b-1) \\ &= 3(a+b)-7 > 0, \end{aligned}$$

which is a contradiction.

Case 2. $h = 0$.

Let $Y = \{x \in T : d_{G-S}(x) = 0\}$. Obviously, $Y \neq \emptyset$ and Y is independent. Thus, we obtain by the condition of the theorem

$$\frac{(b-1)p+|Y|+bn-1}{a+b-1} < |N_G(Y)| \leq |S|. \tag{7}$$

From (7) and $|S|+|T| \leq p$, we get

$$\begin{aligned} \delta_G(S,T) &= f(S)+d_{G-S}(T)-f(T) \\ &\geq a|S|+d_{G-S}(T)-b|T| \\ &\geq a|S|+|T|-|Y|-b|T| \\ &= a|S|-(b-1)|T|-|Y| \\ &\geq a|S|-(b-1)(p-|S|)-|Y| \\ &= (a+b-1)|S|-(p-1)n-|Y| \\ &> (a+b-1)(\frac{(b-1)p+|Y|+bn-1}{a+b-1})-(b-1)p-|Y| \\ &= bn-1, \end{aligned}$$

which contradicts (4).

From the above contradictions we deduce that G is a fractional (f,n)-critical graph. This completes the proof of Theorem 5.

REMARK. Let us show that the condition $|N_G(X)| > \frac{(b-1)p+|X|+bn-1}{a+b-1}$ in Theorem 5 cannot be replaced by $|N_G(X)| \geq \frac{(b-1)p+|X|+bn-1}{a+b-1}$. Let $b = a \geq 1$, $n \geq 0$, $t \geq 3$ be integers such that $\frac{(t-2)a-1}{2}$ is an integer. Since $a = b$, then we have $f(x) = b$ for each $x \in V(G)$. We construct a graph $G = K_{(b-1)t+n+1} \bigvee (2aK_1 \cup \frac{(t-2)a-1}{2}K_2)$. Clearly, $p = (b-1)t+n+1+at-1 = (a+b-1)t+n$. Set $X = V(2aK_1)$. Then $\delta(G) = (b-1)t+n+1 = \frac{(a+b-1)((b-1)t+n+1)}{a+b-1} = \frac{(b-1)p+a+b+bn-1}{a+b-1} > \frac{(b-1)p+a+b+bn-2}{a+b-1}$ and $|N_G(X)| = (b-1)t+n+1 = \frac{(b-1)p+a+b+bn-1}{a+b-1} = \frac{(b-1)p+2a+bn-1}{a+b-1} = \frac{(b-1)p+|X|+bn-1}{a+b-1}$, and it is easy to see from this that $|N_G(X)| \geq \frac{(b-1)p+|X|+bn-1}{a+b-1}$ for every non-empty independent subset X of $V(G)$. Let $S = V(K_{(b-1)t+n+1}) \subseteq V(G)$, $T = V(2aK_1 \cup \frac{(t-2)a-1}{2}K_2) \subseteq V(G)$. Then $|S| = (b-1)t+n+1 > n$, $|T| = at-1$ and $d_{G-S}(T) = (t-2)a-1$. Thus, we obtain

$$\begin{aligned}\delta_G(S,T) &= f(S) + d_{G-S}(T) - f(T)\\ &= b|S| + d_{G-S}(T) - a|T|\\ &= b((b-1)t + n + 1) + (t-2)a - 1 - a(at-1)\\ &= b((b-1)t + n + 1) + (t-2)b - 1 - b(bt-1)\\ &= bn - 1 < bn = f_n(S).\end{aligned}$$

According to Lemma 2.1, G is not a fractional (f, n)-critical graph. In the above sense, the condition $|N_G(X)| > \frac{(b-1)p+|X|+bn-1}{a+b-1}$ in Theorem 5 is the best possible condition.

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